Conceptual Connections of Circularity and Category Theory (Part II)

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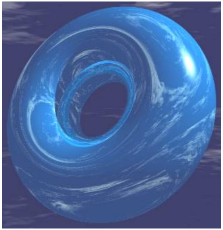
ESSLLI 2012, Opole



set with algebraic operations	set with transitions and observations
algebra for a functor	coalgebra for a functor
initial algebra	final coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence rel'n
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a final coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
useful in syntax	useful in semantics
bottom-up	top-down

Take a roulette wheel labeled with points in $[0, 1]$.
Spin it successively, until the total of the spins is ≥ 1.

It might happen in 2 spins, or 3, or 6238.

What is the average number of spins that it would take to get a total of > 1?

Let $E(t) =$ the average number of spins that it would take to get a total of $> t$.

So $E(0) = 1$.

How can we get a formula for $E(t)$?

Fix a number t.

If we spin the wheel once, we get some number, say x.

If $x > t$, we're done on the first spin.

If $x \leq t$, we need to continue.
How many further spins are needed, on average?
For $x \leq t$, we on average will need $E(t - x)$.

We would want to take the probability of getting x, and then multiply it by $1 + E(t - x)$.

But the probability of getting x exactly is 0, and thus we integrate.

$$\begin{aligned} E(t) &= \int_t^1 1\, dx \quad + \quad \int_0^t 1 + E(t-x)\, dx \\ &= 1 + \int_0^t E(t-x)\, dx \\ &= 1 + \int_0^t E(u)\, du \end{aligned}$$

(We made a substitution $u = t - x$.)

By the Fundamental Theorem of Calculus, $E'(t) = E(t)$.

Combined with $E(0) = 1$, we see that

$$E(t) \quad = \quad e^t,$$

and the answer to the original problem is e.

Let $(A, a : FA \to A)$ and $(B, b : FB \to B)$ be algebras.
A morphism in the algebra category of F is $f : A \to B$ in the same underlying category so that

$$\begin{tikzcd} FA \arrow[r, "a"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FB \arrow[r, "b"'] & B \end{tikzcd}$$

commutes.

Let $(A, a : A \to FA)$ and $(B, b : B \to FB)$ be coalgebras.
A morphism in the coalgebra category of F is $f : A \to B$ in the same underlying category so that

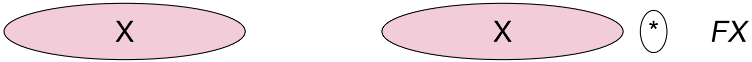
$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "b"'] & FB \end{tikzcd}$$

commutes.

$FX = 1 + X$ on Set

1 here is *any* one-element set, and to emphasize the arbitrariness
one often writes it as $\{*\}$.

$+$ is disjoint union, the categorical coproduct.



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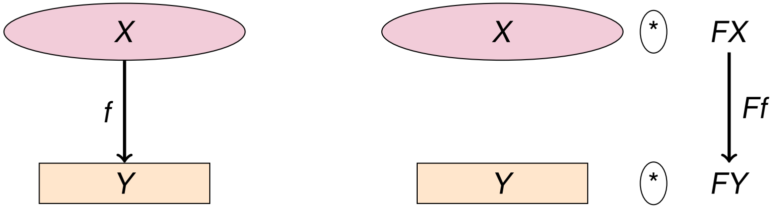

Ff is defined to preserve the new points.

For $FX = 1 + X$ on Set,

- Initial algebra is (N, ν), with N the set of natural numbers, and $\nu : 1 + N \to N$ given by

$$\begin{aligned} * &\mapsto 0 \\ n &\mapsto n + 1 \end{aligned}$$

  Initiality "is" recursion, the most important definition principle for functions on numbers.

- Final algebra is a one point set.
- Initial coalgebra is the empty set.
- Final coalgebra is $N^\infty = N + \{\infty\}$.
  What do you think the structure map $N^\infty \to 1 + N^\infty$ is??

Consider an algebra $(A, a : 1 + A \to A)$, where $A = \{\alpha, \beta, \gamma, \delta\}$, and

$$\begin{array}{lclclcl} a(*) & = & \gamma & \quad & a(\gamma) & = & \delta \\ a(\alpha) & = & \beta & & a(\delta) & = & \alpha \\ a(\beta) & = & \gamma & & & & \end{array}$$

Query: What is the map h below?

$$\begin{tikzcd} 1 + N \arrow[r, "\nu"] \arrow[d, "Fh"'] & N \arrow[d, "h"] \\ 1 + A \arrow[r, "a"'] & A \end{tikzcd}$$

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It is

$$0 \mapsto \gamma \quad 1 \mapsto \delta \quad 2 \mapsto \alpha \quad 3 \mapsto \beta \quad 4 \mapsto \gamma \quad 1 \mapsto \delta \quad \ldots$$

h is defined by recursion.

