Natural Logic Notes
Course Notes for ESSLLI 2007

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Contents

1 Introduction: what is in these notes?

These notes are for my class at ESSLLI 2007 on the topic of Natural Logic. They were sent in on July 1, about six weeks before the class. The notes themselves are a bit like rough lecture notes. I expect to put more polished versions up on a web site: see

`www.indiana.edu/~iulg/moss/nl`

They are also both *too much* and *too little*. The class will be five lectures of 90 minutes each, aimed at an Introductory level. So the notes here cover more of the technical material than I expect to do. I will try to cover much of what is here, but some of the longer proofs will just be sketched. The notes are also too much in the sense that I expect to cover many more subjects, including ones in the paper by Johan van Benthem that is also part of the course reader. In fact, I would say that about half of the ESSLLI course will be on the topics in these notes, and the other half will be material from quite a few other papers.

The main sources for the material in these notes are my papers [9, 10, 11]. What I have done in these notes is to try to organize them into a series of lecture sections, to add exercises and discussions, etc.

I hope that my students will help me improve the notes with suggestions and criticism.

1.1 Getting Started

For most of its history, logic was concerned with *syllogisms*. One simple example, perhaps the most famous one, is:

All men are mortal.
Socrates is a man.
———
Socrates is mortal.

The idea is that the sentences above the line should *semantically entail* the one below the line. Specifically, in every model in which *All men are mortal* and *Socrates is a man* are true, it must be the case that *Socrates is mortal* is also true. We have to say what *semantically entail* means, and this will come in due course. The matter might be clearer with another example. Suppose someone accepts as true the following sentences:

1. *All raredos are slonados.*
2. *John is a raredo.*
3. *Mary is an alphatoric.*
4. *John is Mary.*

Then they should also accept as true the conclusion *Some slonado is an alphatoric*. We have purposely used nonsense words here; the whole point is that the inference depends only on the form of the argument. In this case, the key elements of the form include the worlds *All* and *Some*, and two different uses of *is*. So rather than deal with actual words, we instead consider things schematically. Assuming

1. *All* X *are* Y.
2. J *is an* X.
3. M *is a* Z.
4. J *is* M.

We should accept *Some* Y *is a* Z.

Natural Logic is concerned with a mathematical model of these kinds of inferences. We'd like to know when a given sentence would be a good conclusion to a given argument, and when it would not. (Incidentally, the same question arises for the traditional syllogisms. But those are three-line arguments, and the question of which syllogisms are intuitively valid is a special case of the question we ask in this subject.)

To make life simple here, we are only going to consider a few very restricted forms of English sentences. These are the ones we list in Figure 1 below. We are going to be fairly strict in restricting attention to just sentences of those forms. The only deviation is that we write *a* or *an* following the usual uses in English, as we did in (2) and (3) just above.

To define *validity of an argument*, we first say what the semantics of individual sentences is. This again is given in Figure 1. Here is an example. Let $\mathcal{M}$ be the set $\{1,2,3,4,5\}$. Let $[\![X]\!] = \{1,3,4\}$, $[\![Y]\!] = \{1,5\}$, $[\![Z]\!] = \{5\}$, $[\![J]\!] = 3$, and $[\![M]\!] = 1$. Then $[\![J \textit{ is an } X]\!] = \mathsf{true}$, but for all three other assumptions R, $[\![R]\!] = \mathsf{false}$.

Let Γ be our set of four assumptions, and let S be *Some* Y *is a* Z. Then $\Gamma \models S$ means that all models $\mathcal{M}$ satisfying all sentences in Γ also satisfy S. The example in the previous paragraph did not satisfy all sentences in Γ. But if we change $[\![Y]\!]$ to $\{1,2,3,4\}$, $[\![Z]\!]$ to $\{5\}$, and $[\![M]\!]$ to 3, we would satisfy all the assumptions in Γ. We would also satisfy S.

This last model is just one example, and we want to know whether *all* models of Γ are models of S. The idea is that we cannot determine this by looking at examples; there are "too many". Besides, the reason that someone would accept S on the basis of Γ does not have so much to do with examples as with *reasons*. This is what our proof system intends to model. The second part of Figure 1 defines proof trees. For the same Γ and S, here is a proof tree which shows that $\Gamma \vdash S$:

$$\dfrac{\dfrac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \qquad \dfrac{M \textit{ is a } Z \quad J \textit{ is } M}{J \textit{ is an } Z}}{\textit{Some } Y \textit{ is a } Z} \tag{1}$$

The idea is that what counts as a proof tree is an entirely *syntactic* matter: the meaning of the English words such *All* and *Some* is completely irrelevant. A computer, or a speaker of some other language, could *check* whether a given labeled tree obeyed the conditions in the definition.

