Syllogistic Logic with Complements

Lawrence S. Moss
Department of Mathematics
Indiana University
Bloomington, IN 47405 USA

Abstract

This paper continues the development of complete fragments of natural language begun in [7] by adding a noun-level complement operator to the basic syllogistic language of *All X are Y* and *Some X are Y*: in the fragment of this paper we can also say, for example, *Some non-X are Y*. The main result is an axiomatization of this logic. Our main system uses *ex falso quodlibet* (EFQ) to infer an arbitrary sentence from a contradiction. It is also possible to consider a stronger system, one which uses *reductio ad absurdum* (RAA). It turns out that the weaker system is already complete. Our work on this uses a representation theorem from orthoposets and also a semantic lemma due to Ian Pratt-Hartmann [9]. We also provide a proof-theoretic reduction of the system with (RAA) to the one with (EFQ).

Contents

1 Introduction

This paper presents a logic for statements of the form *All* X *are* Y and *Some* X *are* Y, where the X and Y are intended as (plural) nouns or other expressions whose natural denotation is as subsets of an underlying universe. Languages like this have been studied previously, and the novelty here is to add an *explicit complement operator* to the syntax. So we now can say, for example, *All* X' *are* Y, or *Some non-X are* Y. The point of the paper is to present a sound and complete proof system for the associated entailment relation. In its details, the work is rather different from previous work in the area (for example, [1, 3, 5, 6, 7] and references therein). Our particular system seems new. In addition, the work here builds models using a representation theorem coming from quantum logic.

1.1 Syllogistic logic with complement

We start with the syntax and semantics of a language which we call $\mathcal{L}(all, some, ')$. Let $\mathcal{V}$ be an arbitrary set whose members will be called *variables*. We use X, Y, ..., for variables. The idea is that they represent plural common nouns. We also assume that there is a *complementation operation* $' : \mathcal{V} \to \mathcal{V}$ on the variables such that $X'' = X$ for all X. This *involutive* property implies that complementation is a bijection on $\mathcal{V}$. In addition, to avoid some uninteresting technicalities, we shall always assume that $X \neq X'$. Then we consider sentences *All* X *are* Y and *Some* X *are* Y. Here X and Y are any variables, including the case when they are the same. We call this language $\mathcal{L}(all, some, ')$. We shall use letters like S to denote sentences.

Semantics One starts with a set M and a subset $[\![X]\!] \subseteq M$ for each variable X, subject to the requirement that $[\![X']\!] = M \setminus [\![X]\!]$ for all X. This gives a *model* $\mathcal{M} = (M, [\![\ ]\!])$. We then define

$$\begin{array}{lll} \mathcal{M} \models \textit{All } X \textit{ are } Y & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} \models \textit{Some } X \textit{ are } Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \end{array}$$

We say $\mathcal{M}$ *satisfies* S iff $\mathcal{M} \models S$. We allow $[\![X]\!]$ to be empty, and in this case, recall that $\mathcal{M} \models$ *All* X *are* Y vacuously. (For that matter, we also allow a model to have an empty universe.) And if Γ is a set of sentences, then we write $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$. $\Gamma \models S$ means that every model which satisfies all sentences in Γ also satisfies S.

Example 1 We claim that $\Gamma \models$ *All* A *are* C, where

$$\Gamma \quad = \quad \{\textit{All } B' \textit{ are } X, \textit{All } X \textit{ are } Y, \textit{All } Y \textit{ are } B, \textit{All } B \textit{ are } X, \textit{All } Y \textit{ are } C\}.$$

Here is an informal explanation. Since all B and all B' are X, everything whatsoever is an X. And since all X are Y, and all Y are B, we see that everything is a B. In particular, all A are B. But the last two premises and the fact that all X are Y also imply that all B are C. So all A are C.

No In previous work, we took *No* X *are* Y as a basic sentence in the syntax. There is no need to do this here: we may regard *No* X *are* Y as a variant notation for *All* X *are* Y'. So the semantics would be

$$\mathcal{M} \models \textit{No } X \textit{ are } Y \quad \text{iff} \quad [\![X]\!] \cap [\![Y]\!] = \emptyset$$

In other words, if one wants to add *No* as a basic sentence forming-operation, on a par with *Some* and *All*, it would be easy to do so.

$$\frac{}{\mathit{All}\ X\ \mathit{are}\ X}\ \mathit{Axiom} \qquad \frac{\mathit{Some}\ X\ \mathit{are}\ Y}{\mathit{Some}\ X\ \mathit{are}\ X}\ \mathit{Some}_1 \qquad \frac{\mathit{Some}\ X\ \mathit{are}\ Y}{\mathit{Some}\ Y\ \mathit{are}\ X}\ \mathit{Some}_2$$

$$\frac{\mathit{All}\ X\ \mathit{are}\ Z \quad \mathit{All}\ Z\ \mathit{are}\ Y}{\mathit{All}\ X\ \mathit{are}\ Y}\ \mathit{Barbara} \qquad \frac{\mathit{All}\ Y\ \mathit{are}\ Z \quad \mathit{Some}\ X\ \mathit{are}\ Y}{\mathit{Some}\ X\ \mathit{are}\ Z}\ \mathit{Darii}$$

$$\frac{\mathit{All}\ Y\ \mathit{are}\ Y'}{\mathit{All}\ Y\ \mathit{are}\ X}\ \mathit{Zero} \qquad \frac{\mathit{All}\ Y'\ \mathit{are}\ Y}{\mathit{All}\ X\ \mathit{are}\ Y}\ \mathit{One}$$

$$\frac{\mathit{All}\ Y\ \mathit{are}\ X'}{\mathit{All}\ X\ \mathit{are}\ Y'}\ \mathit{Antitone} \qquad \frac{\mathit{All}\ X\ \mathit{are}\ Y \quad \mathit{Some}\ X\ \mathit{are}\ Y'}{S}\ \mathit{Contrad}$$

Figure 1: Syllogistic logic with complement.

Proof trees We have discussed the meager syntax of $\mathcal{L}(\mathit{all}, \mathit{some}, ')$ and its semantics. We next turn to the proof theory. A *proof tree over* Γ is a finite tree $\mathcal{T}$ whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules for the fragment listed in Figure 1. $\Gamma \vdash S$ means that there is a proof tree $\mathcal{T}$ for over Γ whose root is labeled S.

We attached names to the rules in Figure 1 so that we can refer to them later. We usually do not display the names of rules in our proof trees except when to emphasize some point or other. The only purpose of the axioms $\mathit{All}\ X\ \mathit{are}\ X$ is to derive these sentences from all sets; otherwise, the axioms are invisible. The names "Barbara" and "Darii" are traditional from Aristotelian syllogisms. But the (*Antitone*) rule is not part of traditional syllogistic reasoning. It is possible to drop (Some_2) if one changes the conclusion of (*Darii*) to $\mathit{Some}\ Z\ \mathit{are}\ X$. But at one point it will be convenient to have (Some_2), and so this guides the formulation. The rules (*Zero*) and (*One*) are concerned with what is often called vacuous universal quantification. That is, if $Y' \subseteq Y$, then Y is the whole universe and Y' is empty; so Y is a superset of every set and Y' a subset. It would also be possible to use binary rules instead; in the case of (*Zero*), for example, we would infer $\mathit{All}\ X\ \mathit{are}\ Z$ from $\mathit{All}\ X\ \mathit{are}\ Y$ and $\mathit{All}\ X\ \mathit{are}\ Y'$. The (*Contrad*) rule is *ex falso quodlibet*; it permits inference of any sentence S whatsoever from a contradiction. See also Section 1.2 for a different formulation, and Proposition 1.3 and Theorem 4.8 for their equivalence.

