

Harsanyi Type Spaces and Final Coalgebras Constructed from Satisfied Theories*

Lawrence S. Moss Ignacio Viglizzo
lsm@cs.indiana.edu igvigliz@indiana.edu
Department of Mathematics
Indiana University
Bloomington, IN 47405-5701 USA

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Abstract

This paper connects coalgebra with a long discussion in the foundations of game theory on the modeling of type spaces. We argue that type spaces are coalgebras, that universal type spaces are final coalgebras, and that the modal logics already proposed in the economic theory literature are closely related to those in recent work in coalgebraic modal logic. In the other direction, the categories of interest in this work are usually measurable spaces or compact (Hausdorff) topological spaces. A coalgebraic version of the construction of the universal type space due to Heifetz and Samet [4] is generalized for some functors in those categories. Since the concrete categories of interest have not been explored so deeply in the coalgebra literature, we have some new results. We show that every functor on the category of measurable spaces built from constant functors, products, coproducts, and the probability measure space functor has a final coalgebra. Moreover, we construct this final coalgebra from the relevant version of coalgebraic modal logic. Specifically, we consider the set of theories of points in all coalgebras and endow this set with a measurable and coalgebra structure.

1 Introduction: Type Spaces

This paper is a first exploration of the application of ideas and results from coalgebra to the foundational area of game theory concerned with *type spaces*. Type spaces are mathematical structures used in modeling settings where agents are described by their *types*, and these types give us “beliefs about the world”, “beliefs about each other’s beliefs about the world”, “beliefs about each other’s beliefs about each other’s beliefs about the world”, etc. That is, the formal concept of a type space is intended to capture in one structure an unfolding infinite hierarchy related to *interactive belief*.

The 1994 Nobel Prize in Economic Sciences was awarded to John C. Harsanyi, John F. Nash Jr., and Reinhard Selten “for their pioneering analysis of equilibria in the theory of non-cooperative games.” In addition to his work on equilibria, Harsanyi will also be remembered

*Due to lack of space, we have not included all proofs in this version. Between the CMCS meeting and the posting of the ENTCS volume from it, we expect that a version of the paper will be available on our web pages.

for his introduction of type spaces in a three-part paper published in 1967 and 1968 [3]. He showed how to convert a game with incomplete information into one with complete yet imperfect information. This matter is not relevant to our paper, but three related points are noteworthy. First, Harsanyi’s notion of types goes further than what we described above: an agent’s type gives us their *beliefs about the types of the other agents*. Second, despite this circularity, the informal concept of a *type* (as a “pool” from which all players can be picked) is widespread in areas of non-cooperative game theory and economic theory. And finally, the *formalization* of type spaces was, and to some extent still is, an open area. That is, Harsanyi did not really formalize type spaces in his original paper; this was left to later researchers. Getting back to our very rough informal description above, what exactly are “beliefs”? And how can a structure contain types which give rise to beliefs about other types? What is the relation of this to the infinite hierarchy of beliefs about beliefs about \dots beliefs about the world? Can we characterize the space of all possible types? So there is a collection of papers on this matter, starting with Böge and Eisele’s paper [1] from 1979. Again, we are not much concerned with these conceptual matters in this paper. Most of the important papers for our study are technical contributions dealing with the matter of *universal type spaces*. A universal type space is intended to capture all possible types, so it is an answer to our third question above.

There are some clear conceptual clues that coalgebra could be connected to type spaces. The first is that the notion of “belief” in the game theory literature is typically a probabilistic one. If we replace “belief” with “knowledge” above, then we have a very-well-studied notion, the formalization of knowledge by *possible worlds semantics*. The mathematical structures for possible worlds semantics are sets W of *worlds* with two functions, one giving for each world $w \in W$ some set of “atomic propositions” true at w , and the other giving for each w some set of worlds which are said to be “possible from w ”. These structures are essentially the coalgebras in the category \mathbf{Set} of sets of the functor $F(W) = A \times \mathcal{P}(W)$, where A is the power set of the set of atomic propositions and \mathcal{P} is the power set functor on sets. Perhaps the primary contribution of coalgebra to this area to date is to show that *modal logic*, the natural logical language for the structures, generalizes to *coalgebraic versions of modal logic*. We’ll return to this point shortly.

The second clue has to do with the role of the universal type space in this field. Types are taken to be elements of the universal type space. In the universal type space all possible types are uniquely represented, and the idea is that two types with exactly the same beliefs about the underlying world of “nature” plus the types of the other players are taken to be undistinguishable. This is the same ideology as we find concerning *final coalgebras*, in which (as is well known) points with the same behavior are identified.

Returning to type spaces, we recall that the usual modeling of belief in game theory is via probability. So we would expect that *type spaces should be probabilistic versions of Kripke models*. One should replace the functor \mathcal{P} with something like Δ , where

$$\Delta(W) = \{\mu \mid \mu \text{ is a probability measure on } W\}. \tag{1}$$

Indeed, this is the case: most proposals in the literature do end up studying certain mappings from a space X to some variation of the functor Δ applied to X . This is our third clue of the connection. But note that (1) leaves a lot lacking: if W is just a set, how do we know that it has any probability measures? Does it matter which σ -algebra we use? And if W is an object in some other category, say measurable spaces or compact metric spaces, then what structure do we put on $\Delta(W)$?

We intend this paper to be a contribution to this area by connecting it with coalgebra. Here are the main conceptual claims of the paper, as well as the main results:

- ★ The original notion of a Harsanyi type space may be taken to be a coalgebra of a functor F on \mathbf{Meas}^I , where \mathbf{Meas} is the category of measurable spaces and measurable maps, and I is a discrete category (of agents). But this reformulation is not obvious, because the original notion had an extra condition that the agents “know their own type.” We discuss this in Section 2.
- ★ Universal type spaces are final coalgebras.
- ★ The constructions of universal type spaces in the literature are related to constructions found in coalgebra. However, there are differences due primarily to the fact that the work is going on in categories like \mathbf{Meas} rather than \mathbf{Set} . Putting things differently, the universal type space constructions could be generalized, and this could be of interest in coalgebra.
- ★ There are versions of coalgebraic modal logic for functors of interest, and one can prove the existence of final coalgebras by considering the *satisfied theories* in these logics. This construction parallels the construction of a final coalgebra on \mathbf{Set} for $F(W) = A \times \mathcal{P}(W)$ (or rather $A \times \mathcal{P}_{fin}(W)$, using the finite power set functor) using the set of descriptions of all modal theories of all possible worlds. We work a formulation of coalgebraic modal logic based on the one due to Jacobs [6], following work of Rößiger [9, 10]. But we do differ from these papers in that our final coalgebras are not connected to logical theories, but to the theories realized in coalgebras. In the economic literature, this construction is due to Heifetz and Samet [4].
- ★ We formulate a notion of “measure polynomial functor” on \mathbf{Meas} . We prove a new result in Section 3: every such polynomial has a final coalgebra with carrier consisting of sets of sets of sentences in a logic. The method also works for polynomial functors on \mathbf{Set} , as we show in Section 5.2. Our work also gives a new proof of the result of Kupke, Kurz, and Venema [7] that the Vietoris polynomial endofunctors on the category of Stone spaces have final coalgebras. Our proof is probably implicit in [7]; their paper derives the result from the study of a great deal of structure, and we have not (yet) needed to do the analogous work in our settings.

