Intersecting Adjectives in Syllogistic Logic

Lawrence S. Moss

Department of Mathematics, Indiana University, Bloomington IN 47405 USA.
lsm@cs.indiana.edu.

Abstract. The goal of natural logic is to present and study logical systems for reasoning with sentences of (or which are reasonably close to) ordinary language. This paper explores simple systems of natural logic which make use of intersecting adjectives; these are adjectives whose interpretation does not vary with the noun they modify. Our project in this paper is to take one of the simplest syllogistic fragments, that of *all* and *some*, and to add intersecting adjectives. There are two ways to do this, depending on whether one allows iteration or prefers a "flat" structure of at most one adjective. We present rules of inference for both types of syntax, and these differ. The main results are four completeness theorems: for each of the two types of syntax we have completeness for the *all* fragment and for the full language of this paper.
keywords: syllogistic logic,completeness, adjectives, transitive relations.

1 Introduction: Intersecting Adjectives

By "natural logic" I mean the study of logical systems designed to model linguistic inference in a manner which is as "close to the surface" as possible. The idea is to study inference in language on its own terms, and hopefully to obtain sound and complete systems for linguistic inference which are also decidable. This contrasts with approaches that go via translation to first-order logic because first-order logic is undecidable, and because work done via translation does not yield logical systems in the first place.

Among the simplest kind of logical systems of the type studied in this paper are ones derived from the classical syllogistic. These are extremely small logical systems, containing as sentences only expressions of the form *all p and q* and *some p are q*. The classical syllogistic can be viewed as a logical system, and then one could study its properties. The earliest work on this topic may be found in Łukasiewicz [2], and the goal there was to propose a modern reconstruction of the ancient sources of logic. In contrast, most of the contemporary interest in the topic is aimed at other matters: decidable fragments of language; alternatives to model-theoretic semantics based on proof theory; and logical systems for human reasoning. For examples, see Nishihara et. al [5] as well as [3, 4, 6].

The main new point in this paper concerns a class of adjectives call *intersecting* adjectives. This class includes the color adjectives, also *male* and *female*, and frequently also nationality adjectives such as *Xhosa* and *Yoruba*. Intersecting adjectives have two defining features, and as we shall see these features are closely related. The first is a *proof-theoretic* feature noted by Keenan and Faltz [1], p. 123:

> The sense in which an intersecting adjective determines a property can be described as follows: If Dana is a female student and Dana is also an athlete, then Dana is a female athlete. (1)

The second feature of the intersecting adjectives is more *semantic*: in a standard model-theoretic semantics, the interpretation of a phrase such as *female shopkeeper* would be the intersection of the interpretation of *shopkeeper* (some subset of the underlying universe of discourse) with a set of "female individuals". In this respect, the intersecting adjectives differ from the larger class of adjectives. To recall an oft-made point, consider a (non-intersecting) adjective such as *tall*. It may well be that a person could simultaneously be a tall student but not a tall basketball player. And this would mean that *tall* lacks both the proof-theoretic and model-theoretic features of the intersecting adjectives. That is, the statement of Keenan and Faltz above would be false with *tall* replacing *female*, and it also would not be sensible to interpret *tall student* in a model by intersecting the interpretation of *student* with a fixed set interpreting *tall*.

We are interested in syllogistic inferences using the intersecting adjectives in addition to the determiners *all* and *some*. To make things precise, we must settle on a formal syntax and semantics. However, though the fragment of interest is very small indeed, the syntax already gives us pause. For intersecting adjectives can iterate, as in *The driver was a gay Albanian with a brown-spotted partly-grey white dog.* In fact, although color adjectives do not usually iterate on their own, if one adds words like "partly", then we do get iteration: *The partly blue, partly red, partly green ball was lost in the attic*. For this reason, we propose two versions of the syntax. First, a flat syntax where nouns are either basic or contain an iterating adjective (Section 2). We call the languages of that section $\mathcal{L}(\forall, adj)$ and $\mathcal{L}(\forall, \exists, adj)$. We give a proof theory and completeness theorem for this language before turning to our second syntax, the languages $\mathcal{L}_r(\forall, adj)$ and $\mathcal{L}_r(\forall, \exists, adj)$ (Section 3).

It might be interesting to mention that the rules of inference of our systems are indirectly based on the formulation of Keenan and Faltz from (1). (We say "indirectly" because our logical languages do not have proper nouns.) In a sense, one could state the basic issue of this paper: is (1) *all* that one could generally say about intersecting adjectives with the standard semantics? Does everything else follow from (1), together with more general facts about *all* and *some*? Or are there yet other logical principles waiting to be discovered? We shall return to this point at the end of the paper.

2 $\mathcal{L}(\forall, adj)$ and $\mathcal{L}(\forall, \exists, adj)$: Non-productive Syntax

Our syntax begins with *basic nouns* $x_1, x_2, \ldots$ and then adds *intersecting adjectives* $a_1, a_2, \ldots$ We then define the set of *nouns*, and denote nouns by letters like

n, p, and q, by saying that the basic nouns are nouns, and if x is a noun and a an intersecting adjective, then $a\ x$ is a noun. We call nouns of the form $a\ x$ *complex nouns*. This is a very simple model of predication. It is also *non-productive* in the sense that nouns may contain only zero or one adjective, not more. Later in the paper we shall explore the possibility of re-working the syntax so that predication is productive (see Section 3).

In what follows, we usually "abbreviate" the intersecting adjectives with color adjectives *red*, *blue*, and *green*. This helps to avoid subscripts, and it seems to improve readability.

At first, the only sentences which we consider are those of the form $\forall(p, q)$, read as *all p are q*. The collection of these sentences is called $\mathcal{L}(\forall, adj)$. Later, we'll expand this to a language $\mathcal{L}(\forall, \exists, adj)$ by adding sentences *some p are q*.

Our semantics for $\mathcal{L}(\forall, adj)$ is based on *models* $\mathcal{M}$ for the fragment. A model consists of a set M, subsets $[\![x]\!] \subseteq M$ for each basic noun x, and sets $[\![a]\!]$ for the intersecting adjectives. Then we define the semantics of a noun $a\ x$ by $[\![a\ x]\!] = [\![a]\!] \cap [\![x]\!]$.

We define the relation of truth between models and sentences in the obvious way: $\mathcal{M} \models \forall(p, q)$ iff $[\![p]\!] \subseteq [\![q]\!]$. Then we say that $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models S$ for all sentences $S \in \Gamma$. The main semantic definition is given by $\Gamma \models S$ if for all models $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then also $\mathcal{M} \models S$. The first logical question about this semantic notion is whether there is a matching proof-theoretic counterpart.

2.1 $\mathcal{L}(\forall, adj)$: All and Intersecting Adjectives

The simplest syllogistic fragment "of all" is simply the collection of sentences of the form *All n are p*, where n and p are nouns. We shall call this language $\mathcal{L}(\forall, adj)$, and later we shall expand it to $\mathcal{L}(\forall, \exists, adj)$. Previous work studied the case of nouns without modifiers, and in this paper we allow nouns to be modified by intersecting adjectives. A logical system for $\mathcal{L}(\forall, adj)$ is presented in Figure 1. The rules (T) and (B) are standard in syllogistic logic. (T) reflects our decision to let *all p are p* statements be valid (i.e., true in all models), regardless of whether a given model has p or not. The rule (B) gets its name from the classical syllogism *Barbara*.

We shall not give a precise definition of a *proof tree* in a syllogistic logic, but the idea is that it should be a tree labeled with sentences, all of whose internal nodes match one of the rules of the logic. For a more precise definition, see Pratt-Hartmann and Moss [6]. The examples throughout this paper should help make this clear. If $\Gamma \cup \{S\}$ is a set of sentences in this fragment, we write $\Gamma \vdash S$ to mean that there is a proof tree whose root is labeled S and whose leaves are labeled with sentences in the set Γ.