Example 2 Returning to Example 1, here is a proof tree showing $\Gamma \vdash \mathit{All}\ A\ \mathit{are}\ C$:

$$\dfrac{\dfrac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ X \quad \dfrac{\mathit{All}\ X\ \mathit{are}\ Y \quad \mathit{All}\ Y\ \mathit{are}\ B}{\mathit{All}\ X\ \mathit{are}\ B}}{\mathit{All}\ B'\ \mathit{are}\ B}}{\mathit{All}\ A\ \mathit{are}\ B} \qquad \dfrac{\mathit{All}\ B\ \mathit{are}\ X \quad \dfrac{\mathit{All}\ X\ \mathit{are}\ Y \quad \mathit{All}\ Y\ \mathit{are}\ C}{\mathit{All}\ X\ \mathit{are}\ C}}{\mathit{All}\ B\ \mathit{are}\ C}}{\mathit{All}\ A\ \mathit{are}\ C}$$

Example 3 Readers who desire an exercise might wish to show that

$$\{\mathit{All}\ B\ \mathit{are}\ X, \mathit{All}\ B'\ \mathit{are}\ X, \mathit{All}\ Y\ \mathit{are}\ C, \mathit{Some}\ A\ \mathit{are}\ C'\} \vdash \mathit{Some}\ X\ \mathit{are}\ Y'.$$

A solution is displayed in the proof of Lemma 4.7.

Lemma 1.1 *The following are derivable:*

1. Some X are $X' \vdash S$ *(a contradiction fact)*
2. All X are Z, No Z are $Y \vdash$ No Y are X *(Celarent)*
3. No X are $Y \vdash$ No Y are X *(E-conversion)*
4. Some X are Y, *No Y are Z* $\vdash$ Some X are Z' *(Ferio)*
5. All Y are Z, All Y are $Z' \vdash$ No Y are Y *(complement inconsistency)*

Proof For the assertion on contradictions,

$$\frac{\dfrac{}{\textit{All } X \textit{ are } X}\;\textit{Axiom} \qquad \textit{Some } X \textit{ are } X'}{S}\;\textit{Contrad}$$

(*Celarent*) in this formulation is just a re-phrasing of (*Barbara*), using complements:

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y'}{\textit{All } Y \textit{ are } Z'}\;\textit{Barbara}$$

(*E-conversion*) is similarly related to (*Antitone*), and (*Ferio*) to (*Darii*). For complement inconsistency, use (*Antitone*) and (*Barbara*). $\dashv$

The logic is easily seen to be sound: if $\Gamma \vdash S$, then $\Gamma \models S$. The main contribution of this paper is the completeness of this system.

Some syntactic abbreviations The language lacks boolean connectives, but it is convenient to use an informal notation for it. It is also worthwhile specifying an operation of *duals*.

$$\begin{array}{lcl|lcl}
\neg(\textit{All } X \textit{ are } Y) & = & \textit{Some } X \textit{ are } Y' & (\textit{All } X \textit{ are } Y)^d & = & \textit{All } Y' \textit{ are } X' \\
\neg(\textit{Some } X \textit{ are } Y) & = & \textit{All } X \textit{ are } Y' & (\textit{Some } X \textit{ are } Y)^d & = & \textit{Some } Y \textit{ are } X
\end{array}$$

Here are some uses of this notation. We say that Γ is *inconsistent* if for some S, $\Gamma \vdash S$ and $\Gamma \vdash \neg S$. The first part of Lemma 1.1 tells us that if $\Gamma \vdash$ *Some X is X'*, then Γ is inconsistent. Also, we have the following result:

Proposition 1.2 *If $S \vdash T$, then $\neg T \vdash \neg S$.*

This fact is not needed below, but we recommend thinking about it as a way of getting familiar with the rules.

1.2 The (RAA) System

Frequently the logic of syllogisms is set up as an *indirect* system, where one in effect takes *Reductio Ad Absurdum* to be part of the system instead of (*Contrad*). We formulate a notion $\Gamma \vdash_{RAA} S$ of indirect proof in this section. It is easy to check that the (RAA) system is stronger than the one with (*Contrad*). It will turn out that the weaker system is complete, and then since the stronger one is sound, it is thus complete as well. In the final section of the paper, we even provide a proof-theoretic reduction.

We define $\Gamma \vdash_{RAA} S$ as follows:

1. If $S \in \Gamma$ or S is *All* X *are* X, then $\Gamma \vdash_{RAA} S$
2. For all rules in Figure 1 except the contradiction rule, if S_1 and S_2 are the premises of some instance of the rule, and T the conclusion, if $\Gamma \vdash_{RAA} S_1$ and $\Gamma \vdash_{RAA} S_2$, then also $\Gamma \vdash_{RAA} T$.
3. If $\Gamma \cup \{S\} \vdash_{RAA} T$ and $\Gamma \cup \{S\} \vdash_{RAA} \neg T$, then $\Gamma \vdash_{RAA} \neg S$.

In effect, one is adding hypothetical reasoning in the manner of the sequent calculus.

Proposition 1.3 *If* $\Gamma \vdash S$, *then* $\Gamma \vdash_{RAA} S$.

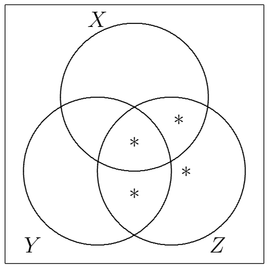
Proof By induction on the heights of proof trees for $\vdash$. The only interesting step is when $\Gamma \vdash S$ via application of the contradiction rule. So for some T, $\Gamma \vdash T$ and $\Gamma \vdash \neg T$. Using the induction hypothesis, $\Gamma \vdash_{RAA} T$ and $\Gamma \vdash_{RAA} \neg T$. Clearly we also have $\Gamma \cup \{\neg S\} \vdash_{RAA} T$ and $\Gamma \cup \{\neg S\} \vdash_{RAA} \neg T$. Hence $\Gamma \vdash_{RAA} S$. $\dashv$

It is natural to ask whether the converse holds. We show that it does in Section 4. using proof theory, and before this via a semantic argument using completeness.

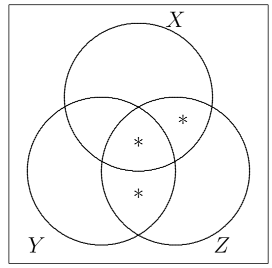
1.3 Comparison with previous work

The proof system in this paper, the one presented by the rules in Figure 1, appears to be new. However, the indirect system appears to be close to the earlier work of Corcoran [3] and Martin [6]. Thus, the fact that the systems turn out to be equivalent is of some interest. In any case, these papers are mostly concerned with modern reconstruction of Aristotleian syllogisms, as is the pioneering work in this area, Łukasiewicz's book [5]. We are not so concerned with this project, but rather our interest lies in logical completeness results for fragments of natural language. The fragment in this paper is obviously quite small, but we believe that the techniques used in studying it may help with larger fragments. This is the main reason for this work.