1.1 Background notions

A measurable space is a pair $M = (M, \Sigma)$, where M is a set and Σ is a σ -algebra of subsets of M . The sets in Σ are called *measurable sets* or *events*. Usually Σ contains all singletons $\{x\}$; we shall almost always assume a weaker condition that for each $x \in M$, $\{x\}$ is the intersection of the measurable subsets of M containing x . A collection \mathcal{B} of subsets of M *generates* a σ -algebra Σ if Σ is the smallest σ -algebra including \mathcal{B} . A *measure on M* is a σ -additive function $\mu : \Sigma \rightarrow [0, \infty]$. The measure μ is a *probability measure* if $\mu(M) = 1$.

A morphism of measurable spaces $f : (M, \Sigma) \rightarrow (N, \Sigma')$ is a function $f : M \rightarrow N$ such that for each $A \in \Sigma'$, $f^{-1}(A) \in \Sigma$. This gives a category \mathbf{Meas} . \mathbf{Meas} has products and coproducts; indeed it has much more structure. There is an endofunctor $\Delta : \mathbf{Meas} \rightarrow \mathbf{Meas}$

defined by: $\Delta(M)$ is the set of probability measures on M with the σ -algebra generated by $\{\beta^p(E) : p \in [0, 1], E \in \Sigma\}$, where

$$\beta^p(E) = \{\mu \in \Delta(M) \mid \mu(E) \geq p\}.$$

Here is how Δ acts on morphisms. If $f : M \rightarrow N$ is measurable, then for $\mu \in \Delta(M)$ and $A \in \Sigma'$, $(\Delta f)(\mu)(A) = \mu(f^{-1}(A))$. That is, $(\Delta f)(\mu) = \mu \circ f^{-1}$. For more on the functorial aspects of Δ , including important points on a related monad, see Giry [2].

We also note some additional structure. First, there is a natural transformation $\delta : Id \rightarrow \Delta$ defined by $\delta_M(m)(E) = 1$ if $m \in E$ and 0 if $m \notin E$. We also write δ_m instead of $\delta_M(m)$; this is the Dirac measure supported at m .

Lemma 1.1 *For each $p \in [0, 1]$, β^p may be regarded as a predicate lifting. That is β^p takes measurable subsets of each space M to measurable subsets of $\Delta(M)$, and it is natural in the sense that if $f : M \rightarrow N$, then for all measurable $E \subseteq Y$, $\beta^p(f^{-1}(E)) = (\Delta f)^{-1}(\beta^p(E))$.*

Lemma 1.2 *Suppose that the collection \mathcal{B} of sets generates Σ . Then the σ -algebra on $\Delta(M)$ is the generated by the sets of the form $\{\beta^p(E) \mid E \in \mathcal{B} \text{ and } p \in [0, 1]\}$.*

We also need at some point to look more closely at products in **Meas**. Given two measurable spaces, A and B , their product is the cartesian product of the sets A and B , endowed with the σ -algebra generated by the “rectangles” obtained as cartesian product of a measurable subset of A times a measurable subset of B . With this σ -algebra, both projections are measurable. For a subset $E \subseteq A \times B$, the sections of E are the sets: $E_a = \{b : (a, b) \in E\}$, $E^b = \{a : (a, b) \in E\}$. Each section of a measurable subset of the product is measurable.

If μ is a probability measure on A and ν a probability measure on B , we can define the probability measure $\mu \times \nu$ on $A \times B$ by $(\mu \times \nu)(E) = \int \mu(E^b) d\nu = \int \nu(E_a) d\mu$.

Going in the other direction, a probability measure μ on $A \times B$ induces via the projections, a measure on each of the factor spaces. These measures are called *marginals*, and denoted by $mar_A \mu = (\Delta \pi_A) \mu = \mu \circ \pi_A^{-1}$; $mar_B \mu = (\Delta \pi_B) \mu = \mu \circ \pi_B^{-1}$.

Lemma 1.3 *Let μ be a probability measure on a product measurable space $A \times B$. If $mar_B \mu = \delta_{b_0}$ for some $b_0 \in B$ then $\mu = mar_A \mu \times \delta_{b_0}$.*

2 Formulation of Type Spaces as Coalgebras

The first constructions of Harsanyi type spaces were based on a hierarchy of beliefs. In this hierarchical approach, as seen for example in [8], types are constructed by detailing the players’ beliefs about nature, about the other players’ beliefs about nature, etc. Since each type gives a probability measure on the set of all types, there is a function from types to probability measures on states of nature and types. As we noted before, this is almost a coalgebra, except that there are some extra conditions imposed on these functions.

At this point, we need to formulate the category of interest. Fix a set I of *players*. We assume that $0 \notin I$ and we then define $I_0 = I \cup \{0\}$. This “0” stands for “nature”, and so I_0 includes nature as one of the players (but one who won’t have beliefs about the other players).

We consider I and I_0 as discrete categories. Usually I is finite, but nothing we do hinges on this.

We shall be interested in Meas^I . The objects here are families $X = (X_i)_{i \in I}$ of measurable spaces, and the morphisms are also tuples of measurable maps. Fix a measurable space M to represent the “states of nature”, and write X_0 for M . Each player should have beliefs about nature and about the beliefs of the other player. This leads to the following definition.

Let $C : \text{Meas}^I \rightarrow \text{Meas}$ be the functor given by

$$CX = \prod_{i \in I_0} X_i$$

At first glance, it might look like what we want is to consider for each $x_i \in X_i$, a probability measure on CX . However, this is *not* what we want because it misses an important intuition concerning type spaces. This is that *players should know their own types*. In other words, each player i should only have beliefs about the (joint distribution on) other players’ beliefs; i ’s own beliefs should not even enter in. Thus we define functors $U_i : \text{Meas}^I \rightarrow \text{Meas}$ given by

$$U_i(X) = \prod\{X_j \mid j \in I_0, j \neq i\}.$$

U_i acts the obvious way on morphisms. Note that U_i depends on the space M of nature, even though our notation does not mention this.

Now $CX = U_i X \times X_i$ up to isomorphisms. The fact we mentioned above, about player i knowing her own type is modeled in [4] by adding the condition that the corresponding measure on CX has marginal δ_{x_i} on X_i . Here is where Lemma 1.3 plays a role allowing us to recast type spaces as coalgebras: it is essentially the same to consider measures on $U_i X$ and measures on CX for which the marginal on X_i is a Dirac measure on a point (and we can easily tell which point that would be).