This same definition works for all our fragments and all logics. All of our systems are *sound*: if $\Gamma \vdash S$, then $\Gamma \models S$. This easy point is shown by induction on derivations.

$$\frac{}{\forall(n,n)}\ (\mathrm{T}) \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ (\mathrm{B})$$

$$\frac{}{\forall(\mathit{red}\ x, x)}\ (\mathrm{Adj}_1) \qquad \frac{\forall(n, \mathit{red}\ x) \quad \forall(n,y)}{\forall(n, \mathit{red}\ y)}\ (\mathrm{Adj}_2)$$

Fig. 1. The logic for the fragment $\mathcal{L}(\forall, \mathit{adj})$ of sentences $\forall(n,p)$, read as *all n are p*. Note that x and y denote basic nouns, and n, p, and q denote noun which are either basic or complex.

Example 2.1. $\forall(x, \mathit{red}\ y) \vdash \forall(x, \mathit{red}\ x)$. In words, if all x are red y, then all x are red x (hence red objects). Here is a derivation:

$$\frac{\forall(x, \mathit{red}\ y) \qquad \dfrac{}{\forall(x,x)}\ (\mathrm{T})}{\forall(x, \mathit{red}\ x)}\ (\mathrm{Adj}_2)$$

Perhaps the most interesting single-premise inference available in this system is the following *monotonicity* fact.

Example 2.2. $\forall(x,y) \vdash \forall(\mathit{red}\ x, \mathit{red}\ y)$.

The derivation is indicated below:

$$\frac{\dfrac{}{\forall(\mathit{red}\ x, \mathit{red}\ x)}\ (\mathrm{T}) \qquad \dfrac{\dfrac{}{\forall(\mathit{red}\ x, x)}\ (\mathrm{Adj}_1) \qquad \forall(x,y)}{\forall(\mathit{red}\ x, y)}\ (\mathrm{B})}{\forall(\mathit{red}\ x, \mathit{red}\ y)}\ (\mathrm{Adj}_2)$$

Theorem 2.3. *The logic of Figure 1 is complete for $\mathcal{L}(\forall, \mathrm{adj})$: if $\Gamma \models \forall(n,p)$, then $\Gamma \vdash \forall(n,p)$.*

Proof. Suppose that $\Gamma \models \forall(n,p)$; we show that $\Gamma \vdash \forall(n,p)$. Consider a model $\mathcal{M}$ whose universe M is a singleton $\{*\}$, and whose structure is given by

$$[\![x]\!] = \begin{cases} \{*\} & \text{if } \Gamma \vdash \forall(n,x) \\ \emptyset & \text{if } \Gamma \nvdash \forall(n,x) \end{cases}$$

$$[\![\mathit{red}]\!] = \begin{cases} \{*\} & \text{if for some basic noun } x,\ \Gamma \vdash \forall(n, \mathit{red}\ x) \\ \emptyset & \text{otherwise} \end{cases}$$

These definitions are made using the specific noun n from our overall statement of the theorem.

We first claim that $\mathcal{M} \models \Gamma$. Take a sentence in Γ such as $\forall(l_1, l_2)$. We have four cases, depending on whether l_1 and l_2 are basic or complex nouns. The most interesting is when l_1 is *red* x and l_2 is *blue* y. Again, we must show that $[\![red\ x]\!] \subseteq [\![blue\ y]\!]$. For this, we may assume that $[\![red\ x]\!] \neq \emptyset$; otherwise, we trivially have the desired conclusion. Hence $[\![x]\!] = \{*\}$, so $\Gamma \vdash \forall(n, x)$. Also, $[\![red]\!]$ must be $\{*\}$, so for some z, $\Gamma \vdash \forall(n, red\ z)$. Using (Adj$_2$), $\Gamma \vdash \forall(n, red\ x)$. Thus $\Gamma \vdash \forall(n, blue\ y)$. Using (Adj$_1$) and (B), we have $\Gamma \vdash \forall(n, y)$. So $* \in [\![blue]\!] \cap [\![y]\!] = [\![blue\ y]\!]$. This completes the proof of our claim.

We have verified that $\mathcal{M} \models \Gamma$. Recalling that $\Gamma \models \forall(n, p)$, we have $\mathcal{M} \models \forall(n, p)$. We again have four cases, and we only mention two of them. First, in case n is a basic noun x, we have $* \in [\![n]\!]$ by (T). Then $* \in [\![p]\!]$ as well, and this means that $\Gamma \vdash \forall(n, p)$. Second, assume that n is of the form *red* z also. Then using (Adj$_1$), $* \in [\![n]\!]$. Hence again $* \in [\![p]\!]$. We only deal with the case that p is of the form *blue* w. So $\Gamma \models \forall(n, w)$; also, for some basic noun z, $\Gamma \vdash \forall(n, blue\ z)$. By (Adj$_2$), $\Gamma \vdash \forall(n, blue\ w)$. That is, $\Gamma \vdash \forall(n, p)$. This completes the proof.

2.2 Proof Rules for *some* and Intersecting Adjectives

Next, we add sentences $\exists(p, q)$ to our fragment. We call the resulting language $\mathcal{L}(\forall, \exists, adj)$. The obvious semantics is to say that in a model $\mathcal{M}$ we have $\mathcal{M} \models \exists(p, q)$ just in case $[\![p]\!] \cap [\![q]\!] \neq \emptyset$. We aim to study the semantic consequence relation $\Gamma \models S$, and especially to associate with it a sound and complete proof system. Figure 2 provides some sound inference rules, and the system that we study has as its rules the rules in Figures 1 and 2.

The first two rules in Figure 2 come from syllogistic logic. Forgetting the adjectives for a moment, the rules (T), (B), (I), and (D) are complete for the language of sentences *all* p *are* q and *some* p *are* q; see [3], Theorem 4. There are related results in Łukasiewicz [2] and Westerståhl [7]. The name (D) comes from its name in classical syllogistics, *Darii*. The “twisted” form our formulation of (D) implies that the *conversion* property of *some* is derivable: $\exists(n, p) \vdash \exists(p, n)$. (For this, take $p = q$ in (D).)

In the remainder of this paper, we use conversion frequently, and usually without mention.

Example 2.4. $\exists(red\ x, blue\ y) \vdash \exists(blue\ x, red\ y)$. Use (Adj$_5$), taking *green* to be *red*, and z to be x.

Example 2.5. $\exists(x, red\ y) \vdash \exists(y, red\ x)$. On the left is a derivation of this *reciprocity rule*:

$$\dfrac{\dfrac{\exists(x, red\ y)}{\exists(red\ x, red\ y)}\ (\mathrm{Adj}_3) \qquad \dfrac{}{\forall(red\ y, y)}\ (\mathrm{Adj}_1)}{\exists(red\ x, y)}\ (\mathrm{D})$$

Example 2.6. $\exists(red\ x, blue\ y) \vdash \exists(blue\ x, red\ x)$. Use (Adj$_4$), taking *green* to be *red*, and z to be x.

$$\frac{\exists(n,p)}{\exists(n,n)}\ (\mathrm{I}) \qquad \frac{\exists(n,q) \quad \forall(q,p)}{\exists(p,n)}\ (\mathrm{D}) \qquad \frac{\exists(x, \mathit{red}\ y)}{\exists(\mathit{red}\ x, \mathit{red}\ y)}\ (\mathrm{Adj}_3)$$

$$\frac{\exists(\mathit{red}\ x, \mathit{blue}\ y) \quad \forall(\mathit{red}\ x, \mathit{green}\ z)}{\exists(\mathit{red}\ x, \mathit{blue}\ z)}\ (\mathrm{Adj}_4)$$

$$\frac{\exists(\mathit{red}\ x, \mathit{blue}\ y) \quad \forall(\mathit{red}\ x, \mathit{green}\ z)}{\exists(\mathit{blue}\ x, \mathit{green}\ y)}\ (\mathrm{Adj}_5)$$

Fig. 2. Additions to Figure 1 for the larger language $\mathcal{L}(\forall, \exists, \mathit{adj})$ which contains sentences $\exists(p,q)$.