Recursion on N: For all sets A, all $a \in A$, and all $f : A \to A$, there is a unique $\varphi : N \to A$ so that $\varphi(0) = a$, and $\varphi(n+1) = f(\varphi(n))$ for all n.

Initiality of N: For all (A, a), there is a unique homomorphism $\varphi : (N, \nu) \to (A, a)$:

$$\begin{tikzcd} 1 + N \arrow[r, "\nu"] \arrow[d, "1+\varphi"'] & N \arrow[d, "\varphi"] \\ 1 + A \arrow[r, "a"'] & A \end{tikzcd}$$

In other words, $1 + N \to N$ is an initial algebra of $FX = 1 + X$.

These are equivalent in set theory without Infinity.

Let's study $FX = 1 + X$ on another category

MS is category of metric spaces (X, d) with distances ≤ 1, and non-expanding maps:

$$d(fx, fy) \leq d(x, y)$$

$FX = 1 + X$ is the disjoint union of X with a one-point space, with distance 1 to the new point.

Initial algebra is the discrete metric space on the natural numbers.

Let's study $FX = 1 + \frac{1}{2}X$ on MS

$\frac{1}{2}X$ is the space X, but with distances scaled by $\frac{1}{2}$.

$$(X, d) \quad \mapsto \quad (X, \tfrac{1}{2}d) \qquad *$$

$d(*, x) = 1$ for $x \in X$

Initial algebra of this F turns out to be

$$((\mathrm{Nat}, d), \varphi),$$

where Nat is the set of natural numbers, and for $n \neq m$,

$$d(n, m) = 2^{-\min(n,m)}.$$

The structure $\varphi : 1 + \frac{1}{2}(\mathrm{Nat}, d) \to (\mathrm{Nat}, d)$ is the same as for N.

CMS is the subcategory of *complete* metric spaces.
(Again, distances bounded by 1, and non-expanding maps.)

Initial algebra of $1 + X$ is same as for MS: $(N, d_{discrete})$.

For $1 + \frac{1}{2}X$: it's N^{∞}, again with

$$d(n, m) = 2^{-\min(n,m)},$$

treating ∞ as larger than all $n \in N$.

So it's a Cauchy sequence together with its limit.

$1 + X$ AND $1 + \frac{1}{2}X$ ON KMS

KMS is the subcategory of compact metric spaces.
(Again, distances bounded by 1, and non-expanding maps.)

Initial algebra of $1 + X$ doesn't exist.

For $1 + \frac{1}{2}X$: it's the same space as for this functor on CMS.

Let's go back to $F(X) = X \times X$ on Set.

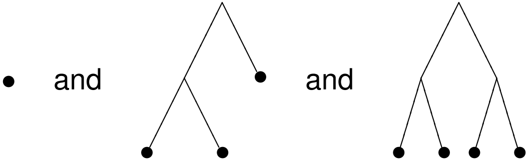
Initial algebra is the empty set $\emptyset \times \emptyset = \emptyset$
together with the empty function $F\emptyset \to \emptyset$.

Final coalgebra is a singleton $1 = \{*\}$.
Note that $1 \times 1 = 1$, and the structure $1 \to 1 \times 1$ is clear.

$F(X) = (X \times X) + 1$ ON Set

This one is more interesting!

Initial algebra is the set T of all finite binary trees.
For example,



What do you think the structure map $(T \times T) + 1 \to T$ is?
What do you think the initiality property comes to?

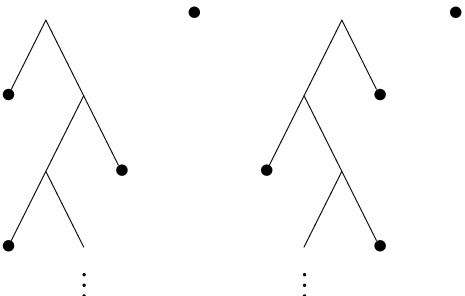
Final coalgebra is the set T of all finite and infinite binary trees.