Finally our main innovation and the one that lets us model type spaces as coalgebras is to work in Meas^I , rather than working with the product of the different spaces considered.

Let $F : \text{Meas}^I \rightarrow \text{Meas}^I$ be defined by

$$F(X) = (\Delta U_i(X))_{i \in I}$$

As before Δ is the probability measure space functor, and once again, our notation elides the underlying space M of nature.

This way, instead of having a family of functions in Meas , each one of them with a condition on one of its marginals, any morphism in Meas^I works as a coalgebra structure. The particular functor we use automatically takes care of the condition on marginals.

Definition A *Harsanyi type space (over M)* is a coalgebra for the functor F in the category Meas^I . A *universal type space* is a final coalgebra for F in Meas^I .

The points of CX are called *states of the world*. A point of X_i is called an *i -type*.

Our main result here is that there is a universal Harsanyi type space. Our proof follows that of Heifetz and Samet [4]. (However, they did not formulate type spaces as coalgebras.) In order to make the ideas more transparent, and also because the method is much more general, we shall temporarily forget about all of the machinery involved in the multi-player setting. Instead, we consider functors on Meas built using Δ . We show in Section 3 below that each such functor has a final coalgebra.

3 Coalgebraic Modal Logic for Functors on Measurable Spaces

3.1 Syntax and Semantics

Definition The class of *measure polynomial functors* is the smallest class of functors on Meas containing the identity, each measurable spaces M and closed under products, coproducts, and Δ .

For a measure polynomial functor T , we define $\text{Ing}(T)$, the *ingredients of T* , by the following recursion: For a “constant” space M , $\text{Ing}(M) = \{M, \text{Id}\}$, $\text{Ing}(\text{Id}) = \{\text{Id}\}$ $\text{Ing}(U \times V) = \{U \times V\} \cup \text{Ing}(U) \cup \text{Ing}(V)$, and similarly for $U + V$; $\text{Ing}(\Delta S) = \{\Delta S\} \cup \text{Ing}(S)$. Each T has only finitely many ingredients.

Syntax We define just below a language $\mathcal{L}(T)$. The language is sorted, and the sorts are the ingredients of T . We write $\varphi : S$ to mean that φ is a formula of sort S ; when we need it, we let Form_S denote the set of such formulas.

1. If $M \in \text{Ing}(T)$ and A is a measurable subset of M , then $A : M$.
2. $\text{true}_{\text{Id}} : \text{Id}$.
3. If $S \in \text{Ing}(T)$ and $\varphi, \psi : S$, then also $\varphi \wedge \psi : S$ and $\neg\varphi : S$.
4. If $U \times V \in \text{Ing}(T)$, $\varphi : U$, and $\psi : V$, then $\langle \varphi, \psi \rangle \in : U \times V$.
5. If $U+V \in \text{Ing}(T)$ ($V+U \in \text{Ing}(T)$) and $\varphi : U$, then $\text{inl}^{U+V}\varphi : \text{Form}_{U+V}$ ($\text{inr}^{V+U}\varphi : V + U$).
6. If $\Delta S \in \text{Ing}(T)$ and $\varphi : S$ and $p \in [0, 1]$, then $\beta^p\varphi : \Delta S$.
7. If $\varphi : T$, then $[\text{next}]\varphi : \text{Id}$.

Semantics Let $c : X \rightarrow TX$ be a coalgebra of T . The semantics assigns to each $S \in \text{Ing}(T)$ and each $\varphi : S$ a subset $\llbracket \varphi \rrbracket_S^c \subseteq SX$.

$$\begin{array}{llll}
 \llbracket A \rrbracket_M^c & = & A & \llbracket \text{inl}^{U+V}\varphi \rrbracket_{U+V}^c & = & \mathcal{P}\text{inl}_{UX+VX}(\llbracket \varphi \rrbracket_U^c) \\
 \llbracket \text{true}_{\text{Id}} \rrbracket_{\text{Id}}^c & = & X & \llbracket \text{inr}^{U+V}\varphi \rrbracket_{U+V}^c & = & \mathcal{P}\text{inr}_{UX+VX}(\llbracket \varphi \rrbracket_V^c) \\
 \llbracket \varphi \wedge \psi \rrbracket_S^c & = & \llbracket \varphi \rrbracket_S^c \cap \llbracket \psi \rrbracket_S^c & \llbracket \beta^p\varphi \rrbracket_{\Delta S}^c & = & \beta^p(\llbracket \varphi \rrbracket_S^c) \\
 \llbracket \neg\varphi \rrbracket_S^c & = & -\llbracket \varphi \rrbracket_S^c & \llbracket [\text{next}]\varphi \rrbracket_{\text{Id}}^c & = & c^{-1}(\llbracket \varphi \rrbracket_T^c) \\
 \llbracket \langle \varphi, \psi \rangle^{U \times V} \rrbracket_{U \times V}^c & = & \llbracket \varphi \rrbracket_U^c \times \llbracket \psi \rrbracket_V^c & & &
 \end{array}$$

The notation $\mathcal{P}f(A)$ indicates throughout the paper the image under f of the set A . We check easily that $\llbracket \varphi \rrbracket_S^c$ is always a measurable subset of SX . In the sequel, we shall omit the superscripts on the pairing, inl , and inr operators, since they are mostly clear from the context. We also will occasionally omit the superscript c and the sort subscript when dealing with the semantics of $\varphi : S$ on a particular coalgebra $c : X \rightarrow TX$.

Remark As we mentioned, if M is a measurable space, then the measurable subsets of M are

taken as formulas. This departs from most of the treatments in coalgebraic modal logic, where one would take the *elements* of M as formulas; these formulas are then interpreted by singletons. Our treatment here makes for a more expressive language. We feel that when dealing with a space like $[0, 1]$, one might want to have a set denoting a subinterval or a measurable subset of it. Also, there is a technical advantage: at various points, it will be good to know that the set of interpretations of formulas of all sorts coincide with the measurable subsets. To get this, one clearly must start with the measurable subsets of the constants. The only price we pay for this is that we require that all spaces be *separative*: that is, for each $x \in M$, $\{x\}$ is the intersection of all measurable A containing x . (However, with more work this requirement may be lifted: see Section 5.1.)

Lemma 3.1 *Coalgebra morphisms preserve the semantics. That is, if $f : b \rightarrow c$ is a morphism of coalgebras $b : X \rightarrow TX$ and $c : Y \rightarrow TY$, and if $\varphi : S$, then $(Sf)^{-1}(\llbracket \varphi \rrbracket_S^c) = \llbracket \varphi \rrbracket_S^b$.*

3.2 The description operations, and the canonical spaces

For each coalgebra $c : X \rightarrow TX$ and each $x \in SX$, we define

$$d_S^c(x) = \{\varphi : S \mid x \in \llbracket \varphi \rrbracket_S^c\}.$$

In the terminology of this paper's title, each such set $d_S^c(x)$ is a *satisfied theory*.