Example 2.7. $\exists(x,y), \forall(x, \mathit{red}\ z) \vdash \exists(x, \mathit{red}\ y)$. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\exists(x,y)}{\exists(y,x)} \quad \dfrac{\forall(x,\mathit{red}\ z)}{\forall(x,\mathit{red}\ x)}\ \mathit{Example}\ 2.1}{\exists(y,\mathit{red}\ x)}\ (\mathrm{D})}{\exists(x,\mathit{red}\ y)}\ \mathit{Example}\ 2.5$$

In what follows, a *sequent* is a pair $\sigma = (\Gamma, S)$ consisting of a set of sentences (of some fragment under discussion) and a sentence S of it. The set Γ is the set of *premises* of the sequent σ. The sequent is *valid* if $\Gamma \models S$. Note that completeness of a logical system is just the statement that every valid sequent is provable in the system.

Proposition 2.8. *Let σ be a sequent in $\mathcal{L}(\forall, \exists, \mathrm{adj})$ with one or two premises. If σ is valid, then it is provable.*

For the verification, we begin with the one-premise sequents. The only valid ones are listed below, along with reasons for why they are provable:

1. $\forall(x,y) \models \forall(x,x)$: use (Adj_1).
2. $\forall(x, \mathit{red}\ y) \models \forall(x, \mathit{red}\ x)$: use (Adj_2) as in Example 2.7.
3. $\exists(x,y) \models \exists(y,x)$: use (D) with $n = x$ and $p = y = q$.
4. $\exists(\mathit{red}\ x, n) \models \exists(x,x)$: use (I) to get $\exists(\mathit{red}\ x, \mathit{red}\ x)$, and then use (D) and (Adj_1) to get $\exists(\mathit{red}\ x, x)$. Then use conversion and (I).
5. $\exists(x, \mathit{red}\ y) \models \exists(y, \mathit{red}\ x)$: see Example 2.5.
6. $\exists(\mathit{red}\ x, \mathit{blue}\ y) \models \exists(\mathit{blue}\ x, \mathit{red}\ y)$: see Example 2.4.
7. $\exists(\mathit{red}\ x, \mathit{blue}\ y) \models \exists(\mathit{blue}\ x, \mathit{red}\ x)$: see Example 2.6.

We mentioned that these are the “only valid” sequents. To show that a given sequent is *not* valid, one may use a semantic argument. We shall see some of these below.

There are a few others, and they are all minor modifications on the list above. For example, one can conclude $\forall(x,x)$ from any premise.

We next turn to the rules with two premises. For rules with two universal premises, the only sound ones are instances of (B), perhaps also involving monotonicity. For example,

$$\forall(x, y), \forall(\textit{red } y, \textit{blue } z) \models \forall(\textit{red } x, \textit{blue } z).$$

And in our system we have a corresponding proof: use monotonicity (Example 2.2) and the first premise to see $\forall(\textit{red } x, \textit{red } y)$, and then use (B) with this and the other premise.

Concerning two existential premises, it is easy to see that if S, T, and U are existential and $S, T \models U$, then either $S \models U$ or $T \models U$. For if not, take a model $\mathcal{M}$ of S but not U, and a model $\mathcal{N}$ of T but not U, and then take the disjoint union. This would satisfy S and T but not U, a contradiction.

The main work concerns the case when one premise is existential and the other is universal. In what follows, we are only going to consider existential consequences.

The first two premise forms are as follows:

1. $\exists(x, y), \forall(\textit{red } x, n)$.
2. $\exists(x, y), \forall(x, \textit{blue } z)$.

There are no sound conclusions of form (1) beyond what one can infer from the existential premise $\exists(x, y)$. The second form has as sound conclusions $\exists(x, \textit{blue } y)$, and this was treated in Example 2.7. The other sound conclusions are $\exists(x, \textit{blue } z)$ (this is easy), and $\exists(y, \textit{blue } z)$ (this comes from (D)).

The next forms would involve an existential sentence with one adjective, say $\exists(\textit{red } x, y)$. But this has the same models as $\exists(\textit{red } x, \textit{red } y)$, and it is interderivable with it. So we may proceed to forms with premises containing a sentence of the form $\exists(\textit{red } x, \textit{blue } y)$. The relevant forms continue thus:

3. $\exists(\textit{red } x, \textit{blue } y), \forall(x, \textit{green } z)$.
4. $\exists(\textit{red } x, \textit{blue } y), \forall(\textit{red } x, z)$.
5. $\exists(\textit{red } x, \textit{blue } y), \forall(\textit{red } x, \textit{green } z)$.

In form (3), one shows that for all existential sentences S,

$$\exists(\textit{red } x, \textit{blue } y), \forall(x, \textit{green } z) \models S \quad \text{iff} \quad \exists(\textit{red } x, \textit{blue } y), \forall(\textit{red } x, \textit{green } z) \models S.$$

Here again, the argument is semantic. The implication from right to left is trivial, so assume the assertion on the right. Let $\mathcal{M}$ satisfy $\exists(\textit{red } x, \textit{blue } y)$ and $\forall(\textit{red } x, \textit{green } z)$. Consider the submodel $\mathcal{M}_{red}$ of $\mathcal{M}$ induced by $[\![\textit{red}]\!]$. Then $\mathcal{M}_{red}$ satisfies $\exists(\textit{red } x, \textit{blue } y)$ and $\forall(x, \textit{green } z)$. So $\mathcal{M}_{red} \models S$. Since S is existential, we also have $\mathcal{M} \models S$, as desired.

The upshot is that form (3) is subsumed by form (5). Form (4) is easier, since $\forall(\textit{red } x, z)$ and $\forall(\textit{red } x, \textit{red } z)$ are inter-derivable.

We consider in full detail the premise form $\exists(\textit{red } x, \textit{blue } y), \forall(\textit{red } x, \textit{green } z)$. Here are all of the sound conclusions of the form $\exists(a\ u, b\ v)$, but omitting ones which are related by a use of Example 2.4. We list all the sound conclusions, together with an accounting of how each is proved in our system:

$\exists(\mathit{red}\ x, \mathit{blue}\ y)$: this is the first premise.

$\exists(\mathit{red}\ x, \mathit{blue}\ z)$: use ($\mathrm{Adj}_4$).

$\exists(\mathit{red}\ y, \mathit{blue}\ z)$: First use the second premise to get $\forall(\mathit{red}\ x, \mathit{red}\ z)$. Then from this and the first premise and (D) get $\exists(\mathit{blue}\ y, \mathit{red}\ z)$. Then use Example 2.4.

$\exists(\mathit{red}\ x, \mathit{green}\ y)$: Use the first premise to get $\exists(\mathit{red}\ y, \mathit{red}\ x)$, and the second premise to get $\forall(\mathit{red}\ x, \mathit{green}\ x)$. Then use (D) to get $\exists(\mathit{red}\ y, \mathit{green}\ x)$. Finally, use Example 2.4.

$\exists(\mathit{red}\ x, \mathit{green}\ z)$: use the second premise, with $\exists(\mathit{red}\ x, \mathit{red}\ x)$ from the first.

$\exists(\mathit{red}\ y, \mathit{green}\ z)$: First use the first premise to get $\exists(\mathit{red}\ y, \mathit{red}\ x)$. Then from this and the first premise and (D) get $\exists(\mathit{red}\ y, \mathit{green}\ z)$.

$\exists(\mathit{blue}\ x, \mathit{green}\ y)$: from ($\mathrm{Adj}_5$).

$\exists(\mathit{blue}\ x, \mathit{green}\ z)$: Use the first premise to get $\exists(\mathit{blue}\ x, \mathit{red}\ x)$; see Example 2.6. Now use (D).