We write $\mathcal{L}(\textit{all}, \textit{some})$ for the fragment of $\mathcal{L}(\textit{all}, \textit{some}, ')$ without the complement. That is, one drops the complementation from the syntax, treating $\mathcal{V}$ as a set (simpliciter) rather than a set with an operation. One also drops the requirement that $[\![X']\!]$ be the complement of $[\![X]\!]$, since this now makes no sense. For the proof theory, use the rules on the top half of Figure 1. In [7] we checked the completeness of this fragment. We also considered the extension $\mathcal{L}(\textit{all}, \textit{some}, \textit{no})$ of $\mathcal{L}(\textit{all}, \textit{some})$ with sentences *No* X *are* Y. For that, one needs the (*Contrad*) rule and also some additional axioms: the equivalence of *No* X *are* Y and *All* X *are* Y' cannot be stated without complements. Indeed, the language $\mathcal{L}(\textit{all}, \textit{some}, ')$ of this paper is more expressive than $\mathcal{L}(all, some, no)$ in the following precise sense: Consider the two models $\mathcal{M}$ and $\mathcal{N}$ shown below:



$\mathcal{M}$



$\mathcal{N}$

They satisfy the same sentences in $\mathcal{L}(all, some, no)$. (They also satisfy the same sentences of the form *Some A are B*$'$.) But let S be *Some X*$'$ *are Y*$'$ so that $\neg S$ is *All X*$'$ *are Y*. $\mathcal{M} \models S$ but $\mathcal{N} \models \neg S$. We conclude from this example is that a logical system for the language with complements cannot simply be a translation into the smaller language.

The work in Section 2 is not new, though after I obtained the results and looked a little, I first thought so. Later I found the paper of Calude, Hertling, and Svozil [2], and this contains Theorem 2.2. It, too, was not the first work to explore the area, mentioning the papers by N. Zierler and M. Schlessinger [11] and F. Katrnoška [4] as having results which seem to be variations on the same representation theorem. We have included the proofs, mainly because we need to know in Theorem 2.2 that the map m preserves the order in *both* directions: the statement in [2] only has m preserving the order and being one-to-one. Still the proof is essentially the same as in [2].

2 Completeness via representation of orthoposets

An important step in our work is to develop an *algebraic semantics* for $\mathcal{L}(all, some, ')$. There are several definitions, and then a representation theorem. As with other uses of algebra in logic, the point is that the representation theorem is also a *model construction technique*.

An *orthoposet* is a tuple $(P, \leq, 0, ')$ such that

1. $(P, \leq)$ is a partial order: $\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.
2. 0 is a minimum element: $0 \leq p$ for all $p \in P$.
3. $x \mapsto x'$ is an antitone map in both directions: $x \leq y$ iff $y' \leq x'$.
4. $x \mapsto x'$ is involutive: $x'' = x$.
5. complement inconsistency: If $x \leq y$ and $x \leq y'$, then $x = 0$.

The notion of an orthoposet mainly appears in papers on quantum logic. (In fact, the stronger notion of an *orthomodular poset* appears to be more central there. However, I do not see any application of this notion to logics of the type considered in this paper.)

Example 4 For example, for all sets X we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $\subseteq$ is the inclusion relation, $\emptyset$ is the empty set, and $a' = X \setminus a$ for all subsets a of X.

Example 5 Let Γ be any set of sentences in $\mathcal{L}(all, some, ')$. Γ need not be consistent. We define a relation $\leq_\Gamma$ on the set $\mathcal{V}$ of variables of our logical system by

$$X \leq_\Gamma Y \quad \text{iff} \quad \Gamma \vdash \textit{All } X \textit{ are } Y.$$

We always drop the subscript Γ because it will be clear from the context which set Γ is used. We have an induced equivalence relation $\equiv$, and we take $\mathcal{V}_\Gamma$ to be the quotient $\mathcal{V}/\!\!\equiv$. It is a partial order under the induced relation. If there is some X such that $X \leq X'$, then for all Y we have $[X] \leq [Y]$ in $\mathcal{V}/\!\!\equiv$. In this case, set 0 to be $[X]$ for any such X. (If such X exists, its equivalence class is unique.) We finally define $[X]' = [X']$. If there is no X such that $X \leq X'$, we add fresh elements 0 and 1 to $\mathcal{V}/\!\!\equiv$. We then stipulate that $0' = 1$, and that for all $x \in \mathcal{V}_\Gamma$, $0 \leq x \leq 1$.

It is not hard to check that we have an orthoposet $\mathcal{V}_\Gamma = (\mathcal{V}_\Gamma, \leq, 0, ')$. The antitone property comes from the axiom with the same name, and the complement inconsistency is verified using the similarly-named part of Lemma 1.1.

A *morphism of orthoposets* is a map m preserving the order (if $x \leq y$, then $mx \leq my$), the complement $m(x') = (mx)'$, and minimum elements ($m0 = 0$). We say m is *strict* if the following extra condition holds: $x \leq y$ iff $mx \leq my$.

A *point* of a orthoposet $P = (P, \leq, 0, ')$ is a subset $S \subseteq P$ with the following properties:

1. If $p \in S$ and $p \leq q$, then $q \in S$ (S is *up-closed*).
2. For all p, either $p \in S$ or $p' \in S$ (S is *complete*), but not both (S is *consistent*).

Example 6 Let $X = \{1, 2, 3\}$, and let $\mathcal{P}(X)$ be the power set orthoposet from Example 4. Then S is a point, where

$$S \quad = \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

(More generally, if X is any finite set, then the collection of subsets of X containing more than half of the elements of X is a point of $\mathcal{P}(X)$.) Also, it is easy to check that the points on this $\mathcal{P}(X)$ are exactly S as above and the three principal ultrafilters. S shows that a point of a boolean algebras need not be an ultrafilter or even a filter. Also, the lemma just below shows that for $\mathcal{P}(X)$, a collection of elements is included in a point iff every pair of elements has a non-empty intersection.

Lemma 2.1 *For a subset S_0 of an orthoposet $P = (P, \leq, ')$, the following are equivalent:*

1. *S_0 is a subset of a point S in P.*
2. *For all $x, y \in S_0$, $x \not\leq y'$.*

Proof Clearly (1) $\implies$ (2). For the more important direction, use Zorn's Lemma to get a

$\subseteq$-maximal superset S_1 of S_0 with the consistency property. Let $S = \{q : (\exists p \in S_1) q \geq p\}$. So S is up-closed. We check that consistency is not lost: suppose that $r, r' \in S$. Then there are $q_1, q_2 \in S_1$ such that $r \geq q_1$ and $r' \geq q_2$. But then $q_2' \geq r \geq q_1$. Since $q_1 \in S_1$, so too $q_2' \in S_1$. Thus we see that S_1 is not consistent, and this is a contradiction. To conclude, we only need to see that for all $r \in P$, either r or r' belongs to S. If $r \notin S$, then $r \notin S_1$. By maximality, there is $q \in S_1$ such that $q_1 \leq r'$. (For otherwise, $S_1 \cup \{r\}$ would be a consistent proper superset of S_1.) And as $r' \notin S$, there is $q_2 \in S_1$ such that $q_2 \leq r$. Then as above $q_1 \leq q_2'$, leading to the same contradiction. $\dashv$

We now present a representation theorem that implies the completeness of the logic. It is due to Calude, Hertling, and Svozil [2]. We also state an additional technical point.