For each $S \in \text{Ing}(T)$, we define S^* , the *canonical domain of sort S* , to be the following measurable space. In each case, the underlying set is

$$S^* = \{d_S^c(x) \mid \text{for some } c : X \rightarrow TX, x \in SX\}.$$

Note that each S^* is a set; indeed it has cardinality at most 2^{c^λ} , where $c = 2^{\aleph_0}$ is the cardinality of the continuum, and λ is the maximum of the cardinalities of the sets of measurable subsets of the constant functors in $\text{Ing}(T)$. Usually we will use letters like s for elements of S^* . For the σ -algebra, we first take the boolean algebra of subsets of S^* of the form

$$|\varphi|_S = \{s \in S^* \mid \varphi \in s\}. \tag{2}$$

This is a boolean algebra because the language has boolean operations interpreted classically. Then the σ -algebra on each S^* is the one generated by this boolean algebra. (Incidentally, each S^* is separative: $s = \bigcap \{|\varphi|_S \mid s \in |\varphi|_S\}$.)

Lemma 3.2 *For all $c : X \rightarrow TX$, all $S \in \text{Ing}(T)$:*

1. *For all $\varphi : S$, $\llbracket \varphi \rrbracket_S^c = (d_S^c)^{-1}(|\varphi|)$.*
2. *$d_S^c : SX \rightarrow S^*$ is measurable.*

3.3 The maps $r_S : S^* \rightarrow S(Id^*)$

We introduce some notation for the statement below and for the sequel. For $\varphi : S$, let

$$\bar{\varphi} = \mathcal{P}r_S(|\varphi|).$$

Lemma 3.3 *There is a family of measurable maps $r_S : S^* \rightarrow S(Id^*)$ indexed by the ingredients of T such that the following hold:*

a. *For all coalgebras $c : X \rightarrow TX$ the diagram below commutes:*

$$\begin{array}{ccc} SX & & \\ d_S^c \downarrow & \searrow Sd_{Id}^c & \\ S^* & \xrightarrow{r_S} & S(Id^*) \end{array} \quad (3)$$

b. *r_S is injective.*

c. *The σ -algebra on $S(Id^*)$ is generated by the sets $\bar{\varphi}$ for $\varphi : S$.*

Proof The maps $r_S : S^* \rightarrow S(Id^*)$ are defined by recursion on $\text{Ing}(T)$.

For $S = Id$, we take r_S to be the identity on Id^* .

For the constant functor M , recall that we are assuming that M is separative. It follows that for each $m \in M^*$ there is a unique $x \in M$ such that $m \supseteq \{A \in M : x \in A\}$. (Note that since we have boolean connectives even in the formulas of constant sort, we in fact will not have equality here.) We define $r_M : M^* \rightarrow M$ so that $r_M(m)$ is the unique x with this property. Then r_M is a bijection preserving measurability in both directions.

Products We define

$$r_{U \times V}(s) = \langle r_U(\pi_1(s)), r_V(\pi_2(s)) \rangle. \quad (4)$$

Before going further, we must explain this notation a bit. If $s \in (U \times V)^*$, then s contains formulas of the form $\langle \varphi, \psi \rangle$ where $\varphi : U$ and $\psi : V$. But s contains formulas not of this shape: boolean combinations of formulas of the form $\langle \varphi, \psi \rangle$. So by $\pi_1(s)$ we mean the set of $\varphi : U$ such that for some $\psi : V$, $\langle \varphi, \psi \rangle \in s$. Similar remarks apply to $\pi_2(s)$, of course. We omit all the details of the verification of (a)–(c), only noting one equation that we establish here and use later:

$$r_{U \times V}^{-1}(\bar{\varphi} \times \bar{\psi}) = |\langle \varphi, \psi \rangle|. \quad (5)$$

Coproducts We omit in this extended abstract all details concerning the coproduct construction.

From descriptions of measures to measures on descriptions The inductive step for ΔS is the most involved in this lemma. So assume that we have r_S for S .

Let $s \in (\Delta S)^*$. To define $r_{\Delta S}s$, it is enough by the induction hypothesis, part (c), to define its value on each subset of $S(Id^*)$ of the form $\overline{\varphi}$, where φ here ranges over formulas of sort S .

We *provisionally* define $r_{\Delta S}s$ on each set $\overline{\varphi}$ by

$$r_{\Delta S}s(\overline{\varphi}) = \max\{p \mid \beta^p \varphi \in s\}. \quad (6)$$

We now establish an important property of this provisional definition. Let $c : X \rightarrow T(X)$ be a coalgebra, and let $\mu \in \Delta S(X)$ be such that $s = d_{\Delta S}^c(\mu)$. We claim that

$$\mu(\{y \in SX \mid d_S^c(y) \in |\varphi|\}) = r_{\Delta S}s(\overline{\varphi}). \quad (7)$$

Note that $\{y \in SX \mid d_S^c(y) \in |\varphi|\}$ is measurable by Lemma 3.2. Let q be the value on the left, and let p be the value on the right. By Lemma 3.2, $q = \mu(\llbracket \varphi \rrbracket_S^c)$. So $\mu \in \llbracket \beta^q \varphi \rrbracket_{\Delta S}^c$. Since $s = d_{\Delta S}^c(\mu)$, $\beta^q \varphi \in s$. Thus $q \leq p$. But in the other direction, note that $\beta^p \varphi \in s$ by definition of p . And again since $s = d_{\Delta S}^c(\mu)$, $\mu \in \llbracket \beta^p \varphi \rrbracket_{\Delta S}^c$. That is, $\mu(\llbracket \varphi \rrbracket_S^c) \geq p$. Thus $q \geq p$. We conclude that $p = q$, establishing (7).

We now use the induction hypothesis $Sd_{Id}^c = r_S \circ d_S^c$ and the injectivity of r_S to re-write (7) as

$$\Delta Sd_{Id}^c(\mu)(\overline{\varphi}) = \mu((Sd_{Id}^c)^{-1}(\overline{\varphi})) = r_{\Delta S}s(\overline{\varphi}).$$

So we may now *officially* define $r_{\Delta S}s = \Delta Sd_{Id}^c(\mu)$ for *any* μ with $s = d_{\Delta S}^c(\mu)$. We are sure to get a measure this way. Moreover, the definition is independent of the particular μ . (For suppose that if ν , too, is such that $s = d_{\Delta S}^c(\nu)$, then for all φ , $\Delta Sd_{Id}^c(\nu)(\overline{\varphi}) = \max\{p \mid \beta^p \varphi \in s\}$. But then $\Delta Sd_{Id}^c(\mu)$ and $\Delta Sd_{Id}^c(\nu)$ agree on the generators of the σ -algebra on $\Delta S(Id^*)$. So we would have $\Delta Sd_{Id}^c(\mu) = \Delta Sd_{Id}^c(\nu)$, as desired.)