$\exists(\mathit{blue}\ y, \mathit{green}\ z)$: use (D) after inferring $\exists(\mathit{blue}\ y, \mathit{red}\ x)$ from the first premise.

This concludes our discussion of valid two-premise sequents and Proposition 2.8.

2.3 The Completeness Theorem

At this point, we have examined the proof system and know that it is strong enough to prove all of the valid two-premise sequents. We are ready to prove that the system is complete.

Notation If Γ is a set of sentences, we write $\Gamma_\forall$ for the subset of Γ containing only sentences of the form $\forall(n,p)$. We do this for $\Gamma_\exists$, mutatis mutandis.

Theorem 2.9. *The logic of Figures 1 and 2 is complete for* $\mathcal{L}(\forall, \exists, \mathrm{adj})$*: if* $\Gamma \models S$*, then* $\Gamma \vdash S$.

Proof. Suppose that $\Gamma \models S$. There are two overall cases, depending on whether S is of the form $\forall(n,m)$ or of the form $\exists(n,m)$. In the first case, we claim that $\Gamma_\forall \models S$. To see this, let $\mathcal{M} \models \Gamma_\forall$. We get a new model $\mathcal{M}' = \mathcal{M} \cup \{*\}$ via $[\![x]\!]' = [\![x]\!] \cup \{*\}$. The model $\mathcal{M}'$ so obtained satisfies $\Gamma_\forall$ and all $\exists$ sentences whatsoever in the fragment. Hence $\mathcal{M}' \models \Gamma$. So $\mathcal{M}' \models S$. And since S is a universal sentence, $\mathcal{M} \models S$ as well. This proves our claim that $\Gamma_\forall \models S$. By Theorem 2.3, $\Gamma_\forall \vdash S$. Hence $\Gamma \vdash S$.

The second case, where S is an existential sentence, is more interesting. Consider the following model $\mathcal{M} = \mathcal{M}(\Gamma)$. Let M be the set of all unordered pairs $\{p,q\}$ such that p and q are nouns, and $\Gamma \vdash \exists(p,q)$. (We may well have $p = q$ in such a pair.) For each basic nouns x and each intersecting adjective *red* we define sets $[\![x]\!]_i$ and $[\![\mathit{red}]\!]_i$ for $i = 0, 1, \ldots$; then sets we are after are $\bigcup_i [\![x]\!]_i$ and $\bigcup_i [\![\mathit{red}]\!]_i$; we take these to be $[\![x]\!]$ and $[\![\mathit{red}]\!]$. The sets are defined by:

1. If $\{p,q\} \in M$ and p is basic, then $\{p,q\} \in [\![p]\!]_0$.
2. If $\{p,q\} \in M$ and p is *red* x, then $\{p,q\} \in [\![x]\!]_0 \cap [\![\mathit{red}]\!]_0$.

3. If $\{p,q\} \in [\![x]\!]_i$ and $\Gamma \vdash \forall(x,y)$, then $\{p,q\} \in [\![y]\!]_{i+1}$.
4. If $\{p,q\} \in [\![x]\!]_i \cap [\![red]\!]_i$ and $\Gamma \vdash \forall(red\ x, y)$, then $\{p,q\} \in [\![y]\!]_{i+1}$.
5. If $\{p,q\} \in [\![x]\!]_i$ and $\Gamma \vdash \forall(x, blue\ y)$, then $\{p,q\} \in [\![y]\!]_{i+1} \cap [\![blue]\!]_{i+1}$.
6. If $\{p,q\} \in [\![x]\!]_i \cap [\![red]\!]_i$ and $\Gamma \vdash \forall(red\ x, blue\ y)$, then $\{p,q\} \in [\![y]\!]_{i+1} \cap [\![blue]\!]_{i+1}$.

An easy induction shows that if $\Gamma \vdash \forall(x,y)$, then $[\![x]\!] \subseteq [\![y]\!]$. Moreover, this same fact is true for nouns containing adjectives. These facts imply that if a universal sentence $\forall(p,q)$ belongs to Γ (so that $\Gamma \vdash \forall(p,q)$), then indeed $[\![p]\!] \subseteq [\![q]\!]$. We also want to check the analogous fact for sentences $\exists(p,q)$. As usual, we have a number of cases, and we'll only mention the one when p is *red* x and q is *blue* y. Then $\{x,y\}$ belongs to $[\![x]\!] \cap [\![red]\!] \cap [\![y]\!] \cap [\![blue]\!]$. Hence $\mathcal{M} \models \exists(p,q)$.

As a result of these observations, $\mathcal{M} \models \Gamma$. Since we began with the assumption that $\Gamma \models S$, we see that $\mathcal{M} \models S$. Now S is an existential sentence, say $\exists(n,m)$, and our goal is to show that $\Gamma \vdash \exists(n,m)$. In fact, we show the following facts:

1. If $\{u,v\} \in [\![x]\!] \cap [\![y]\!]$, then $\Gamma \vdash \exists(x,y)$.
2. If $\{u,v\} \in [\![x]\!] \cap [\![y]\!] \cap [\![red]\!] \cap [\![blue]\!]$, then $\Gamma \vdash \exists(red\ x, blue\ y)$.

It is at this point that we use the fact that our semantics of nouns and intersecting adjectives was the *least* fixed point of a monotone inductive definition, so that we can argue by induction it. That is, we show by induction on i that (a) if $\{u,v\} \in [\![x]\!]_i \cap [\![y]\!]_i$, then $\Gamma \vdash \exists(x,y)$; and similarly for the other assertion.

The first base cases of this induction is when $[\![u,v]\!] \in [\![x]\!]_0 \cap [\![y]\!]_0$ via clause (1) in the definition. Then we have a number of subcases. To mention one, it might be that $u = x$ and $v = y$. Since $\{u,v\} \in M$, we have $\Gamma \vdash \exists(u,v)$. And thus $\Gamma \vdash \exists(x,y)$. For another subcase, it might be that $u = x$ and also $u = y$. Now as we have seen, $\Gamma \vdash \exists(u,v)$, and by our logic, we also have $\Gamma \vdash \exists(u,u)$. So in this case, we again have $\Gamma \vdash \exists(x,y)$.

Another base case in the induction is when $[\![u,v]\!] \in [\![x]\!]_0 \cap [\![y]\!]_0 \cap [\![red]\!]_0$ via clauses (1) and (2) in the definition of the semantics. For example, we might have $u = red\ y$ so that $[\![u,v]\!] \in [\![y]\!]_0 \cap [\![red]\!]_0$, and also $v = x$. Then $\Gamma \vdash \exists(x, red\ y)$. And by the reciprocity fact noted in Example 2.5 we see that indeed $\Gamma \vdash \exists(red\ x, y)$.

The last base case in the induction is when $[\![u,v]\!] \in [\![x]\!]_0 \cap [\![y]\!]_0 \cap [\![red]\!]_0 \cap [\![blue]\!]_0$ via clauses (1) and (2) in the definition of the semantics. The arguments would be similar, and Example 2.4 would also be used.

Next, we turn to the induction steps proper. Here is an example. Suppose that

$$\{u,v\} \in [\![x]\!]_{i+1} \cap [\![y]\!]_{i+1} \cap [\![red]\!]_{i+1} \cap [\![blue]\!]_{i+1}$$

because

$$\{u,v\} \in [\![x]\!]_i \cap [\![w]\!]_i \cap [\![red]\!]_i \cap [\![green]\!]_{i+1}$$

and also $\Gamma \vdash \forall(\textit{green } x, \textit{blue } z)$ and $\Gamma \vdash \forall(w, y)$. By induction hypothesis, $\Gamma \vdash \exists(\textit{green } x, \textit{red } w)$. We have the following derivation from Γ:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \exists(\textit{green } x, \textit{red } w)\end{array} \quad \begin{array}{c}\vdots \\ \forall(\textit{green } x, \textit{blue } z)\end{array}}{\exists(\textit{red } x, \textit{blue } w)}\,(\mathrm{Adj}_5) \quad \begin{array}{c}\vdots \\ \forall(w, y)\end{array}}{\exists(\textit{red } x, \textit{blue } y)}\ \textit{Proposition } 2.8$$

(We are quoting Proposition 2.8 mostly because we did all the work to obtain that result.) There are, of course, many more induction steps. These all go through, and the main reason was mentioned before we started in on the proof of this theorem: we have included in the rules all of the sound two-premise rules that are expressible in the language. This fact is not directly used, but all of the reasoning that we have already seen would be used in the full verification here.