Theorem 2.2 ([2]; see also [4, 11]) *Let $P = (P, \leq, ')$ be an orthoposet. There is a set* points(P) *and a strict morphism of orthoposets $m : P \to \mathcal{P}(\text{points}(P))$.*

Moreover, if $S \cup \{p\} \subseteq P$ has the following two properties, then $m(p) \setminus \bigcup_{q \in S} m(q)$ is non-empty:

1. *For all $q \in S$, $p \not\leq q$.*

2. *For all $q, r \in S$, $q \not\geq r'$.*

Proof Let *points*(P) be the collection of points of P. The map m is defined by $m(p) = \{S : p \in S\}$. The preservation of complement comes from the completeness and consistency requirement on points, and the preservation of order from the up-closedness. Clearly $m0 = \emptyset$. We must check that if $q \not\geq p$, then there is some point S such that $p \in S$ and $q \notin S$. For this, take $S = \{q\}$ in the "moreover" part. And for that, let $T = \{p\} \cup \{q' : q \in S\}$. Lemma 2.1 applies, and so there is some point $U \supseteq T$. Such U belongs to $m(p)$. But if $q \in S$, then $q' \in T \subseteq U$; so U does not belong to $m(q)$. $\dashv$

2.1 Completeness

The completeness theorem is based on algebraic machinery that we have just seen.

Lemma 2.3 *Let $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some}, ')$. There is a model $\mathcal{M} = (M, [\![\ ]\!])$ such that*

1. *$\mathcal{M} \models \Gamma_{all}$.*

2. *If T is a sentence in* All *and $\mathcal{M} \models T$, then $\Gamma \vdash T$.*

3. *If Γ is consistent, then also $\mathcal{M} \models \Gamma_{some}$.*

Proof Let $\mathcal{V} = \mathcal{V}_\Gamma$ be the orthoposet from Example 5 for Γ. Let n be the natural map of

$\mathcal{V}$ into $\mathcal{V}_\Gamma$, taking a variable X to its equivalence class $[X]$. If $X \leq Y$, then $[X] \leq [Y]$ by definition of the structure. In addition, n preserves the order in both directions. We also apply Theorem 2.2, to obtain a strict morphism of orthoposets m as shown below:

$$\mathcal{V} \xrightarrow{\quad n \quad} \mathcal{V}_\Gamma \xrightarrow{\quad m \quad} \mathit{points}(\mathcal{V}_\Gamma)$$

Let $M = \mathit{points}(\mathcal{V}_\Gamma)$, and let $[\![\ ]\!] : \mathcal{V} \to \mathcal{P}(M)$ be the composition $n \circ m$. We thus have a model $\mathcal{M} = (\mathit{points}(\mathcal{V}_\Gamma), [\![\ ]\!])$.

We check that $\mathcal{M} \models \Gamma$. Note that n and m are strict monotone functions. So the semantics has the property that the *All* sentences holding in $\mathcal{M}$ are exactly the consequences of Γ. We turn to a sentence in Γ_{some} such as *Some U are V*. Assuming the consistency of Γ, $U \not\leq V'$. Thus $[\![U]\!] \not\subseteq ([\![V]\!])'$. That is, $[\![U]\!] \cap [\![V]\!] \neq \emptyset$. ⊣

Unfortunately, the last step in this proof is not reversible, in the following precise sense. $U \not\leq V'$ does not imply that $\Gamma \vdash$ *Some U are V*. (For example, if Γ is the empty set we have $U \not\leq V'$, and indeed $\mathcal{M}(\Gamma) \models$ *Some U are V*. But Γ only derives valid sentences.

Lemma 2.4 (Pratt-Hartmann [9]) *Suppose that* $\Gamma \models$ Some X are Y. *Then there is some existential sentence in* Γ*, say* Some A are B*, such that*

$$\Gamma_{all} \cup \{\text{Some } A \text{ are } B\} \models \text{Some } X \text{ are } Y.$$

Proof If not, then for every $T \in \Gamma_{some}$, there is a model $\mathcal{M}_T \models \Gamma_{all} \cup \{T\}$ and $\mathcal{M} \models$ *All X are Y'*. Take the disjoint union of the models $\mathcal{M}_T$ to get a model of $\Gamma_{all} \cup \Gamma_{some} = \Gamma$ where S fails. ⊣

Theorem 2.5 $\Gamma \vdash S$ *iff* $\Gamma \models S$.

Proof As always, the soundness half is trivial. Suppose that $\Gamma \models S$; we show that $\Gamma \vdash S$. We may assume that Γ is consistent.

If S is a sentence in *All*, consider $\mathcal{M}(\Gamma)$ from Lemma 2.3. It is a model of Γ, hence of S; and then by the property the second part of the lemma, $\Gamma \vdash S$.

For the rest of this proof, let S be *Some X are Y*. From Γ and S, we find A and B satisfying the conclusion of Lemma 2.4.

We again use Lemma 2.3 and consider the model $\mathcal{M} = \mathcal{M}(\mathcal{V}_{\Gamma_{all}})$ of points on $\mathcal{V}_{\Gamma_{all}}$. $\mathcal{M} \models \Gamma_{all}$.

Consider $\{[A], [B], [X']\}$. If this set were a subset of a point x, then consider $\{x\}$ as a one-point submodel of $\mathcal{M}$. In the submodel, $\Gamma_{all} \cup \{$*Some A are B*$\}$ would hold, and yet *Some X are Y* would fail since $[\![X]\!] = \emptyset$.

We use Lemma 2.1 to divide into cases:

1. $A \leq A'$.

2. $A \leq B'$.

3. $A \leq X$.

1. All substitution instances of propositional tautologies.
2. *All* X *are* X
3. $(\textit{All } X \textit{ are } Z) \wedge (\textit{All } Z \textit{ are } Y) \to \textit{All } X \textit{ are } Y$
4. $(\textit{All } Y \textit{ are } Z) \wedge (\textit{Some } X \textit{ are } Y) \to \textit{Some } Z \textit{ are } X$
5. $\textit{Some } X \textit{ are } Y \to \textit{Some } X \textit{ are } X$
6. $\neg(\textit{Some } X \textit{ are } X) \to \textit{All } X \textit{ are } Y$
7. $\textit{Some } X \textit{ are } Y' \leftrightarrow \neg(\textit{All } X \textit{ are } Y)$

Figure 2: Axioims for a system which adds sentential boolean connectives

4. $B \leq B'$.
5. $B \leq X$.
6. $X' \leq X$.

(More precisely, the first case would be $[A] \leq [A']$. By strictness of the natural map, this means that $A \leq A'$; that is, $\Gamma_{all} \vdash \textit{All } A \textit{ are } A'$.) In cases (1), (2), and (4), we easily see that Γ is inconsistent, contrary to the assumption at the outset. Case (6) implies that both (3) and (5) hold. Thus we may restrict attention to (3) and (5).

Next, consider $\{A, B, Y'\}$. The same analysis gives two other cases, independently: $A \leq Y$, and $B \leq Y$. Putting these together with the other two gives four pairs. The following are representative:

$\underline{A \leq X \text{ and } B \leq Y}$: Using *Some* A *are* B, we see that $\Gamma \vdash \textit{Some } X \textit{ are } Y$.

$\underline{A \leq X \text{ and } A \leq Y}$: We first derive *Some* A *are* A, and then again we see $\Gamma \vdash \textit{Some } X \textit{ are } Y$.