And we now know that each triangle as in (3) commutes. We emphasize that (6) holds for all $\varphi : S$. We continue by checking that $r_{\Delta S}$ is measurable. The σ -algebra on $\Delta S(Id^*)$ is generated by the sets

$$\beta^p(\overline{\varphi}) = \{\mu \in \Delta S(Id^*) \mid \mu(\overline{\varphi}) \geq p\},$$

for $\varphi : S$ and $p \in [0, 1]$. The inverse image under $r_{\Delta S}$ of this set is $\{s \in (\Delta S)^* \mid \beta^p \varphi \in s\} = |\beta^p \varphi|_{\Delta S}$, and this is measurable in $(\Delta S)^*$. Reading this the other way, we can see that the σ -algebra on $\Delta S(Id^*)$ is generated by the sets $\mathcal{P}r_{\Delta}(|\psi|)$, as ψ runs through formulas of sort ΔS .

We conclude by checking the injectivity of $r_{\Delta S}$. Suppose that $r_{\Delta S}s = r_{\Delta S}t$. Then the two sets s and t agree on all formulas of the form $\beta^p \varphi$. An easy induction shows that they agree on all boolean combinations of such formulas. So s and t agree on all formulas of sort ΔS . Thus they are the same set. \dashv

3.4 The canonical model of $\mathcal{L}(T)$

At this point, we are almost ready to define the canonical model. We need a preliminary concept first. For each $s \in Id^*$, let

$$\text{next}^{-1}(s) = \{\varphi : T \mid [\text{next}] \varphi \in s\}. \quad (8)$$

Lemma 3.4 For each $s \in Id^*$, $\text{next}^{-1}(s) \in T^*$. Moreover, this defines a measurable injective function $\text{next}^{-1} : Id^* \rightarrow T^*$.

We define $c^* : Id^* \rightarrow T(Id^*)$ to be

$$r_T \circ \text{next}^{-1} : Id^* \rightarrow T^* \rightarrow T(Id^*) \quad (9)$$

Note that c^* is injective. We shall show that c^* is a final T -coalgebra. As our title indicates, we build final coalgebras from satisfied theories.

In the statement and proof of the Truth Lemma below, recall that for $\varphi : S$, $\overline{\varphi}$ denotes $\mathcal{P}r_S(|\varphi|)$.

Lemma 3.5 (Truth Lemma) For all formulas φ of $\mathcal{L}(T)$, $\overline{\varphi} = \llbracket \varphi \rrbracket_S^{c^*}$. That is, the diagram below commutes:

$$\begin{array}{ccc} \text{Form}_S & & \\ \downarrow |\cdot|_S & \searrow \llbracket \cdot \rrbracket_S^{c^*} & \\ \mathcal{P}(S^*) & \xrightarrow{\mathcal{P}r_S} & \mathcal{P}(S(Id^*)) \end{array}$$

Proof By induction on φ .

The base case concerns a measurable subset A of some $M \in \text{Ing}(T)$. Recall that $r_M : M^* \rightarrow M$ has the property that $r_M \circ d_M = Id_M$ and that $|A|_M = \{d_M(x) \mid x \in A\}$. So $\mathcal{P}r_M(|A|_M) = \{x \mid x \in A\} = A = \llbracket A \rrbracket_M$.

The steps for true_{Id} and the boolean connectives are easy. We omit the inductive step for $\text{inl } \varphi : U + V$.

The inductive step for $\langle \varphi, \psi \rangle : U \times V$ is similar. Our induction hypothesis is that $\overline{\varphi} = \llbracket \varphi \rrbracket_U^{c^*}$ and $\overline{\psi} = \llbracket \psi \rrbracket_V^{c^*}$. Equation (5) tells us that $r_{U \times V}^{-1}(\overline{\varphi} \times \overline{\psi}) = |\langle \varphi, \psi \rangle|$. This means that $\overline{\varphi} \times \overline{\psi} = \overline{\langle \varphi, \psi \rangle}$. Hence

$$\overline{\langle \varphi, \psi \rangle} = \overline{\varphi} \times \overline{\psi} = \llbracket \varphi \rrbracket_U^{c^*} \times \llbracket \psi \rrbracket_V^{c^*} = \llbracket \langle \varphi, \psi \rangle \rrbracket_{U \times V}^{c^*}.$$

Here is the inductive step for sentences $\beta^p \varphi$ of sort ΔS . For all $s \in (\Delta S)^*$, we have the following equivalences:

$$\begin{aligned} & s \in |\beta^p \varphi|_{\Delta S} \\ \text{iff} & \beta^p \varphi \in s && \text{by (2)} \\ \text{iff} & \max\{q \mid \beta^q \varphi \in s\} \geq p \\ \text{iff} & r_{\Delta S}(s)(\overline{\varphi}) \geq p && \text{by (6)} \\ \text{iff} & r_{\Delta S}(s)(\llbracket \varphi \rrbracket_S^{c^*}) \geq p && \text{by induction hypothesis} \\ \text{iff} & r_{\Delta S}(s) \in \llbracket \beta^p \varphi \rrbracket_{\Delta S}^{c^*} && \text{by the semantics of } \beta^p \varphi \end{aligned}$$

From the overall equivalence, we see that $\llbracket \beta^p \varphi \rrbracket_{\Delta S}^{c^*} = (\mathcal{P}r_{\Delta S})(|\beta^p \varphi|_{\Delta S})$, as desired.

We omit the inductive step for $[\text{next}] \varphi$. The argument is similar to what we have seen, and it uses equations (2 and (8), and also the semantics of $[\text{next}] \varphi$ in c^* .

This completes the proof. \dashv

3.5 The Final Coalgebra Theorem

We end this section by proving that each measure polynomial functor T has a final coalgebra.

Lemma 3.6 $d_{Id}^{c^*} = Id_{Id^*}$.

Proof If $\varphi : Id$, then by the Truth Lemma, $\llbracket \varphi \rrbracket_{Id}^{c^*} = \overline{\varphi}$. And since the map r_{Id} in Lemma 3.3 is also the identity, this is exactly $|\varphi|$. So we see that

$$d_{Id}^{c^*}(s) = \{\varphi \mid s \in \llbracket \varphi \rrbracket_{Id}^{c^*}\} = \{\varphi \mid s \in |\varphi|\} = s.$$

⊣

Lemma 3.7 For each coalgebra $c : X \rightarrow TX$, the diagrams below commute:

$$\begin{array}{ccccc} X & \xrightarrow{c} & TX & & \\ d_{Id}^c \downarrow & & \downarrow d_T^c & \searrow Td_{Id}^c & \\ Id^* & \xrightarrow{\text{next}^{-1}} & T^* & \xrightarrow{r_T} & T(Id^*) \end{array}$$

Hence d_{Id}^c is a morphism of coalgebras.