This completes the proof.

2.4 A Note on ($\mathbf{Adj_4}$) and ($\mathbf{Adj_5}$)

At this point, we digress from our main line and make a comment on (Adj_4) and (Adj_5). We shall check that they are not derivable from the other rules in our system.

To see this, take the premises $\exists(\textit{red } x, \textit{blue } y)$ and $\forall(\textit{red } x, \textit{green } z)$, and call them Γ. An easy induction on derivations shows that if S is universal and $\Gamma \vdash S$ without (Adj_4) or (Adj_5), then S must be of one of the following three forms: $\forall(u, u)$ for some u, $\forall(\textit{red } u, u)$ for some u, or $\forall(\textit{red } x, \textit{green } z)$.

Now assume that $\Gamma \vdash \exists(\textit{red } x, \textit{blue } z)$, or that $\Gamma \vdash \exists(\textit{blue } z, \textit{red } x)$; again without ($\mathrm{Adj}_4$) or ($\mathrm{Adj}_5$). Take a derivation of minimal height. The last step in the derivation must be an application of (D). There are two cases, depending on the conclusion. They are similar, and we only go into details concerning $\exists(\textit{blue } z, \textit{red } x)$. For some noun q, we must have $\Gamma \vdash \exists(\textit{red } x, q)$ and $\forall(q, \textit{blue } z)$. By our observation in the last paragraph, q must be *blue* z or *red* x. If q is *blue* z, we contradict the minimality assertion. And if q is *red* x, have $\Gamma \vdash \forall(\textit{red } x, \textit{blue } z)$, contradicting what we showed in the last paragraph.

This shows that without (Adj_4) or (Adj_5), we cannot derive $\exists(\textit{red } x, \textit{blue } z)$ from our premises. Similar work shows the same thing about $\exists(\textit{blue } x, \textit{green } y)$. The upshot is that neither $\exists(\textit{red } x, \textit{blue } y)$ nor $\forall(\textit{red } x, \textit{green } z)$ can be proved on the basis of (T), (B), (I), (D), (Adj_1), (Adj_2), and (Adj_3).

$$\frac{}{\forall(n,n)}\ (\mathrm{T}) \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ (\mathrm{B}) \qquad \frac{\exists(n,p)}{\exists(n,n)}\ (\mathrm{I}) \qquad \frac{\exists(n,q) \quad \forall(q,p)}{\exists(p,n)}\ (\mathrm{D})$$

$$\frac{}{\forall(\mathit{red}\ n,n)}\ (\mathrm{Adj}_1) \qquad \frac{\forall(n,\mathit{red}\ p) \quad \forall(n,q)}{\forall(n,\mathit{red}\ q)}\ (\mathrm{Adj}_2) \qquad \frac{\exists(p,\mathit{red}\ q)}{\exists(\mathit{red}\ p,\mathit{red}\ q)}\ (\mathrm{Adj}_3)$$

Fig. 3. The logical system for $\mathcal{L}_r(\forall, \exists, \mathit{adj})$. Note that the x and y denote basic nouns, and n, p, and q denote complex nouns in the sense of this section.

3 $\mathcal{L}_r(\forall, \exists, \mathit{adj})$: Productive Predication

We start with basic nouns x, y, ..., and (intersecting) adjectives a_1, a_2, ..., and then say that basic nouns are nouns, and if n is a noun and *red* an adjective, then *red* n is a noun.

The semantics of nouns in then is given by recursion, using the main clause

$$[\![a\ n]\!] \quad = \quad [\![a]\!] \cap [\![n]\!].$$

Then we define $\mathcal{M} \models S$, $\mathcal{M} \models \Gamma$, and $\Gamma \models S$ as earlier. (See the end of Section 1.) Our main goal again is to provide a proof system, thereby defining a relation $\Gamma \vdash_r S$ in a syntactic way, and then to show the connection in a soundness/completeness theorem.

The proof system itself is listed in Figure 3. To keep straight the distinction between the proof system for $\mathcal{L}(\forall, \exists, \mathit{adj})$ and the one for $\mathcal{L}_r(\forall, \exists, \mathit{adj})$, we write $\Gamma \vdash_r S$ for the derivation relation in this section.

Our first examples concern iterated adjectives. Example 3.1 shows that applying the same adjective twice gives nothing new; perhaps this is a justification for why we never see phrases like *red red ball* in natural language. The two adjectives in Example 3.2 are likewise odd, but we encourage the reader to read *red* and *blue* as *partly red* and *partly blue*, or to remember our semantics.

Example 3.1. For all n, $\vdash_r \forall(\mathit{red}\ n, \mathit{red}\ \mathit{red}\ n)$ and $\vdash_r \forall(\mathit{red}\ \mathit{red}\ n, \mathit{red}\ n)$. For the first point, we have the following derivation:

$$\dfrac{\dfrac{}{\forall(\mathit{red}\ n,\mathit{red}\ n)}\ (\mathrm{T}) \quad \dfrac{}{\forall(\mathit{red}\ n,\mathit{red}\ n)}\ (\mathrm{T})}{\forall(\mathit{red}\ n,\mathit{red}\ \mathit{red}\ n)}\ (\mathrm{Adj}_2)$$

The second point is an instance of (Adj_1).

Example 3.2. $\vdash_r \forall(\mathit{red}\ \mathit{blue}\ n, \mathit{blue}\ \mathit{red}\ n)$. Here is the derivation:

$$\dfrac{\dfrac{}{\forall(\mathit{red}\ \mathit{bl}\ n,\mathit{bl}\ n)}\ (\mathrm{Adj}_1) \quad \dfrac{\dfrac{}{\forall(\mathit{red}\ \mathit{blue}\ n,\mathit{red}\ \mathit{blue}\ n)}\ (\mathrm{T}) \quad \begin{array}{c}\vdots\\ \forall(\mathit{red}\ \mathit{blue}\ n,n)\end{array}}{\forall(\mathit{red}\ \mathit{blue}\ n,\mathit{red}\ n)}\ (\mathrm{Adj}_2)}{\forall(\mathit{red}\ \mathit{blue}\ n,\mathit{blue}\ \mathit{red}\ n)}\ (\mathrm{Adj}_2)$$

The point at the top which is not shown consists of two applications of (Adj$_1$) and also (B).