This completes the proof. $\dashv$

3 Going Further: Boolean Connectives Inside and Out

The main work of this paper has been completed. However, we wish to continue a little further, mentioning a larger system whose completeness can be obtained by using our the results which we have seen.

We have in mind the language of *boolean compounds of* $\mathcal{L}(\text{all}, \text{some}, ')$ *sentences.* This language is just propositional logic built over $\mathcal{L}(\textit{all}, \textit{some}, ')$ as the set of atomic propositions. We call this larger language $\mathcal{L}(\textit{all}, \textit{some}, ', \textit{bc})$. For the semantics, we use the same kind of structures $\mathcal{M} = (M, [\![\ ]\!])$ as we have been doing in this paper. The semantics treats the boolean connectives classically. So we have notions like $\mathcal{M} \models S$ and $\Gamma \models S$.

The system is a Hilbert-style one, with axioms listed in Figure 2. The only rule is modus ponens.

We define $\vdash_{bc} S$ in the usual way, and then we say that $\Gamma \vdash_{bc} S$ if there are $T_1, \ldots, T_n$ from Γ such that $\vdash_{bc} (T_1 \wedge \cdots \wedge T_n) \to S$.

Rules 1–6 are essentially the system SYLL from [5]. Łukasiewicz and Słupecki proved a completeness and decidability result for SYLL, and different versions of this result may be found in Westerståhl [10] and in [7].

Proposition 3.1 $\vdash_{bc}$ All X are $Y \rightarrow$ All Y' are X'.

Lemma 3.2 *Let* $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some}, ')$*. If* $\Gamma \vdash S$*, then* $\Gamma \vdash_{bc} S$*.*

Proof By induction on the relation $\Gamma \vdash S$. (*Some*$_2$) comes from axioms 2 and 4. For (*Antitone*), use the point about (*Some*$_2$) and also axiom 7 twice. For (*One*), use axiom 7 to see that *All* X' *are* $X \leftrightarrow \neg$(*Some* X *are* X'). This along with axiom 6 gives *All* X' *are* $X \leftrightarrow$ *All* X' *are* Y'. Now we use our point about (*Antitone*) to see that *All* X' *are* $X \leftrightarrow$ *All* Y *are* X. For (*Contrad*), use the Deduction Theorem and a propositional tautology. ⊣

Theorem 3.3 *The logical system above is sound and complete for* $\mathcal{L}(\text{all}, \text{some}, ', bc)$*:* $\Gamma \models S$ *iff* $\Gamma \vdash_{bc} S$*.*

Proof The soundness being easy, here is a sketch of the completeness. We use compactness and disjunctive normal forms to reduce completeness to the verification that every consistent conjunction of $\mathcal{L}(all, some, ')$ sentences and their negations has a model. But $\mathcal{L}(all, some, ')$ essentially has a negation, so we need only consider consistent conjunctions. Now consistency here is in the propositional sense ($\vdash_{bc}$). But by Lemma 3.2, this implies consistency in the sense of our earlier logic ($\vdash$). And here we use Lemma 2.3. ⊣

A second proof We have another proof of this result, one which uses the completeness of SYLL and also the algebraic work from earlier in this paper.

The completeness of a system with classical negation reduces to the matter of showing that consistent sets Γ in $\mathcal{L}(all, some, ', bc)$ are satisfiable. Fix such a set Γ. We assume that Γ is maximal consistent. This implies first of all that Γ is closed under deduction in our logic, and so it contains all instances of the sentence in Proposition 3.1. It also implies that we need only build a model of $\Gamma \cap \mathcal{L}(all, some, ')$.

In our setting, we need a model in the style of this paper; in particular, we need the interpretation of complementary variables to be complementary sets. However, let us forget about this requirement for a moment, and pretend that U and U' are unrelated variables, except for what is dictated by the logic. That is, we consider Γ to be set of sentences in the boolean closure of syllogistic logic taken over a set of variables which is two copies of ours set $\mathcal{V}$. By completeness of that logic, $\Gamma \cap \mathcal{L}(all, some, ')$ has a model. Call it $\mathcal{M}$.

The problem is that since we forgot a key requirement, it probably will not hold in $\mathcal{M}$. So $[\![U]\!]_{\mathcal{M}}$ and $[\![U']\!]_{\mathcal{M}}$ need not be complementary sets: the best we can say is that these sets are disjoint by axiom 7. In other words, $\mathcal{M}$ is *not* the kind of model which we use to define the semantic notions of the language, and thus we cannot use it is not directly of use in the completeness result. We must adjust the model in such a way as to (1) put each point in either $[\![U]\!]$ or $[\![U']\!]$, and at the same time (2) not changing the truth value of any sentence in the language $\mathcal{L}(all, some, ')$. The key is to use some of the algebraic work from earlier in the paper.

Consider the following orthoposet which we call $\mathcal{V}_{\mathcal{M}}$. Let $\mathcal{V}$ be the variables, let $\leq$ be defined by $U \leq V$ iff $[\![U]\!]_{\mathcal{M}} \subseteq [\![V]\!]_{\mathcal{M}}$. The points of $\mathcal{V}_{\mathcal{M}}$ are the equivalences classes of variables under the associated equivalence relation, and the order is the inherited one. Proposition 3.1 implies that if $U \leq V$, then also $V' \leq U'$. So we can define the complementation on $\mathcal{V}_{\mathcal{M}}$ by $[U]' = [U']$. Further, Proposition 3.1 also implies that the order is antitone. The complement inconsistency property comes from the fact that $[\![V]\!]$ and $[\![V']\!]$ are disjoint sets. (We may also need to add a 0 and 1 to $\mathcal{V}_{\mathcal{M}}$ if the structure so far has no minimum element. This is as in Example 5.)

For each $x \in M$, let

$$S_0(x) \quad = \quad \{[U] : x \in [\![U]\!]_{\mathcal{M}}\}.$$

This set is well-defined. We wish to apply Lemma 2.1 to each set $S_0(x)$. To be sure that the lemma applies, we must check that for $[U]$, $[V] \in S_0(x)$, $[U] \not\leq [V']$. The point x itself belongs to $[\![U]\!]_{\mathcal{M}} \cap [\![V]\!]_{\mathcal{M}}$, and as $[\![V]\!] \cap [\![V']\!] = \emptyset$, we have $[\![U]\!] \not\subseteq [\![V']\!]$.

For each x, let $S(x)$ be a point of $\mathcal{V}_{\mathcal{M}}$ including $S_0(x)$. Define a model $\mathcal{N} = (M, [\![\ \ ]\!])$ by using the same universe M as underlies $\mathcal{M}$, and then

$$[\![U]\!]_{\mathcal{N}} \quad = \quad \{x \in M : U \in S(x)\}.$$

Since each $S(x)$ is a point, $\mathcal{N}$ is a bona fide structure for the language; that is, the semantic evaluation map preserves complements.

We claim that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences. Let $\mathcal{M} \models$ *All* U *are* V. Thus $U \leq V$ in $\mathcal{V}_{\mathcal{M}}$. Then since points are closed upwards, $[\![U]\!]_{\mathcal{N}} \subseteq [\![V]\!]_{\mathcal{N}}$.