Let's see an example.
Consider $S = \{x, y, z, w\}$ and $m : S \to FS$

$$\begin{array}{rclcrcl} m(x) & = & \langle y, z \rangle & \quad & m(z) & = & \langle x, w \rangle \\ m(y) & = & \bullet & & m(w) & = & \bullet \end{array}$$

The semantics here is $m^\dagger : S \to Trees$

$m^\dagger(x)$ $m^\dagger(y)$ $m^\dagger(z)$ $m^\dagger(w)$



$F(X) = A \times X$ on Set

Here A is a fixed set.

The initial algebra is the empty set.

The final coalgebra is the set of streams on A.

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If we change to $F(X) = (A \times X) + 1$, then what would we get?

$\mathbb{R}$ here is the set of real numbers.
For $FX = \mathbb{R} \times X$ on Set,

- Initial algebra is the empty set
- Another representation of the terminal coalgebra:
  Let RA be the set of functions which are real analytic at 0:
  $f^{(n)}(0)$ exists for all n, and f agrees with its Taylor series in a neighborhood of 0.

  The coalgebra structure $\varphi : RA \to \mathbb{R} \times RA$ is given by

$$f \quad \mapsto \quad (f(0), f').$$

Consider a coalgebra $(A, a : A \to \mathbb{R} \times A)$, where $A = \{\alpha, \beta, \gamma, \delta\}$, and

$$\begin{array}{rclcrcl} a(\alpha) & = & (0, \beta) & \qquad & a(\gamma) & = & (0, \delta) \\ a(\beta) & = & (1, \gamma) & \qquad & a(\delta) & = & (-1, \alpha) \end{array}$$

Query: What is the map h below?

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] & \mathbb{R} \times A \arrow[d, "Fh"] \\ RA \arrow[r, "\varphi"'] & \mathbb{R} \times RA \end{tikzcd}$$

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It is

$$\alpha \mapsto \sin\ x, \quad \beta \mapsto \cos\ x, \quad \gamma \mapsto -\sin\ x, \quad \gamma \mapsto -\cos x$$

h is defined by corecursion.

Let $FX = A \times X \times X$.

The final coalgebra is the set C of infinite binary trees with all points labeled by an element of A.

We have a structure $c : C \to A \times C \times C$.

The trees are ordered, with a left child and a right child.

Let $swap : T \times T \to T \times T$ be

$$swap(\langle t, u\rangle) = \langle u, t\rangle.$$

Now consider

$$T \xrightarrow{c} A \times T \times T \xrightarrow{id \times swap} A \times T \times T$$

Still working with $FX = A \times X \times X$

$$T \xrightarrow{c} A \times T \times T \xrightarrow{id \times swap} A \times T \times T$$

It's a coalgebra for F.

So what is its map into the final coalgebra??

$$\begin{tikzcd}
T \arrow[r, "c"] \arrow[d, "mirror"'] & A \times T \times T \arrow[r, "id \times swap"] & A \times T \times T \arrow[d, "id_A \times mirror \times mirror"] \\
T \arrow[rr, "c"'] & & A \times T \times T
\end{tikzcd}$$

How would we define the left and right branch of a tree?

Let $F(X) = A \times X \times X$, with final coalgebra $c : C \to FC$.

Let $G(X) = A \times X$, with final coalgebra A^∞, the set of streams on A.

How can we define $lb : C \to A$?

$$\begin{tikzcd}
T \arrow[r, "c"] \arrow[d, "lb"'] & A \times T \times T \arrow[r, "id \times \pi_1"] & A \times T \arrow[d, "id_A \times lb"] \\
C \arrow[rr, "c"'] & & A \times C
\end{tikzcd}$$

We write down the coalgebra on the top, and then lb comes automatically
by finality of C as a G-coalgebra.

What is the relation of *lb*, *rb* (right branch), and *mirror*?

$[0,1]$ is a final coalgebra, but one has to consider the right category.

$[0,1]$ is a bi-pointed set, with 0 as $\perp$ and 1 as $\top$.

We use the "smash" functor $F(X) = X \oplus X$ from yesterday.

The structure $[0,1] \to [0,1] \oplus [0,1]$ is

$$\begin{array}{ccl} x < \frac{1}{2} & \mapsto & 2x \text{ on left} \\ x > \frac{1}{2} & \mapsto & 2x - 1 \text{ on right} \\ \frac{1}{2} & \mapsto & 1 \text{ on left} = 0 \text{ on right} \end{array}$$

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THEOREM (FREYD)

$[0,1]$ is a final coalgebra of F.