Syntax: We start with *variables* X, Y, ..., representing plural common nouns of English. We also also *names* J, M, ... Then we consider sentences S of the following very restricted forms:
All X are X, *Some X are X*, *No X are X*, *J is an X*, *J is M*.

Semantics: One starts with a set $\mathcal{M}$, a subset $[\![X]\!] \subseteq \mathcal{M}$ for each variable X, and an element $[\![J]\!] \in \mathcal{M}$ for each name J. This gives a *model* $\mathcal{M} = (\mathcal{M}, [\![\ ]\!])$.
We then assign a semantics $[\![S]\!] \in \{\mathsf{true}, \mathsf{false}\}$ to the sentence S in a model $\mathcal{M}$, as follows:

$$\begin{array}{lll} [\![\text{All } X \text{ are } Y]\!] = \mathsf{true} & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ [\![\text{Some } X \text{ are } Y]\!] = \mathsf{true} & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ [\![\text{No } X \text{ are } Y]\!] = \mathsf{true} & \text{iff} & [\![X]\!] \cap [\![Y]\!] = \emptyset \\ [\![J \text{ is an } X]\!] = \mathsf{true} & \text{iff} & [\![J]\!] \in [\![X]\!] \\ [\![J \text{ is } M]\!] = \mathsf{true} & \text{iff} & [\![J]\!] = [\![M]\!] \end{array}$$

We write $\mathcal{M} \models S$ if $[\![S]\!] = \mathsf{true}$. And if Γ is a set of sentences, then we write $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$.

Main semantic definition: $\Gamma \models S$ means that every model which makes all sentences in the set Γ true also makes S true. We say Γ *semantically implies* S.

Inference rules of the logical system: Rules for various fragments are presented as needed.

Proof trees: A *proof tree over* Γ is a finite tree whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules.

Formal proofs: $\Gamma \vdash S$ means that there is a proof tree over Γ whose root is labeled S. We say Γ *proves*, or *derives*, S.

Figure 1: The main definitions in syllogistic logic.

This is probably a good place to mention the ways in which we are (and are not) strict with rules. In writing these notes, I have tried to be completely strict about the left-right match in rules. So since one of the rules is

$$\frac{M\ \textit{is an}\ X \quad J\ \textit{is}\ M}{J\ \textit{is an}\ X} \qquad (2)$$

I would not make a tree like

$$\frac{J\ \textit{is}\ M \quad M\ \textit{is an}\ X}{J\ \textit{is an}\ X}$$

This kind of strictness is not essential, however. You should feel free to loosen it. It is more important to note that the rules are to be read *schematically*: one is allowed to substitute other variables or names for the ones in the statement of the rules. We already did this in (1): into the actual rule in (2) we substituted Z for X (and kept the other variables as they are in (2)).

Here is another example, chosen to make some different points: Let Γ be

$$\{\textit{All}\ A\ \textit{are}\ B, \textit{All}\ Q\ \textit{are}\ A, \textit{All}\ B\ \textit{are}\ D, \textit{All}\ C\ \textit{are}\ D, \textit{All}\ A\ \textit{are}\ Q\}$$

Let S be *All* Q *are* D. Here is a proof tree showing that $\Gamma \vdash S$:

$$\frac{\textit{All}\ Q\ \textit{are}\ A \qquad \dfrac{\dfrac{\textit{All}\ A\ \textit{are}\ B \quad \textit{All}\ B\ \textit{are}\ B}{\textit{All}\ A\ \textit{are}\ B} \qquad \textit{All}\ B\ \textit{are}\ D}{\textit{All}\ A\ \textit{are}\ D}}{\textit{All}\ Q\ \textit{are}\ D}$$

Note that all of the leaves belong to Γ except for one that is *All* B *are* B. Note also that some elements of Γ are not used as leaves. This is permitted according to our definition. The proof tree above shows that $\Gamma \vdash S$. Also, there is a smaller proof tree that does this, since the use of *All* B *are* B is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All* A *are* A.)