Proof The verification of the square is easy, and the triangle comes from Lemma 3.3. ⊣

Theorem 3.8 $c^* : Id^* \rightarrow T(Id^*)$ is a final coalgebra of T .

Proof Let $c : X \rightarrow TX$ be a T -coalgebra. By Lemma 3.7, d_{Id}^c is a coalgebra morphism. For the uniqueness, suppose that f is any morphism. Since f preserves descriptions, $d_{Id}^{c^*} \circ f = d_{Id}^c$. But by Lemma 3.6, $d_{Id}^{c^*} = Id_{Id^*}$. So $f = d_{Id}^{c^*} \circ f = d_{Id}^c$, just as desired. ⊣

4 The Universal Harsanyi Type Space

In this section, we show how to adapt our work to the case of Harsanyi type spaces considered as coalgebras on Meas^I . To make the notation more manageable, we assume that $I = \{1, 2, 3\}$. Let M be a fixed separative measurable space. Let Pr_1, Pr_2 , and Pr_3 be the obvious projections $Pr_i : \text{Meas}^I \rightarrow \text{Meas}$. Let U_1, U_2 , and U_3 be the functors $U_i : \text{Meas}^I \rightarrow \text{Meas}$ given as follows: $U_1(X_1, X_2, X_3) = M \times Pr_2 \times Pr_3$, $U_2(X_1, X_2, X_3) = M \times Pr_1 \times Pr_3$, $U_3(X_1, X_2, X_3) = M \times Pr_1 \times Pr_2$. Let $T_i = \Delta U_i$. So we are interested in the functor $F : \text{Meas}^I \rightarrow \text{Meas}^I$ given by (T_1, T_2, T_3) .

We write $\text{Ing}(F)$ for $\{M, U_i, T_i, Pr_i \mid i = 1, 2, 3\}$.

We formulate our language \mathcal{L} to have formulas of sort S for $S \in \text{Ing}(F)$. \mathcal{L} is defined as follows: $\text{true}_{Pr_i} : Pr_i$. If A is a measurable subset of M , then $A : M$. If $\varphi_0 : M$, $\varphi_2 : Pr_2$, $\varphi_3 : Pr_3$, then $\langle \varphi_0, \varphi_2, \varphi_3 \rangle : U_1$. We have similar clauses for U_2 and U_3 . We also have clauses for the Δ functors: if $\varphi : U_i$, then $\beta^p \varphi : T_i$. If $\varphi : T_i$, then $[\text{next}] \varphi : Pr_i$. We also take boolean conjunction and negation on all sorts. (This is not really needed for the M sorts.)

Let $X = (X_1, X_2, X_3)$, and let $c : X \rightarrow FX$ be a coalgebra of F . The semantics assigns to each $S \in \text{Ing}(F)$ and each $\varphi : S$ a subset $\llbracket \varphi \rrbracket_S^c \subseteq SX$. Here are some representative cases. $\llbracket \text{true}_{Pr_i} \rrbracket = X_i$. Suppose that $\varphi_0 : M, \varphi_2 : Pr_2$ and $\varphi_3 : Pr_3$, then $\langle \varphi_0, \varphi_2, \varphi_3 \rangle : U_1$. We set $\llbracket \langle \varphi_0, \varphi_2, \varphi_3 \rangle \rrbracket_{U_1}^c = \llbracket \varphi_0 \rrbracket_M^c \times \llbracket \varphi_2 \rrbracket_{Pr_2}^c \times \llbracket \varphi_3 \rrbracket_{Pr_3}^c$. Finally, suppose that $\varphi : U_3$ so that $\beta^p \varphi : T_3$. Given $\llbracket \varphi \rrbracket_{U_3}^c \subseteq U_3(X)$, we set $\llbracket \beta^p \varphi \rrbracket_{T_3}^c = \beta^p(\llbracket \varphi \rrbracket_{U_3}^c) \subseteq T_3(X)$.

For a F -coalgebra X , and $S \in \text{Ing}(F)$, we define d_S^c so that $d_S^c(x) = \{\varphi : S \mid x \in \llbracket \varphi \rrbracket_S^c\}$. We also define $|\varphi|_S$ and S^* just as before, using the sets $|\varphi|_S$ as the generators of the σ -algebra on S^* . Each d_S^c is again measurable.

Much of the remaining constructions are similar to what we have already seen. The role of Id^* is played by (Pr_1^*, Pr_2^*, Pr_3^*) . This turns out to be the carrier of the final coalgebra for F .

Lemma 4.1 *There is a family of measurable maps $r_S : S^* \rightarrow S(Pr_1^*, Pr_2^*, Pr_3^*)$ indexed by $\text{Ing}(F)$ such that the following hold:*

- a. For all coalgebras $c : X \rightarrow FX$, $r_S \circ d_S^c = Sd_{(Pr_1^*, Pr_2^*, Pr_3^*)}^c$.
- b. r_S is injective.
- c. The σ -algebra on $S(Pr_1^*, Pr_2^*, Pr_3^*)$ is generated by the sets $\overline{\varphi}$ for $\varphi : S$.

The proof follows closely the work outlined in Lemma 3.3. Notice that as defined here, all the ingredients of F are functors from Meas^I to Meas .

As the reader has probably guessed, the work here can be generalized to *polynomials on Meas^I* . These are I -indexed family of functors $T_i : \text{Meas}^I \rightarrow \text{Meas}$ built from the projections, constants for separative spaces, products and sums, and Δ . The work in this section generalizes to show that each polynomial on Meas^I has a final coalgebra. The details are not much more than what we have seen.

5 Further Variations and Extensions of the Basic Construction

We have already seen the main construction of final coalgebras for the polynomials on Meas built from separative spaces. We also saw (by example) how to generalize this to systems, thereby building the universal Harsanyi type spaces. The rest of the paper offers variations and extensions of the basic technique.

5.1 Extension: Non-separative Spaces

The main point of working with separative spaces is that the points of such a space may be recovered from the σ -algebra (as the set of all singleton intersections) and from the satisfied theories in our language. This allowed us to have a language for the measurable *subsets* of various spaces. In the absence of separativity, we need to do more work. For each $S \in \text{Ing}(T)$, we need two different sorts of formulas. We call these $\text{Form}^{pt}(S)$ and $\text{Form}^{set}(S)$, for the points and sets of sort S , respectively. If $M \in \text{Ing}(T)$ is a constant functor, then $\text{Form}^{pt}(M)$ is the set of elements $m \in M$. In the semantics, we have $\llbracket m \rrbracket = \{m\}$. (This is not in general measurable.) Just as before, $\text{Form}^{set}(M)$ is the set of measurable subsets of M . For the product construction, if $\varphi \in \text{Form}^{pt}(S)$, then $[\pi_1]\varphi \in \text{Form}^{pt}(S \times T)$. There is also a $[\pi_2]\varphi$ construction,

and also constructions for coproducts. These are as expected, as is the semantics in each case. Where things are a little different is with Δ . Here if $\varphi \in \text{Form}^{set}(S)$ and $p \in [0, 1]$, then $\beta^p \varphi \in \text{Form}^{pt}(\Delta S)$. The reason for this is that a measure μ on S is specified by the values $\mu(E)$ for *events* E , not points. We also take $\text{Form}^{pt}(\Delta S)$ and $\text{Form}^{set}(\Delta S)$.