4 Models $\mathcal{M}(\Gamma)$

Let Γ be a set of sentences in this language. We define a model $\mathcal{M} = \mathcal{M}(\Gamma)$ as follows. The universe of the model is

$$M \quad = \quad \{\{n, m\} : \Gamma \vdash \exists(n, m)\}.$$

The semantics of the nouns and adjectives is given in stages. We'll arrange that for each basic noun x,

$$[\![x]\!]_0 \quad \subseteq \quad [\![x]\!]_1 \quad \subseteq \quad [\![x]\!]_2 \quad \subseteq \quad \cdots,$$

and then we define $[\![x]\!] = \bigcup_i [\![x]\!]_i$. We do the same thing for each color c. In addition, if p is the noun $c_1 c_2 \cdots c_k x$, then we write

$$[\![p]\!]_i \quad = \quad [\![c_1]\!]_i \cap [\![c_2]\!]_i \cap \cdots [\![x]\!]_i.$$

The definition is as follows:

$$\begin{array}{lcl}
[\![x]\!]_0 & = & \{\{n, m\} : \Gamma \vdash \forall(n, x) \text{ or } \Gamma \vdash \forall(m, x)\} \\
[\![x]\!]_{i+1} & = & [\![x]\!]_i \cup \{\{n, m\} : \text{for some noun } p,\ \Gamma \vdash \forall(p, x) \text{ and } \{n, m\} \in [\![p]\!]_i\} \\
[\![c]\!]_0 & = & \{\{n, m\} : \text{for some noun } p,\ \Gamma \vdash \forall(n, cp) \text{ or } \Gamma \vdash \forall(m, cp)\} \\
[\![c]\!]_{i+1} & = & [\![c]\!]_i \cup \{\{n, m\} : \text{for some nouns } p \text{ and } x,\ \Gamma \vdash \forall(p, cx) \text{ and } \{n, m\} \in [\![p]\!]_i\}
\end{array}$$

Lemma 4.1. *For all nouns p, $[\![p]\!] = \bigcup_i [\![p]\!]_i$.*

Proof. By induction on p. If p is a basic noun x, then we just have the definition of $[\![x]\!]$ as $\bigcup_i [\![x]\!]_i$. Assume for a noun p that $[\![p]\!] = \bigcup_i [\![p]\!]_i$. We show the same thing for cp.

$$\begin{array}{rcl}
[\![cp]\!] & = & [\![c]\!] \cap [\![p]\!] \\
 & = & [\![c]\!] \cap \bigcup_i [\![p]\!]_i \\
 & = & \bigcup_i ([\![c]\!] \cap [\![p]\!]_i)
\end{array}$$

So clearly $\bigcup_i [\![cp]\!]_i = \bigcup_i ([\![c]\!]_i \cap [\![p]\!]_i) \subseteq \bigcup_i ([\![c]\!] \cap [\![p]\!]_i) = [\![cp]\!]$. In the other direction, let $\{n, m\} \in [\![cp]\!] = [\![c]\!] \cap [\![p]\!]$. We may find i and j so that $\{n, m\} \in [\![c]\!]_i \cap [\![p]\!]_j$. Without loss of generality, $j \leq i$. Since $[\![p]\!]_j \subseteq [\![p]\!]_i$. Hence

$$\{n, m\} \quad \in \quad [\![c]\!]_i \cap [\![p]\!]_i \quad = \quad [\![cp]\!]_i \quad \subseteq \quad \bigcup_i [\![cp]\!]_i\ .$$

This completes the proof.

Lemma 4.2. *If $\Gamma \vdash \forall(n, m)$, then $[\![n]\!]_i \subseteq [\![m]\!]_i$.*

Proof. Write m as $c_1 \cdots c_k x$. Let $\{p, q\} \in [\![n]\!]_i$. First, for $1 \leq j \leq k$, $\Gamma \vdash \forall(n, c_j x)$. So $\{p, q\} \in [\![c_j]\!]_{i+1} \cap [\![x]\!]_{i+1}$. This for all j between 1 and k implies that

$$\{p, q\} \in [\![c_1]\!]_{i+1} \cap [\![c_2]\!]_{i+1} \cap \cdots \cap [\![c_k]\!]_{i+1} \cap [\![x]\!]_{i+1} \quad = \quad [\![m]\!]_{i+1}.$$

Lemma 4.3. *For all c and n, and all i, $[\![cn]\!]_i \subseteq [\![c]\!]_i$.*

Proof. This is obvious from the fact that $[\![cn]\!]_i = [\![c]\!]_i \cap [\![n]\!]_i$.

Lemma 4.4. *If $\{n,m\} \in M$, then $\{n,m\} \in [\![n]\!] \cap [\![m]\!]$.*

Proof. One only has to check that $\{n,m\} \in [\![n]\!]$, of course. Let n be $c_1 c_2 \cdots c_k x$. (It is possible that $k = 0$, and in this case the argument simplifies.) Then the proof system has

$$\Gamma \vdash \forall(c_1 c_2 \cdots c_k x, c_i x)$$

for all i, and thus $n = c_1 c_2 \cdots c_k x$ belongs to $[\![c_i]\!]_0$. Further,

$$\Gamma \vdash \forall(c_1 c_2 \cdots c_k x, x),$$

and so $n \in [\![x]\!]_0$ as well. Thus $n \in [\![n]\!]_0 \subseteq [\![n]\!]$.

Lemma 4.5. *If $\{n,m\} \in M$ and $\{n,m\} \in [\![p]\!]_0 \cap [\![q]\!]_0$, then $\Gamma \vdash \exists(p,q)$.*

Proof. Write p as $c_1 \cdots c_k x$ and q as $d_1 \cdots d_l y$. To make the notation more manageable, we shall assume that $k = 2 = l$ here. So we have $p = c_1 c_2 x$ and $q = d_1 d_2 y$. We assume that

$$\{n,m\} \in [\![c_1]\!]_0 \cap [\![c_2]\!]_0 \cap [\![x]\!]_0 \cap [\![d_1]\!]_0 \cap [\![d_2]\!]_0 \cap [\![y]\!]_0.$$

Now we have 2^6 cases here, and so we shall only give one of them. Suppose that we have nouns u, v, u', and v' such that the following are all provable from Γ in our system:

1. $\forall(n, c_1 u)$.
2. $\forall(m, c_2 v)$.
3. $\forall(n, x)$.
4. $\forall(m, d_1 u')$.
5. $\forall(n, d_2 v')$.
6. $\forall(m, y)$.

Facts 1, 3, and 5 together with (Adj$_2$) show that $\Gamma \vdash \forall(n, c_1 d_2 x)$. Similarly, facts 2, 4, and 6 together with (Adj$_2$) show that $\Gamma \vdash \forall(m, d_1 c_2 y)$. Since $\Gamma \vdash \exists(n,m)$, we easily get

$$\Gamma \vdash \exists(c_1 d_2 x, d_1 c_2 y) \ .$$

And then we can rearrange things as in Example 3.2, to see that

$$\Gamma \vdash \exists(c_1 c_2 x, d_1 d_2 y) \ .$$

This shows that $\Gamma \vdash \exists(p,q)$, just as desired.

Lemma 4.6. *If $\{n,m\} \in M$ and $\{n,m\} \in [\![p]\!] \cap [\![q]\!]$, then $\Gamma \vdash \exists(p,q)$.*

Proof. In view of Lemma 4.1, we need only show that if $\{n,m\} \in M$ and $\{n,m\} \in [\![p]\!]_i \cap [\![q]\!]_i$, then $\Gamma \vdash \exists(p,q)$. We argue by induction on i.

The case $i = 0$ is Lemma 4.5.

Assume this lemma for i, and also suppose that $\{n,m\} \in M$ and $\{n,m\} \in [\![p]\!]_{i+1} \cap [\![q]\!]_{i+1}$. To avoid a lot of messy details, we shall assume that p is a noun of the form c_1c_2x, and q is d_1d_2y. So we know that

$$\begin{array}{c}\{n,m\} \in [\![c_1]\!]_{i+1} \cap [\![c_2]\!]_{i+1} \cap [\![x]\!]_{i+1} \\ \{n,m\} \in [\![d_1]\!]_{i+1} \cap [\![d_2]\!]_{i+1} \cap [\![y]\!]_{i+1}\end{array}$$

We can find nouns p_1, u_1, p_2, u_2, p_3, q_1, v_1, q_2, v_2, and q_3 so that all of the following facts are provable in our system from Γ:

1. $\forall(p_1, c_1u_1)$
2. $\forall(p_2, c_2u_2)$
3. $\forall(p_3, x)$
4. $\forall(q_1, d_1v_1)$
5. $\forall(q_2, d_2v_2)$
6. $\forall(q_3, y)$

and also so that

$$\{n,m\} \in [\![p_1]\!]_i \cap [\![p_2]\!]_i \cap [\![p_3]\!]_i \cap [\![q_1]\!]_i \cap [\![q_2]\!]_i \cap [\![q_3]\!]_i \ .$$

In view of the lemmas above and the facts that we just saw, we have

$$\{n,m\} \in [\![c_1]\!]_i \cap [\![c_2]\!]_i \cap [\![p_3]\!]_i \cap [\![d_1]\!]_i \cap [\![d_2]\!]_i \cap [\![q_3]\!]_i \ .$$

That is,

$$\{n,m\} \in [\![c_1c_2p_3]\!]_i \cap [\![d_1d_2q_3]\!]_i$$

By induction hypothesis, we have

$$\Gamma \vdash \exists(c_1c_2p_3, d_1d_2q_3).$$

Since also $\Gamma \vdash \forall(p_3, x)$ and $\Gamma \vdash \forall(q_3, y)$, we see that

$$\Gamma \vdash \exists(c_1c_2x, d_1d_2y),$$

and this is to say that $\Gamma \vdash \exists(p,q)$.