The main technical question for this subject is: what is the relation the semantic notion $\Gamma \models S$ with the proof-theoretic notion $\Gamma \vdash S$? This kind of question will present itself for all of the logical systems in this course. Probably the first piece of work for you is to *be sure you understand the question.*

Lemma 1.1 (Soundness) *If* $\Gamma \vdash S$, *then* $\Gamma \models S$.

Proof By *strong induction on the number of nodes* of proof trees $\mathcal{T}$ over Γ. If $\mathcal{T}$ is a tree with one node, let S be the label. Either S belongs to Γ, or else S is of the form *All* A *are* A or J *is* J. In the first case, every model satisfying every sentence in Γ clearly satisfies S, as S belongs to Γ. And in the second case, every model whatsoever satisfies S.

Let's suppose that we know our result for all proof trees over Γ with fewer than n nodes, and let $\mathcal{T}$ be a proof tree over Γ with n nodes. The argument breaks into cases depending on which rule is used at the root. Suppose the root and its parents are labeled

$$\frac{\textit{All}\ X\ \textit{are}\ Z \quad \textit{All}\ Z\ \textit{are}\ Y}{\textit{All}\ X\ \textit{are}\ Y}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the subtrees ending at *All* X *are* Y and *All* Y *are* Z. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are proof trees over Γ themselves. For some variable Y, the root of $\mathcal{T}_1$ is labeled *All* X *are* Y, and the root of $\mathcal{T}_2$ is labeled *All Y are Z*. Now $\mathcal{T}_1$ and $\mathcal{T}_2$ both have fewer nodes than $\mathcal{T}$. By our induction hypothesis, $\Gamma \models$ *All X are Y*, and also $\Gamma \models$ *All Y are Z*. We claim that $\Gamma \models$ *All X are Z*. To see this, take any model $\mathcal{M}$ in which all sentences in Γ are true. Then $[\![X]\!] \subseteq [\![Y]\!]$ by our first point above. And $[\![Y]\!] \subseteq [\![Z]\!]$ by second. So $[\![X]\!] \subseteq [\![Z]\!]$ by transitivity of the inclusion relation on sets. Since the model $\mathcal{M}$ here is arbitrary, we conclude that $\Gamma \models$ *All X are Z*.

The other cases on the label of the root of $\mathcal{T}$ are similar. Of special interest might be the case for the rule

$$\frac{\textit{Some } X \textit{ are } Y \quad \textit{No } X \textit{ are } Y}{R}$$

(The intuitive point here is that *every* sentence R follows from the contradictory hypotheses.) Let $\mathcal{T}$ be a proof tree over Γ ending up with an application of this rule. We claim that there are no models of Γ. To see this, suppose toward a contradiction that $\mathcal{M} \models \Gamma$. By induction hypothesis, the sentences *Some X are Y* and *No X are Y* are true in $\mathcal{M}$. That is, $[\![X]\!] \cap [\![Y]\!] = \emptyset$, and also $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. This is a contradiction, and from it, we see that there are no models of Γ. So vacuously, every model of Γ is a model of the root S. ⊣

So at this point, we know that our logic is *sound*: If we have a tree showing that $\Gamma \vdash S$, then S follows semantically from Γ. This means that the formal logical system is not going to give us any bad results. Now this is a fairly weak point. If we dropped some of the rules, it would still hold. Even if we decided to be conservative and say that $\Gamma \vdash S$ *never* holds, the soundness fact would still be true. So the more interesting question to ask is whether the logical *strong* enough to prove everything it should. We want to know if $\Gamma \models S$ implies that $\Gamma \vdash S$; if it does for all Γ and S, then we say that our system is *complete*. As it happens, our system *is* complete. We show this in Section 2.7. There are several reasons why we do not present the completeness result in one fell swoop. First of all, doing so would not give you any idea of what is going on in the proof. So we have divided things up into smaller steps. And second, considering fragments of the logic gives us additional information (that is, additional completeness theorems) that we would not be able to obtain from the overall completeness result.

Exercise 1 Check that

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \not\models \textit{Some } X \textit{ are } Z$$

by building a model in which $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ and $[\![Y]\!] \cap [\![Z]\!] \neq \emptyset$, but $[\![X]\!] \cap [\![Z]\!] = \emptyset$.

Exercise 2 Check that

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \nvdash \textit{Some } X \textit{ are } Z$$

by examining proofs.

Exercise 3 This exercise asks you to come up with definitions and to check their properties.