The initial algebra is the set of dyadic rationals: ($a/2^n$, with $0 \leq a < 2^n$) with the evident structure.

We have seen that the final coalgebra of $F(X) = A \times X \times X$ is the set of infinite binary trees labeled in A.

There is another presentation of this coalgebra, using a different carrier set.

We take the set A^∞ of streams (!) on A.
The structure

$$A^\infty \to A \times A^\infty \times A^\infty$$

is the inverse of

$$A \times A^\infty \times A^\infty \to A^\infty$$

given by

$$(a, s, t) \quad \mapsto \quad (a, zip(s, t))$$

Back to $F(X) = A \times X \times X$

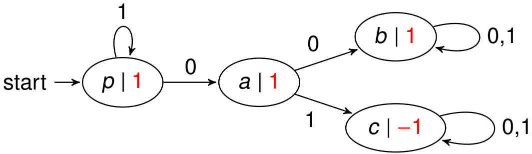
For $A = \{1, -1\}$, consider the coalgebra $e : X \to FX$
where $X = \{p, a, b, c\}$,
and

$$\begin{array}{rcl} e(p) & = & (1, a, p) \\ e(a) & = & (1, b, c) \\ e(b) & = & (1, b, b) \\ e(c) & = & (-1, c, c) \end{array}$$

By finality, we get a unique $e^\dagger : X \to A^\infty$ so that

$$\begin{array}{rcl} e^\dagger(p) & = & (1, \mathrm{zip}(a, p)) \\ e^\dagger(a) & = & (1, \mathrm{zip}(b, c)) \\ e^\dagger(b) & = & (1, \mathrm{zip}(b, b)) \\ e^\dagger(c) & = & (-1, \mathrm{zip}(c, c)) \end{array}$$

And of course $e^\dagger(p)$ is paperfolding sequence.



LEMMA (LAMBEK'S LEMMA)

Let C be a category, let $F : C \to C$ be a functor, and let (a, f) be an initial algebra for F.

Then f is an isomorphism: there is a morphism $g : Fa \to a$ such that $g \cdot f = id_a$ and $f \cdot g = id_{Fa}$.

The same statement holds for final coalgebras of F.

Isomorphisms in Set are exactly the bijections.

And there are no maps from any set onto its power set (Cantor's Theorem).

Together with Lambek's Lemma, we see that $\mathcal{P}$ on Set has no initial algebra and no final coalgebra.

To get around this, one either

1. moves from Set to the category Class.
2. moves from $\mathcal{P}$ to $\mathcal{P}_{fin}$, the functor giving the finite subsets.

Consider the category $\mathcal{A}$ of classes.
$\mathcal{P} : \mathcal{A} \to \mathcal{A}$ gives the class of sub*sets* of a given class.
Note that $\mathcal{P}V = V$.

Work in *ZF* – *Foundation*

The Foundation Axiom is equivalent to the assertion that

$$(V, id : \mathcal{P}V \to V)$$

is an initial algebra of $\mathcal{P}$.

The Anti-Foundation Axiom is equivalent to the assertion that

$$(V, id : V \to \mathcal{P}V)$$

is a final coalgebra of $\mathcal{P}$.

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One way to construct a final coalgebra of $\mathcal{P}_{fin}$
is to take the disjoint union of all coalgebras
and then take the quotient by some equivalence relation.

$\mathcal{P}_{fin}$-coalgebras are finitely branching graphs, but with a different
notion of morphism than one might at first expect.

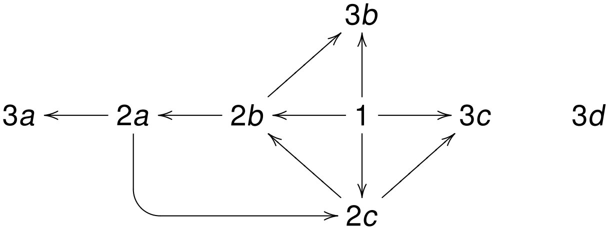
The natural equivalence notion is maximal bisimulation.

Let $(G, \to)$ be a graph. A relation R on G is a bisimulation iff the following holds: whenever xRy,

A. If $x \to x'$, then there is some $y \to y'$ such that $x'Ry'$.

B. If $y \to y'$, then there is some $x \to x'$ such that $x'Ry'$.

For an example, let's look at the following graph G:



The largest bisimulation is the one that relates 1 to itself,
all 2-points to all 2-points,
and all 3-points to all 3-points.