Finally, suppose that $\mathcal{M} \models$ *Some* U *are* V. Let $x \in [\![U]\!]_{\mathcal{M}} \cap [\![V]\!]_{\mathcal{M}}$. Then $\{U, V\} \subseteq S_0(x) \subseteq S(x)$, so $x \in [\![U]\!]_{\mathcal{N}} \cap [\![V]\!]_{\mathcal{N}}$.

We now know that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences. Since $\mathcal{N}$ is the kind of model we consider in the semantics, $\Gamma \cap \mathcal{L}(\textit{all}, \textit{some}, ')$ is satisfiable.

It is possible to add boolean compounds of the NPs in this fragment. Once one does this, the axiomatization and completeness result become quite a bit simpler, since the system becomes a variant of boolean algebra.

4 *Ex Falso Quodlibet* versus *Reductio ad Absurdum*

Our proof system employs the (*Contrad*) rule, also known as *Ex Falso Quodlibet*. We also formulated the stronger system using *Reductio ad Absurdum* in Section 1.2. It follows trivially from the completeness of the (EFQ) system and the fact that it is stronger than the (RAA) system that the latter is complete. The argument is semantic. It might be of interest to those working on *proof-theoretic semantics* (see Ben Avi and Francez [1] and other papers by Nissim Francez) to see the explicit reduction.[1]

Until the very end of this section, the only proof system used is for $\vdash_{RAA}$, and all trees shown are for that system.

[1] And then again, perhaps it would not be of any interest. As I submit this paper in November 2007, I am undecided about whether to drop this entire section.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{G}\\ \vdots\\ \textit{All } B' \textit{ are } B\end{array}}{\textit{All } A \textit{ are } B}\,\textit{One} \qquad \begin{array}{c}\mathcal{H}\\ \vdots\\ \textit{All } B \textit{ are } C\end{array}}{\textit{All } A \textit{ are } C} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{G}_1\\ \vdots\\ \textit{All } B' \textit{ are } B\end{array}}{\textit{All } A \textit{ are } B}\,\textit{One} \qquad \dfrac{\begin{array}{c}\mathcal{G}_0\\ \vdots\\ \textit{All } B \textit{ are } B'\end{array}}{\textit{All } B \textit{ are } C}\,\textit{Zero}}{\textit{All } A \textit{ are } C}$$

$$\dfrac{\begin{array}{c}\mathcal{H}\\ \vdots\\ \textit{All } A \textit{ are } B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{G}\\ \vdots\\ \textit{All } B \textit{ are } B'\end{array}}{\textit{All } B \textit{ are } C}\,\textit{Zero}}{\textit{All } A \textit{ are } C}$$

Figure 3: Normal forms. The principal subtrees must be simple and injective, but we allow degenerate cases where one of these is absent.

4.1 Injective proofs and normal forms

Let $\mathcal{T}$ be a proof tree over a set Γ, that is, a finite tree whose nodes are labeled with sentences according to the rules of our logic.

$\mathcal{T}$ is *injective* if different nodes have different labels.

$\mathcal{T}$ is *simple* if it is either an axiom alone, or if the leaves are labeled with sentences from Γ and the only rules used are (*Antitone*) at the leaves, and (*Barbara*).

Injectivity is a natural condition on proof trees. Unfortunately, the system does not have the property that all deductions have injective proofs. The simplest counterexample that I could find is the one in Example 2: there is no injective tree for the assertion shown there.

However, the system does admit normal forms which are basically two injective and simple trees put side-by-side. We say that $\mathcal{T}$ is a *normal form* if it is of one of the three forms shown in Figure 3. In the first form the root of $\mathcal{G}$ is labeled *All A are B*, and the root of $\mathcal{H}$ is labeled *All A are D*; $\mathcal{G}$ and $\mathcal{H}$ are the *principal subtrees*. Similar definitions apply to the other two forms. We require/permit that

1. The principal subtrees must be injective and simple.
2. One of the principal subtrees in each form might be missing. So any injective, simple tree is automatically a normal form.
3. The label on the root of $\mathcal{T}$ does not occur as the label of any other node. In the first and third forms, we require that the label on the root of $\mathcal{G}$ not label anywhere on $\mathcal{H}$.

As a consequence of the second point, an injective and simple tree counts as normal, as does an injective tree that uses (*Zero*) or (*One*) at the root and is otherwise simple. A normal form tree need not be injective, because it might contain two principal subtrees which (though each is injective) have node labels in common.

The main advantage of working with simple trees is that the following results holds for them. Adding rules like (*Zero*) and (*One*) destroys Lemma 4.1, as easy counterexamples show.

Lemma 4.1 *Let* $\Gamma \cup \{\text{All } X \text{ are } Y\} \vdash_{RAA} \text{All } U \text{ are } V$ *via a simple proof tree* $\mathcal{T}$. *Then one of the following holds:*

1. $\Gamma \vdash_{RAA}$ All U are X, *and* $\Gamma \vdash_{RAA}$ All Y are V.

2. $\Gamma \vdash_{RAA}$ All U are Y', *and* $\Gamma \vdash_{RAA}$ All X' are V.

Proof By induction on $\mathcal{T}$. If $\mathcal{T}$ is a one-point tree, then $U = X$, and $Y = V$, and the statements in (1) are axioms. If $\mathcal{T}$ is one point followed by an application of the (*Antitone*) rule, we use (2). The induction step for (*Barbara*) is easy. ⊣

In the next two lemmas, we consider a graph $G = (\mathcal{V}, \rightarrow)$ related to Γ and to the overall logic. Here as before, $\mathcal{V}$ is the set of variables, and $U \rightarrow V$ iff Γ contains either *All* U *are* V or *All* V *are* U. Then to say that "$\Gamma \vdash_{RAA}$ *All* U *are* V via a proof tree which is simple" is exactly to say that Y is reachable from X in this graph, or that $X = Y$.

Lemma 4.2 *Let* $\Gamma \vdash_{RAA}$ All U are V *via a proof tree which is simple.*

1. *There is a injective, simple proof tree for* $\Gamma \vdash_{RAA}$ All U are V.

2. *Moreover, there is a tree in which (for all* T*) none of the nodes are labeled* All T are U.

Proof If $X = Y$, then a one-node tree is injective. Otherwise, take a path p of minimal length in G from X to Y. This path p contains no repeated edges, and if $A \rightarrow B$ is on p, then $B \rightarrow A$ is not on p. Moreover, X is not the target of any edge on p; this takes care of the second point in our lemma. There are several ways to turn p into an injective simple proof tree, using induction on the length of the path p. ⊣

Lemma 4.3 *Let* V *be such that* $\Gamma \vdash_{RAA}$ All V are V' *and* $\Gamma \vdash_{RAA}$ All V' are V *via simple proof trees. Then for all variables* A *and* B*, there is a normal form proof tree for* $\Gamma \vdash_{RAA}$ All A are B.

Proof Again we consider the graph $G = (\mathcal{V}, \rightarrow)$. The hypothesis tells us that there is a cycle $V \rightarrow \cdots \rightarrow V' \rightarrow \cdots \rightarrow V$. Consider the following proof tree:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } V' \textit{ are } V\end{array}}{\textit{All } A \textit{ are } V}\,\textit{One} \qquad \dfrac{\begin{array}{c}\vdots\\ \textit{All } V \textit{ are } V'\end{array}}{\textit{All } V \textit{ are } B}\,\textit{Zero}}{\textit{All } A \textit{ are } B}$$

For the unshown parts, we use injective simple trees from the last lemma. We can also arrange that the tree overall be injective, by starting with a node V which minimizes the length of the cycle.) ⊣

In the lemma below, and also in the next section, we again use $U \leq V$ to mean that $\Gamma \vdash_{RAA}$ *All* U *are* V. Γ should be clear from context when we do so.