In this setting, we have for each S and each coalgebra c *two* description functions. We define the spaces S^* by taking the points to be the satisfied theories of sort $\text{Form}^{pt}(S)$ and then by endowing this set with a σ -algebra derived from the satisfied theories of sort $\text{Form}^{set}(S)$. To keep this abstract short, we omit all the details here.

5.2 Variation: Finite Kripke Polynomials on Set

In this section, we check that the same method gives representations for final coalgebras for functors on **Set** built from the identity functor, the finite power set functor, product and coproduct, fixed (finite or infinite) sets, and functions from a fixed set.

To avoid double subscripts or confusion with our notation \mathcal{P} for the power set functor, we shall use \mathcal{Q} for the *finite power set* functor on **Set**. Being a functor, we shall apply \mathcal{Q} to functions as well as sets, writing, e.g., $\mathcal{Q}r(X)$ for the image $r[X]$ of the finite set X under r .

Our syntax is constructed so that if $A \in \text{Ing}(T)$, then each *element* $a \in A$ is a formula of sort A . Further, if $\mathcal{Q}S \in \text{Ing}(T)$ and $\varphi : S$, then $\Box\varphi : \mathcal{Q}S$.

In our semantics, we define $\llbracket a \rrbracket_A^c = \{a\}$, and also

$$\llbracket \Box\varphi \rrbracket_{\mathcal{Q}S}^c = \mathcal{P}(\llbracket \varphi \rrbracket_S^c). \quad (10)$$

One then checks that on each coalgebra, $\llbracket \varphi \rrbracket$ is always a set. Most of the rest of the results go through with only minor changes, dropping the word “measurable” and anything having to do with the measure space structure. The only differences are in Lemma 3.3 and the Truth Lemma 3.5.

Lemma 5.1 *For each $S \in \text{Ing}(T)$, there is a bijective map $r_S : S^* \rightarrow S(\text{Id}^*)$ such that for all coalgebras $c : X \rightarrow TX$, $r_S \circ d_S^c = Sd_{\text{Id}^*}^c$.*

The proof here is *not* an immediate adaptation from Lemma 3.3 above. The argument for Lemma 5.1 is more along the lines of classical work in modal logic, as is the argument for the key inductive step in the Truth Lemma below.

Lemma 5.2 (Truth Lemma) *For all $S \in \text{Ing}(T)$ and all $\varphi : S$, $\overline{\varphi} = \llbracket \varphi \rrbracket_S^c$.*

Theorem 5.3 *For each Kripke polynomial functor T , $c^* : \text{Id}^* \rightarrow T(\text{Id}^*)$ is a final coalgebra.*

5.3 Variation: Vietoris Polynomial Functors on Stone

We next consider the same general results for the Vietoris polynomial functors on the category of Stone spaces. For all relevant definitions, see Kupke, Kurz, and Venema [7]. We shall use the following definitions. If X is a Stone space, we let $K(X)$ denote the set of compact subsets of X . For any $K \subseteq X$, $\Box K$ denotes is the set of all compact subsets of K . Further, $\Diamond K$ denotes the set of all compact subsets of X whose intersection with K is non-empty.

Lemma 5.4 *Let \mathcal{S} be a basis of the Stone space X . Then the collection of sets*

$$\{\Box K \mid K \in \mathcal{S}\} \cup \{\Diamond K \mid K \in \mathcal{S}\}$$

is a basis of the topology on $\mathbb{V}(X)$.

We take a language having, for each Stone space X , all clopen subsets of X as formulas of type X . We also take a modal operator \Box so that if φ is of type S , $\Box\varphi$ is of type $\mathbb{V}X$. The semantics is

$$\llbracket \Box\varphi \rrbracket_{\mathbb{V}S}^c = \Box \llbracket \varphi \rrbracket_S^c. \quad (11)$$

Since Stone spaces are closed under topological function spaces, we may also add the obvious syntax for functors from a given space D . One then checks that on each coalgebra X , $\llbracket \varphi \rrbracket_S$ is always a clopen subset of SX .

We define S^* as before, and we give it the topology generated by the sets $|\varphi|_S$. So each set $|\varphi|_S$ is clopen. This results in a Stone space.

Lemma 5.5 *There is a family of continuous maps $r_S : S^* \rightarrow S(Id^*)$ indexed by the ingredients of T such that the following hold:*

a. *For all coalgebras $c : X \rightarrow TX$ the diagram below commutes:*

$$\begin{array}{ccc} SX & & \\ d_S^c \downarrow & \searrow Sd_{Id}^c & \\ S^* & \xrightarrow{r_S} & S(Id^*) \end{array} \quad (12)$$

b. *r_S is injective.*

c. *The sets $\bar{\varphi} = Pr_S(|\varphi|)$ for $\varphi : S$ form a basis of the topology on $S(Id^*)$.*

Proof By induction on S . We shall only give the induction step for $\mathbb{V}S$. So assume that we have r_S so that all the properties in our lemma hold. We need $r_{\mathbb{V}S}$. Let $y \in (\mathbb{V}S)^*$. We set

$$r_{\mathbb{V}S}(y) = Pr_S\left(\bigcap\{|\varphi| \mid \varphi : S \text{ and } y \in |\Box\varphi|\}\right). \quad (13)$$

For the remainder of this proof, we only use φ to range over formulas of type S . Using the $\bar{\varphi}$ notation and the injectivity of r_S , we may rewrite (13) as

$$r_{\mathbb{V}S}(y) = \bigcap_{\Box\varphi \in y} \bar{\varphi}.$$

First of all, we must check for each $y \in (\mathbb{V}S)^*$ that that $\bigcap\{|\varphi| : y \in |\Box\varphi|\}$ is a *compact* subset of S^* ; then the fact that r_S is continuous implies that the image of this set is also compact. Fix y and also $c : X \rightarrow TX$, and let $Y \in \mathbb{V}S(X)$ be such that $y = d_{\mathbb{V}S}^c(Y)$. Notice

that whenever φ is such that $\Box\varphi \in y$, we have $Y \subseteq \llbracket \varphi \rrbracket_S^c$. (To see this, note that $y \in |\Box\varphi|$. And then $Y \in \llbracket \Box\varphi \rrbracket_{\mathbb{V}S}^c$. By our semantics in (11), $Y \subseteq \llbracket \varphi \rrbracket_S^c$.) We claim that