Take the disjoint union of all finitely branching graphs
and then take the quotient by the largest bisimulation.

It can be shown that finitely branching graph sits inside
in a unique way,
and so we have a final coalgebra.

There are generalizations of this result, of course.
If the functor has nice properties
(it commutes with weak pullbacks)
the theory of bisimulation generalizes nicely.
(The resulting subject is called universal coalgebra.)

Take the disjoint union of all finitely branching graphs
with node set included in the natural numbers
and then take the quotient by the largest bisimulation.

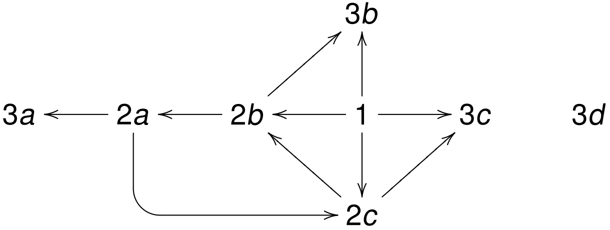
It can be shown that finitely branching graph sits inside
in a unique way,
and so we have a final coalgebra.

There are generalizations of this result, of course.
If the functor has nice properties
(it commutes with weak pullbacks)
the theory of bisimulation generalizes nicely.
(The resulting subject is called universal coalgebra.)

The modal sentences are the smallest collection containing a constant true and closed under the boolean $\neg$, $\wedge$, and $\vee$ and a unary modal operator $\Box$.
That is, the modal sentences are the initial algebra of a functor related to the signature $H_{\Sigma_{\mathrm{modal}}}$, where Σ_{modal} contains true, $\neg$, $\wedge$, $\Box$.
Given a $\mathcal{P}$-coalgebra (X, e), we define $x \models \varphi$, by recursion on $\mathcal{L}$ as follows:

$x \models \mathrm{true}$		always
$x \models \mathrm{false}$		never
$x \models \neg\varphi$	iff	it is not the case that $x \models \varphi$
$x \models \varphi \wedge \psi$	iff	$x \models \varphi$ and $x \models \psi$
$x \models \Box\varphi$	iff	for all $y \in e(x)$, $y \models \varphi$

Modal logic: examples



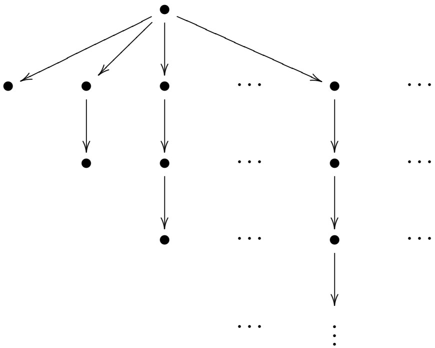
$3a \models \text{true} \wedge \Box\text{false}$
$2b \models (\Diamond\Box\text{false}) \wedge \Diamond\Diamond\text{true}$.

The theory of a point is the set of modal sentences it satisfies.
Bisimilar points have the same theory (but not conversely)
But in finitely branching graphs, points with the same theory are bisimilar.
(the Hennessey-Milner property).

It is the set of all theories of all points in all coalgebras.
This means: the theories of all points in finitely branching models.

The structure is familiar from modal logic:
take a theory T to the set of theories U such that
if $\Box\varphi \in T$, then $\varphi \in U$.

It is the set of all theories of all points in all coalgebras.
This means: the theories of all points in finitely branching models.
So it would exclude the theory of the top point in



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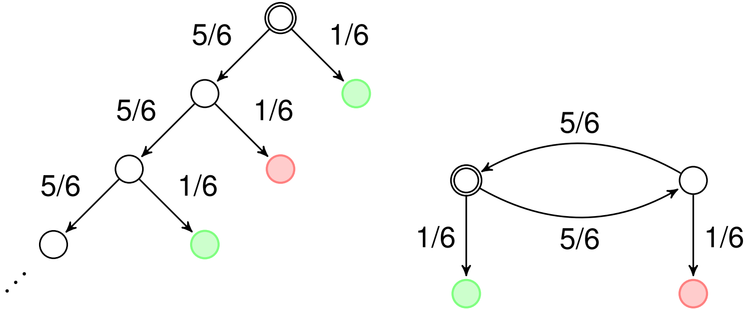

The second is the quotient of the first
by the maximum coalgebraic bisimulation.

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