Lemma 4.4 *Let* $\Gamma \vdash_{RAA}$ All X are Y *via a proof tree without contradiction nodes. Then there is a normal form proof tree* $\mathcal{T}$ *for this deduction.*

Proof One way to show this would be to use induction on proof trees. This is not how we proceed, but the details would not be substantially different. Our method is to modify a proof tree $\mathcal{T}$ without contradictions to obtain a normal tree. First, push all applications of (*Antitone*) to the leaves, out of the way. This is not hard, and we leave the details to the reader. Next, prune $\mathcal{T}$ so that all applications of (*Zero*) are on the right branch as we go up from the root, and all applications of (*One*) are on the left branch. This is accomplished by transformations such as

$$\frac{\mathit{All}\ A\ \mathit{are}\ C \qquad \dfrac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ B}{\mathit{All}\ C\ \mathit{are}\ B}\ \mathit{One} \qquad \mathit{All}\ B\ \mathit{are}\ D}{\mathit{All}\ C\ \mathit{are}\ D}}{\mathit{All}\ A\ \mathit{are}\ D} \qquad \Longrightarrow \qquad \frac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ B}{\mathit{All}\ A\ \mathit{are}\ B}\ \mathit{One} \qquad \mathit{All}\ B\ \mathit{are}\ D}{\mathit{All}\ A\ \mathit{are}\ D}$$

Once this is done, we now eliminate multiple applications of (*Zero*) on the right branch and (*One*) on the left. For example, we transform a tree whose root is as on the left to the tree whose root is as on the right.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ B}{\mathit{All}\ A\ \mathit{are}\ B}\ \mathit{One} \qquad \mathit{All}\ B\ \mathit{are}\ D}{\mathit{All}\ A\ \mathit{are}\ D} \qquad \mathit{All}\ D\ \mathit{are}\ A'}{\mathit{All}\ A\ \mathit{are}\ A'}}{\mathit{All}\ E\ \mathit{are}\ A'}\ \mathit{One} \qquad \Bigg| \qquad \frac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ B}{\mathit{All}\ E\ \mathit{are}\ B} \qquad \dfrac{\mathit{All}\ B\ \mathit{are}\ D \qquad \mathit{All}\ D\ \mathit{are}\ A'}{\mathit{All}\ B\ \mathit{are}\ A'}}{\mathit{All}\ E\ \mathit{are}\ A'}$$

We may then arrange all subtrees without (*Zero*) or (*One*) to be injective, using Lemma 4.2. Now we either have a tree as in Figure 3, or else we have a tree such as

$$\frac{\dfrac{\dfrac{\begin{matrix}\mathcal{G}\\ \vdots\\ \mathit{All}\ B'\ \mathit{are}\ B\end{matrix}}{\mathit{All}\ A\ \mathit{are}\ B}\ \mathit{One} \qquad \begin{matrix}\mathcal{H}\\ \vdots\\ \mathit{All}\ B\ \mathit{are}\ C\end{matrix}}{\mathit{All}\ A\ \mathit{are}\ C} \qquad \dfrac{\begin{matrix}\mathcal{I}\\ \vdots\\ \mathit{All}\ C\ \mathit{are}\ C'\end{matrix}}{\mathit{All}\ C\ \mathit{are}\ D}\ \mathit{Zero}}{\mathit{All}\ A\ \mathit{are}\ D}$$

This tree uses (*One*) and (*Zero*) and also a subsidiary derivation in the middle. In the notation of the figure, we would then have proofs from Γ of $B' \leq B$, $B \leq C$, and $C \leq C'$. These proofs only use (*Antitone*) and (*Barbara*). Then we put these together to get a similar proof of $C' \leq B' \leq B \leq C$. So at this point we may forget about our current tree altogether and use Lemma 4.3. This shows that we may assume that only two of $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ are present.

Finally, we need to arrange the last requirement on normal trees. If this fails, then we replace $\mathcal{H}$ by a simple, normal proof tree of the same assertion with the "moreover" condition of Lemma 4.2. This automatically insures that *All A are B* does not occur in $\mathcal{H}$. $\dashv$

4.2 Proofs with and without contradiction

Lemma 4.5 *Let* $\Gamma \vdash_{RAA}$ Some C are D *without the contradiction rule. Then there is a proof tree* $\mathcal{T}$ *for this deduction such that* $\mathcal{T}$ *has exactly one use of (*Darii*), and this is preceded and followed by zero or one applications of (*Some_1*) or (*Some_2*).*

Proof We eliminate successive applications of (*Darii*) at the root, using (*Barbara*):

$$\dfrac{\textit{All B are C} \quad \dfrac{\textit{All A are B} \quad \textit{Some C are A}}{\textit{Some C are B}}}{\textit{Some C are D}} \quad\Longrightarrow\quad \dfrac{\dfrac{\textit{All A are B} \quad \textit{All B are C}}{\textit{All A are C}} \quad \textit{Some C are A}}{\textit{Some C are D}}$$

The only other rules that involve *Some* sentences are ($Some_1$) and ($Some_2$). If more than one is used in either place, the series of uses may be collapsed to just one. $\dashv$

Recall Γ is *inconsistent* if for some S, $\Gamma \vdash S$ and $\Gamma \vdash \neg S$. (That is, this notion is defined using $\vdash$ and not $\vdash_{RAA}$.)

Lemma 4.6 *Let* Γ *be inconsistent. Then for some sentence* Some U are V *in* Γ*,* $\Gamma_{all} \vdash_{RAA}$ All U are V'.

Proof There is a tree $\mathcal{T}$ for a derivation of an inconsistency that itself does not use the contradiction rule. By using Lemma 4.5, we have a tree such as

$$\dfrac{\begin{array}{c}\mathcal{G}\\ \vdots\\ \textit{All U are W}\end{array} \quad \dfrac{\begin{array}{c}\mathcal{H}\\ \vdots\\ \textit{All V are W}'\end{array} \quad \textit{Some U are V}}{\textit{Some U are W}'}\,\textit{Darii}}{S}\,\textit{Contrad}$$

where $\mathcal{G}$ and $\mathcal{H}$ consist entirely of *All* sentences, so they are trees over Γ_{all}. This tree shows the general situation, except that *Some U are V* might be the result of one or two applications of *Some* rules from a *Some* sentence in Γ, and below *Some U are W*$'$ we might similarly find one or two steps leading to *Some W*$'$ *are U*, *Some W*$'$ *are W*$'$, or *Some U are U*. These extra possibilities are easy to handle. So we assume that *Some U are V* belongs to Γ. Using $\mathcal{G}$ and $\mathcal{H}$, we see that $\Gamma_{all} \vdash_{RAA}$ *All U are V*$'$. In the case of these "extra possibilities", we might need to add one or two steps afterwards. $\dashv$

Lemma 4.7 *If* $\Gamma \cup \{$All X are $Y\}$ *is inconsistent, then* $\Gamma \vdash_{RAA}$ Some X are Y'.