Theorem 4.7. *The rules (T), (B), (Adj$_1$), and (Adj$_2$) give a complete proof system for $\mathcal{L}(\forall, \mathrm{adj})_r$: if $\Gamma \models \forall(n,p)$, then $\Gamma \vdash_r \forall(n,p)S$.*

Proof. Suppose that $\Gamma \models \forall(n,p)$; we show that $\Gamma \vdash_r \forall(n,p)$. Consider a model $\mathcal{M}$ whose universe M is a singleton $\{*\}$, and whose structure is given by

$$[\![x]\!] \quad = \quad \begin{cases} \{*\} & \text{if } \Gamma \vdash_r \forall(n,x) \\ \emptyset & \text{if } \Gamma \not\vdash_r \forall(n,x) \end{cases}$$

$$[\![\mathit{red}]\!] \quad = \quad \begin{cases} \{*\} & \text{if for some basic noun } x,\ \Gamma \vdash_r \forall(n, \mathit{red}\ x) \\ \emptyset & \text{otherwise} \end{cases}$$

These definitions are made using the specific noun n from our overall assumption in this proof.

Claim. For all nouns p,

$$[\![p]\!] \quad = \quad \begin{cases} \{*\} & \text{if } \Gamma \vdash_r \forall(n,p) \\ \emptyset & \text{if } \Gamma \not\vdash_r \forall(n,p) \end{cases} \tag{2}$$

The proof is by induction on p. For p a basic noun, the result is immediate. Assume (2) for p; we show (2) for $\mathit{red}\ p$. If $* \in [\![\mathit{red}\ p]\!] = [\![\mathit{red}]\!] \cap [\![p]\!]$, then $\Gamma \vdash_r \forall(n,p)$ and for some x, $\Gamma \vdash_r \forall(n, \mathit{red}\ x)$ By (Adj$_2$), $\Gamma \vdash_r \forall(n, \mathit{red}\ p)$, as desired.

We now argue the converse. If $\Gamma \vdash_r \forall(n, \mathit{red}\ p)$, then $\Gamma \vdash_r \forall(n,p)$ using (B) and (Adj$_1$). We write p as $a_1 \ \cdots \ a_j\ x$, so that $\mathit{red}\ p$ is $\mathit{red}\ a_1 \cdots a_j\ x$, and then we argue by induction on j that $\vdash_r \forall(\mathit{red}\ p, \mathit{red}\ x)$. If $n = 0$, this is immediate. If $n \geq 1$, we show that $\vdash_r \forall(\mathit{red}\ a_1\ a_2 \cdots a_n\ x, \mathit{red}\ a_2 \cdots a_n\ x)$, using (Adj$_1$) and (B). (See Example 3.2.) And then by induction hypothesis, we have $\vdash_r \forall(\mathit{red}\ a_2 \cdots a_n\ x, \mathit{red}\ x)$. This concludes the induction showing that $\vdash_r \forall(\mathit{red}\ p, \mathit{red}\ x)$, and from this we see that $\Gamma \vdash_r \forall(n, \mathit{red}\ x)$. Therefore $* \in [\![\mathit{red}]\!]$. Overall, $* \in [\![\mathit{red}\ p]\!]$.

This completes the induction on p, hence the proof of this claim.

Continuing with the proof of Theorem 4.7, we next observe that $\mathcal{M} \models \Gamma$. Take a sentence in Γ such as $\forall(l_1, l_2)$. We must show that $[\![l_1]\!] \subseteq [\![l_2]\!]$. For this, we may assume that $[\![l_1]\!] \neq \emptyset$. Hence $[\![l_1]\!] = \{*\}$, so $\Gamma \vdash_r \forall(n, l_1)$. Using (B), $\Gamma \vdash_r \forall(n, l_2)$, so again $* \in [\![l_2]\!]$.

We have verified that $\mathcal{M} \models \Gamma$. Recalling that $\Gamma \models \forall(n,p)$, we have $\mathcal{M} \models \forall(n,p)$. So by our claim, we have the desired conclusion that $\Gamma \vdash_r \forall(n,p)$. This completes the proof.

4.1 Simulation of $\mathcal{L}(\forall, \exists, \mathit{adj})$ in $\mathcal{L}_r(\forall, \exists, \mathit{adj})$

Our goal in the next section is to prove the completeness of $\mathcal{L}_r(\forall, \exists, \mathit{adj})$ using the proof system defined in Figure 3. Here are two ways that one could go about this. First, one could basically repeat the proof of Theorem 2.9. This would be a fairly direct modification. At the same time, it would be uninteresting to read. Instead, we shall present a different approach.

For each of the rules in Figure 2, except possibly (Adj_4) and (Adj_5), the corresponding sequent is provable in the logic for $\mathcal{L}_r(\forall, \exists, \textit{adj})$. In fact, this holds with the basic nouns in Figure 2 replaced by *arbitrary* nouns.

Proposition 4.8. *Every instance of (Adj_4) and (Adj_5) in Figure 2 is provable in the logical system for* $\mathcal{L}_r(\forall, \exists, \mathrm{adj})$*. Moreover, this holds with the basic nouns replaced by arbitrary nouns.*

Proof. Here is the derivation for (Adj_4), omitting some routine conversion steps:

$$\dfrac{\dfrac{\dfrac{\exists(\textit{red } n, \textit{blue } p)}{\exists(\textit{red } n, \textit{blue } n)}\ \textit{Example } 2.6}{\exists(\textit{red } n, \textit{blue red } n)}\ (\mathrm{Adj}_3) \qquad \dfrac{\dfrac{\forall(\textit{red } n, \textit{green } q) \quad \dfrac{}{\forall(\textit{green } q, q)}\ (\mathrm{Adj}_1)}{\forall(\textit{red } n, q)}\ (\mathrm{B})}{\forall(\textit{blue red } n, \textit{blue } q)}\ \textit{Example } 2.2}{\exists(\textit{red } n, \textit{blue } q)}\ (\mathrm{D})$$

What we mean by Examples 2.6 and 2.2 are the obvious versions of those results for the language $\mathcal{L}_r(\forall, \exists, \textit{adj})$: a look back at both derivations shows that they did not use (Adj_4) or (Adj_5).

For (Adj_5), we have

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\exists(\textit{red } n, \textit{blue } p) \qquad \dfrac{\forall(\textit{red } n, \textit{green } q) \quad \dfrac{}{\forall(\textit{red } n, n)}\ (\mathrm{Adj}_1)}{\forall(\textit{red } n, \textit{green } n)}\ (\mathrm{Adj}_2)}{\exists(\textit{blue } p, \textit{green } n)}\ (\mathrm{D})}{\exists(\textit{blue } p, \textit{blue green } n)}\ (\mathrm{Adj}_3)}{\exists(\textit{blue } p, \textit{green blue } n)}\ \textit{using Example 3.2}}{\exists(\textit{green blue } p, \textit{green blue } n)}\ (\mathrm{Adj}_3) \\ \vdots \\ \exists(\textit{green } p, \textit{blue } n)\end{array}$$

We have left out some routine steps at the bottom.