$$\mathcal{P}d_S(Y) = \bigcap_{y \in |\Box\varphi|} |\varphi|. \quad (14)$$

To see this, let $z \in Y$. As we know, whenever φ is such that $\Box\varphi \in y$, we have $z \in \llbracket \varphi \rrbracket_S^c$; in other words, $d_S(z) \in |\varphi|$. Since φ is arbitrary, $d_S(z)$ belongs to the right side of (14). In the other direction, suppose that $z \in S^*$ is such that z belongs to $|\varphi|$ whenever $x \in |\Box\varphi|$. We claim that for some $w \in Y$, $z = d_S(w)$. For if not, then for each $w \in Y$ there is some $\psi_w : S$ such that $\psi_w \in z$ and $\neg\psi_w \in d_S(w)$. It follows that

$$Y \subseteq \bigcup_{w \in Y} \llbracket \neg\psi_w \rrbracket_S^c.$$

The semantics of each sentence is a clopen, hence open, set. Since Y is compact, there is a finite set $Y_0 \subseteq Y$ such that $Y \subseteq \llbracket \bigvee_{w \in Y_0} \neg\psi_w \rrbracket$. In other words, $Y \in \llbracket \Box \bigvee_{w \in Y_0} \neg\psi_w \rrbracket_{\mathbb{V}S}$. Since $y = d_{\mathbb{V}S}^c(Y)$, we see that y contains $\Box \bigvee_{w \in Y_0} \neg\psi_w$. But then by the definition of z , $d_S(z) \in |\bigvee_{w \in Y_0} \neg\psi_w|$. That is, for some $w \in Y_0$, $d_S(z) \in |\neg\psi_w|$. This contradicts the fact that $\psi_w \in d_S(z)$.

So at this point, we know (14). Recall the general fact that the image of a compact set under a continuous map is compact. These imply that the definition of $r_{\mathbb{V}S}(x)$ in (13) is a compact subset of S^* . We still must show that for all coalgebras c , $r_{\mathbb{V}S} \circ d_{\mathbb{V}S}^c = \mathbb{V}Sd_{Id}^c$. The argument at this point is the same as the one in Lemma 5.1.

Next, we show that $r_{\mathbb{V}S}$ is injective. Suppose that $y \neq y'$ belong to $(\mathbb{V}S)^*$. We claim first that there must be some $\psi : S$ such that $\Box\psi \in y'$ and $\Box\psi \notin y$ (or vice-versa). For if y and y' agreed on all $\Box\varphi$ formulas, then they would agree on all boolean combinations of such formulas, and hence they would be equal. So fix $\Box\psi \in y' \setminus y$. Also, fix $c : X \rightarrow TX$, and let $Y \in \mathbb{V}S(X)$ be such that $y = d_{\mathbb{V}S}^c(Y)$. Then since $\Box\psi \notin y$, let $z \in Y$ be such that $z \notin \llbracket \psi \rrbracket_S^c$. By (14), $d_S(z) \in \bigcap_{\Box\varphi \in y} |\varphi|$. So $r_S(d_S(z)) \in r_{\mathbb{V}S}(y)$. However, since $\Box\psi \in y'$, we have $\bigcap_{\Box\varphi \in y'} |\varphi| \subseteq |\psi|$. Recall that $d_S(z) \notin |\psi|$. A fortiori, $d_S(z) \notin \bigcap_{\Box\varphi \in y'} |\varphi|$. Finally, since r_S is injective, it follows that $r_S(d_S(z)) \notin r_{\mathbb{V}S}(y')$. We conclude that $r_{\mathbb{V}S}(y) \neq r_{\mathbb{V}S}(y')$. This concludes the verification that $r_{\mathbb{V}S}$ is injective.

We check now that $r_{\mathbb{V}S}$ is continuous. By Lemma 5.4, we need only consider the sets $\Box\bar{\varphi}$ and $\Diamond\bar{\varphi}$ for $\varphi : S$ and check that the inverse images under $r_{\mathbb{V}S}$ of such sets are open in $(\mathbb{V}S)^*$. Then for all $y \in (\mathbb{V}S)^*$,

$$\begin{aligned} r_{\mathbb{V}S}(y) \in \Box\bar{\varphi} & \quad \text{iff} \quad r_{\mathbb{V}S}(y) \subseteq \bar{\varphi} \\ & \quad \text{iff} \quad y \in |\Box\varphi| \end{aligned}$$

So the inverse image of $\Box\bar{\varphi}$ is the open set $|\Box\varphi|$. It also follows from the calculation above that

$$\begin{aligned} r_{\mathbb{V}S}(y) \in \Diamond\bar{\varphi} & \quad \text{iff} \quad r_{\mathbb{V}S}(y) \not\subseteq \Box\neg\bar{\varphi} \\ & \quad \text{iff} \quad y \notin |\Box\neg\varphi| \\ & \quad \text{iff} \quad y \in |\neg\Box\neg\varphi| \end{aligned}$$

We conclude that the inverse image of $\Diamond\bar{\varphi}$ is open.

Finally, the collection \mathcal{S} of sets $\bar{\varphi}$ for $\varphi : \mathbb{V}S$ includes the collections of sets $\Box\bar{\varphi}$ and $\Diamond\bar{\varphi}$. Also, \mathcal{S} is closed under intersection. So it forms a basis of the Vietoris topology on $\mathbb{V}(S(Id^*)) = (\mathbb{V}S)(Id^*)$. \dashv

5.4 Other spaces

We are confident that the technique here extends to other kinds of concrete categories, such as compact Hausdorff spaces and the Borel probability measure with the weak $*$ -topology. This case had been studied in relation to type spaces beginning with [1]. Indeed, we hope to expand the technique as much as possible in the coming months.

5.5 Conclusion and future directions

This paper has had two overall points. First, we connect work in the economics/game theory area with coalgebras. We feel that most, if not all, of the constructions of universal type spaces and related objects may be obtained by our method. We intend that the final version of our paper will show this. The more common method in the area is to construct an ω^{op} -limit and then follow this by a subspace construction. Though we did not discuss the matter here, we feel that our method is somewhat simpler. As we have indicated, it generalizes easily to other settings. We also already generalized known results from particular functors to polynomials. Since the theories here are connected to logic, it should be possible to formulate logical systems for all the functors involved and prove completeness theorems. But this we leave to future work.

The construction of using satisfied theories to obtain final coalgebras is quite old in the area of coalgebra and its predecessors. For coalgebras per se, one can find it in Rutten [11]. The technique is perhaps implicit in Kurz, Kupke, and Venema [7]; this paper also contains a note that Jacobs' final coalgebra result in [6] for Kripke polynomials on Set contains an error. Our result does not use maximal consistent sets in a logical system but rather the satisfied theories. This is simpler, though of course one must do more work to get completeness results. From the side of coalgebra, perhaps what is most original here is taking a language for *sets* as opposed to (or, in addition to) a language of points. The interplay of the two languages is well worth studying.

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