Proof By what we have seen, there is a normal form proof tree $\mathcal{T}$ over $\Gamma_{all} \cup \{\textit{All X are Y}\}$ to the negation of a sentence in Γ of the form *Some U are V*. We may assume that *All X are Y* labels a leaf; for if not, Γ is inconsistent. We argue by induction on the length n of a path from a leaf labeled *All X are Y* to the root of such a tree.

If the length is 0, then the leaf *All X are Y* is the root of $\mathcal{T}$. Therefore $X = U$ and $Y = V'$. So since Γ contains *Some U are V* $=$ *Some X are Y*$'$, we are done.

This is the base case of the induction. Assume our lemma for n (for all sets and all sentences as in the statement of our theorem, of course). There are two overall cases: (a) *All X are Y* labels one leaf of either $\mathcal{G}$ or $\mathcal{H}$ (but not both); (b)*All X are Y* labels one leaf of $\mathcal{G}$ and one leaf of $\mathcal{H}$. (Recall that $\mathcal{G}$ and $\mathcal{H}$ are injective, so *All X are Y* labels at most one leaf in each.)

Here is the argument in case (a). Suppose that *All X are Y* labels a leaf in $\mathcal{T}$ as in

$$\begin{array}{c}\dfrac{\textit{All X are Y} \quad \textit{All Y are Z}}{\textit{All X are Z}}\\ \vdots\end{array}$$

so that the length of the path from *All* X *are* Y to the root is n. Drop *All* X *are* Y and *All* Y *are* Z from $\mathcal{T}$ to get a tree $\mathcal{T}'$. So *All* X *are* Z is a leaf of $\mathcal{T}'$, and the length of the path in $\mathcal{T}'$ from this leaf to the root is $n-1$, and $\mathcal{T}'$ is a normal form tree over $\Gamma_{all} \cup \{\textit{All } X \textit{ are } Z\}$. By induction hypothesis, $\Gamma \vdash_{RAA}$ *Some* X *are* Z'. But *All* Y *are* Z belongs to Γ, as only one node of $\mathcal{T}$ is labeled with *All* X *are* Y. (We are using the hypothesis of case (a).) So $\Gamma \vdash_{RAA}$ *Some* X *are* Y'. The case when *All* X *are* Y labels a leaf of $\mathcal{T}$ participating in (*Barbara*) on the right is similar.

There are other subcases: *All* X *are* Y might label a leaf on $\mathcal{T}$ that participates in (*One*), (*Zero*), or the (*Antitone*) rule. We shall go into details on (*One*); (*Zero*) is basically the same, and (*Antitone*) is the easiest since it only occurs at the leaves. For (*One*), the tree $\mathcal{T}$ is

$$\dfrac{\dfrac{\textit{All } B' \textit{ are } B}{\textit{All } A \textit{ are } B}\;\textit{One} \qquad \begin{array}{c}\mathcal{H}\\ \vdots \\ \textit{All } B \textit{ are } C\end{array}}{\textit{All } A \textit{ are } C}$$

So $X = B'$ and $Y = B$. By our third requirement on normal proof trees, $\mathcal{H}$ is a proof tree over Γ, and it does not contain *All* A *are* B anywhere. If we remove *All* B' *are* B, the tree is normal, but not because it matches the first type in Figure 3 (it does not match it, since it lacks an application of (*One*)); instead, it is a simple, injective tree. So by what we just did for (*Barbara*), we see that $\Gamma \vdash_{RAA}$ *Some* A *are* B'. Hence $\Gamma \vdash_{RAA}$ *Some* B' *are* B'; that is, $\Gamma \vdash_{RAA}$ *Some* X *are* Y'.

This concludes our work in case (a). Things are more interesting in case (b). There are three subcases, corresponding to which of the normal form trees in Figure 3 $\mathcal{T}$ exemplifies.

Subcase (i): $\mathcal{T}$ has an instance of (*One*) but not of (*Zero*) as on the top left in Figure 3. Its root *All* A *are* C is the negation of a sentence in Γ. Therefore, Γ contains *Some* A *are* C'. Recall that $\mathcal{G}$ and $\mathcal{H}$ are simple. Now since both $\mathcal{G}$ and $\mathcal{H}$ have leaves labeled *All* X *are* Y, we have four possibilities:

(i.a) $B' \leq X$, $Y \leq B$, $B \leq X$, $Y \leq C$.

(i.b) $B' \leq X$, $Y \leq B$, $B \leq Y'$, $X' \leq C$.

(i.c) $B' \leq Y'$, $X' \leq B$, $B \leq X$, $Y \leq C$.

(i.d) $B' \leq Y'$, $X' \leq B'$, $B \leq Y'$, $X' \leq C$.

Note that (i.a) and (i.c) are the same: the first two assertions in (i.a) are the duals of those in (i.c). Similar results apply to (i.b) and (i.d). So we only go into details on the (i.a) and (i.b). For (i.a), we have proofs from Γ of the three subtrees indicated by $\vdots$ below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } B \textit{ are } X\end{array}}{\textit{All } X' \textit{ are } B'} \qquad \begin{array}{c}\vdots\\ \textit{All } B' \textit{ are } X\end{array}}{\textit{All } X' \textit{ are } X}}{\textit{All } Y' \textit{ are } X} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } Y \textit{ are } C\end{array}}{\textit{All } C' \textit{ are } Y'} \qquad \dfrac{\textit{Some } A \textit{ are } C'}{\textit{Some } C' \textit{ are } C'}}{\textit{Some } C' \textit{ are } Y'}}{\textit{Some } Y' \textit{ are } Y'}}{\dfrac{\textit{Some } Y' \textit{ are } X}{\textit{Some } X \textit{ are } Y'}}$$

Thus, the tree above is a tree over Γ, with the desired conclusion.

(The displayed tree (without the three omitted subproofs) is the solution to the exercise mentioned in Example 3 in Section 1.1.)

In possibility (i.b), one gets $Y \leq B \leq Y'$, so informally, there are no Y. Also, $X \leq C'$, so $C' \leq X$. Since some A are C', some X are Y'.

For (ii), Γ contains *Some* A *are* C'. We again have four possibilities, similar to what we saw in (i) above. We'll only go into brief details on the first and third possibilities. The first would be when Γ derives $B' \leq X$, $Y \leq B$, $B \leq X$, and $Y \leq B'$. In this case, we have *All* B' *are* B and *All* B *are* B'. Hence we also get *All* A *are* X and *All* C' *are* Y' (see Lemma 4.3). Hence we have *Some* X *are* Y'. The third possibility finds $Y \leq B' \leq Y'$ and $X' \leq B \leq X$. So we get *All* A *are* X and *All* C *are* Y' again.

Subcase (iii) is entirely parallel to (i).

This conclude our proof. $\dashv$

Theorem 4.8 *If* $\Gamma \vdash_{RAA} S$*, then* $\Gamma \vdash S$.

Proof By induction on the indirect proof relation $\vdash_{RAA}$. The key step is when $\Gamma \vdash_{RAA} S$ via proofs of $\Gamma \cup \{\neg S\} \vdash_{RAA} T$ and $\Gamma \cup \{\neg S\} \vdash_{RAA} \neg T$. By induction hypothesis, $\Gamma \cup \{\neg S\}$ is inconsistent in the direct logic. When S is a *Some* sentence, Lemma 4.7 tells us that $\Gamma \vdash S$. When S is an *All* sentence, we use Lemma 4.6. $\dashv$

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