4.2 Completeness of $\mathcal{L}_r(\forall, \exists, \textit{adj})$

Our final result is the completeness of $\mathcal{L}_r(\forall, \exists, \textit{adj})$. We aim to reduce this fact to our earlier completeness result for $\mathcal{L}(\forall, \exists, \textit{adj})$. Some of the work was done in Proposition 4.8, but there are a few steps to go.

Throughout this paper, we have been working with fixed sets of basic nouns and intersecting adjectives. That is, the languages in the paper have been defined in terms of those sets, but we suppressed the sets in our notation. At this time, we must be a little more explicit. Let N be our set of basic nouns and A our set of adjectives. We'll call our languages $\mathcal{L}(\forall, \exists, \textit{adj})_{N,A}$. Let N^* be the set all adjectives, allowing for recursion. Let X be a new set, and assume that X is in bijective correspondence with N^*. Write $N + X$ for the disjoint union of N and X. The language $\mathcal{L}(\forall, \exists, \mathit{adj})_{N+X,A}$ then has as basic nouns the elements of N together with new basic nouns in X. To be explicit, for every noun n of $\mathcal{L}_r(\forall, \exists, \mathit{adj})_{N,A}$, we have a basic noun v_n of $\mathcal{L}(\forall, \exists, \mathit{adj})_{N+X,A}$.

We translate $\mathcal{L}_r(\forall, \exists, \mathit{adj})_{N,A}$ into $\mathcal{L}(\forall, \exists, \mathit{adj})_{N+X,A}$ via a map $S \mapsto S^*$. For example, if S is

$$\exists(\mathit{red\ blue\ green\ } x, y),$$

then S^* is $\exists(v_{\mathit{red\ blue\ green\ } x}, v_y)$.

Theorem 2.9, the completeness theorem from earlier in this paper, holds for $\mathcal{L}(\forall, \exists, \mathit{adj})_{N+X,A}$, since it holds for the flat syntax language built from any set of basic nouns.

The translation also works in the other direction, taking each v_n to the corresponding n, and also each *red* v_n to the corresponding *red* n.

Theorem 4.9. *The logic of Figures 3 is complete for* $\mathcal{L}_r(\forall, \exists, \mathrm{adj})$*: if* $\Gamma \models S$*, then* $\Gamma \vdash S$*.*

Proof. Assume that $\Gamma \vdash S$. Let $\Gamma^* = \{S^* : S \in \Gamma\}$. Let

$$\Delta \quad = \quad \{\forall(v_{\mathit{red\ } n}, \mathit{red\ } v_n) : n \in N\} \cup \{\forall(\mathit{red\ } v_n, v_{\mathit{red\ } n}) : n \in N\};$$

again, N is the set of nouns with which we started.

We claim that $\Gamma^* \cup \Delta \models S^*$. To see this, let $\mathcal{M} \models \Gamma^* \cup \Delta$. Then an induction on nouns n in the recursive language shows that $[\![v_n]\!]$ (the interpretation of v_n) is the same as $[\![n]\!]$. This is where we use the clauses in Δ. As a result, truth values of sentences in $\mathcal{M}$ are preserved under translation in both directions. Hence $\mathcal{M} \models \Gamma$. Since $\Gamma \models S$, we have $\mathcal{M} \models S$ also. And then $\mathcal{M} \models S^*$.

Having shown the claim, we see that by completeness, $\Gamma^* \cup \Delta \vdash S^*$. Let D be a derivation for this in the sense of Section 2. D is in $\mathcal{L}(\forall, \exists, \mathit{adj})_{N+X,A}$, and therefore we must translate it back to a derivation in $\mathcal{L}_r(\forall, \exists, \mathit{adj})_{N,A}$. For this, replace each v_n with the corresponding noun n. Most instances of the proof rules in D translate to the same steps in $\mathcal{L}_r(\forall, \exists, \mathit{adj})$. This is true for (T), (B), (Adj$_1$), (Adj$_2$), (I), (D), and (Adj$_3$). For example,

$$\frac{\forall(u_{\mathit{red\ } y}, \mathit{red\ } u_{\mathit{red\ } x}) \quad \forall(u_{\mathit{red\ } y}, u_{\mathit{blue\ } x})}{\forall(u_{\mathit{red\ } y}, \mathit{red\ } u_{\mathit{blue\ } x})}\ (\mathrm{Adj}_2)$$

translates to

$$\frac{\forall(\mathit{red\ } y, \mathit{red\ red\ } x) \quad \forall(\mathit{red\ } y, \mathit{blue\ } x)}{\forall(\mathit{red\ } y, \mathit{red\ blue\ } x)}\ (\mathrm{Adj}_2)$$

However, some of the steps in D might use (Adj$_4$) or (Adj$_5$). Take these, and replace them with derivations which do not use them, following Proposition 4.8. Finally, the leaves of D which happen to belong to Δ translate to instances of (T). The conclusion is that we have a derivation in $\mathcal{L}_r(\forall, \exists, \mathit{adj})$, as desired.

Conclusion

The results in this paper are complete logical systems for some very simple syllogistic systems, those extending the basic syllogistic logic of *all* and *some* with intersecting adjectives. These are some of the simplest logical systems of all, all of the work has been completely elementary. This is not to say that it was obvious: I have found that it is easy in this kind of work to omit "obvious" cases and thereby fail to have a complete system, and on the other hand it is also easy to state redundant rules. My point is that the results here do not depend on any facts from other papers.

At the end of the Introduction, we raised the question of whether the principle in (1) was essentially the only new one concerning intersecting adjectives. That is, if one adds it to the logic of *all* and *some*, is the resulting system complete? For the purposes of this point, we take (1) to be formalized as (Adj_2) and (Adj_3). We also assume the extensionality of adjectives, and this is (Adj_1). Our results in Section 2.4 indicate that if one adheres to a flat syntax, then two more logical principles are needed to prove completeness: (Adj_4) and (Adj_5). On the other hand, moving to the larger language that admits recursive modification using intersecting adjectives allows us to prove (Adj_4) and (Adj_5). So in this sense, Keenan and Faltz' (1) is indeed all that there is to the logic of intersecting adjectives.

My feeling is that the results here should extend to many other syllogistic systems without much change. For example, they should extend to all of the systems in Pratt-Hartmann and Moss [6]. The details on this have yet to be worked out. Another worthwhile project would be to investigate what natural logic would look like for adjectives which are not intersecting.

References

[1] Keenan, E.L., Faltz, L.M.: Boolean Semantics for Natural Language, Synthese Language Library, vol. 23. D. Reidel Publishing Co., Dordrecht (1985)

[2] Łukasiewicz, J.: Aristotle's Syllogistic from the Standpoint of Modern Formal Logic. Oxford, at the Clarendon Press (1951)

[3] Moss, L.S.: Completeness Theorems for Syllogistic Fragments. In: Logics for Linguistic Structures, vol. 29, pp. 143–173. Mouton de Gruyter (2008)

[4] Moss, L.S.: Logics for Two Fragments Beyond the Syllogistic Boundary (August 2009), to appear in A. Blass et al (eds.), Studies in Honor of Yuri Gurevich, Lecture Notes in Computer Science, Springer-Verlag, Berlin, 2010

[5] Nishihara, N., Morita, K., Iwata, S.: An Extended Syllogistic System with Verbs and Proper nouns, and its Completeness Proof. Systems and Computers in Japan 21(1), 760–771 (1990)

[6] Pratt-Hartmann, I., Moss, L.S.: Logics for the Relational Syllogistic. Review of Symbolic Logic 2(4), 647–683 (2009)

[7] Westerståhl, D.: Aristotelian Syllogisms and Generalized Quantifiers. Studia Logica XLVIII(4), 577–585 (1989)