1. Define the appropriate notions of *submodel* and *homomorphism of models*.
2. Which sentences S in our language have the property that if $\mathcal{M}$ is a submodel of $\mathcal{M}'$ and $\mathcal{M}' \models S$, then also $\mathcal{M} \models S$?
3. Which sentences S in our language have the property that if $\mathcal{M}$ is a surjective homomorphic image of $\mathcal{M}'$ and $\mathcal{M} \models S$, then also $\mathcal{M}' \models S$?
4. Would anything change if we changed "if" to "iff"?

Fragments As small as our language is, we shall be interested in a number of fragments of it. These include $\mathcal{L}(all)$, the fragment with *All* (and nothing else); and with obvious notation $\mathcal{L}(all, some)$, $\mathcal{L}(all, some, \text{names})$, and $\mathcal{L}(all, no)$. We also will be interested in extensions of the language and variations on the semantics.

Semantics One starts with a set M, a subset $[\![X]\!] \subseteq M$ for each variable X, and an element $[\![J]\!] \in \mathcal{M}$ for each name J. This gives a *model* $\mathcal{M} = (M, [\![\ ]\!])$. We then define

$$\begin{array}{lll} \mathcal{M} \models \textit{All } X \textit{ are } Y & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} \models \textit{Some } X \textit{ are } Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ \mathcal{M} \models \textit{No } X \textit{ are } Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] = \emptyset \\ \mathcal{M} \models J \textit{ is an } X & \text{iff} & [\![J]\!] \in [\![X]\!] \\ \mathcal{M} \models J \textit{ is } M & \text{iff} & [\![J]\!] = [\![M]\!] \end{array}$$

We allow $[\![X]\!]$ to be empty, and in this case, recall that $\mathcal{M} \models \textit{All } X \textit{ are } Y$ vacuously. And if Γ is a finite or infinite set of sentences, then we write $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$.

Main semantic definition $\Gamma \models S$ means that every model which makes all sentences in the set Γ true also makes S true. This is the relevant form of semantic entailment for this work.

Notation If Γ is a set of sentences, we write Γ_{all} for the subset of Γ containing only sentences of the form *All X are Y*. We do this for other constructs, writing Γ_{some}, Γ_{no} and Γ_{names}.

Inference rules of the logical system The complete set of rules for the syllogistic fragment may be found in Figure 7 below. But we are concerned with other fragments, especially in Sections 7 and onward. Rules for other fragments will be presented as needed.

Proof trees A *proof tree over* Γ is a finite tree $\mathcal{T}$ whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules. $\Gamma \vdash S$ means that there is a proof tree $\mathcal{T}$ for over Γ whose root is labeled S.

Example 1.2 Here is a proof tree:

$$\dfrac{\dfrac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \quad \dfrac{M \textit{ is a } Z \quad J \textit{ is } M}{J \textit{ is a } Z}}{\textit{Some } Y \textit{ are } Z}$$

Example 1.3 We take

$$\Gamma = \{\textit{All } A \textit{ are } B, \textit{All } Q \textit{ are } A, \textit{All } B \textit{ are } D, \textit{All } C \textit{ are } D, \textit{All } A \textit{ are } Q\}.$$

Let S be *All Q are D*. Here is a proof tree showing that $\Gamma \vdash S$:

$$\dfrac{\textit{All } Q \textit{ are } A \qquad \dfrac{\dfrac{\textit{All } A \textit{ are } B \quad \textit{All } B \textit{ are } B}{\textit{All } A \textit{ are } B} \quad \textit{All } B \textit{ are } D}{\textit{All } A \textit{ are } D}}{\textit{All } Q \textit{ are } D}$$

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

Figure 2: The logic of *All* X *are* Y.

Note that all of the leaves belong to Γ except for one that is *All* B *are* B. Note also that some elements of Γ are not used as leaves. This is permitted according to our definition. The proof tree above shows that $\Gamma \vdash S$. Also, there is a smaller proof tree that does this, since the use of *All* B *are* B is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All* A *are* A.)

Proof By induction on proof trees. ⊣

Example 1.4 One easy semantic fact is

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \not\models \textit{Some } X \textit{ are } Z.$$

The smallest countermodel is $\{1,2\}$ with $[\![X]\!] = \{1\}$, $[\![Y]\!] = \{1,2\}$, and $[\![Z]\!] = \{2\}$. Even if we ignore the soundness of the logical system, an examination its proofs shows that

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \nvdash \textit{Some } X \textit{ are } Z$$

Indeed, the only sentences which follow from the hypotheses are those sentences themselves, the sentences *Some* X *are* X, *Some* Y *are* Y, *Some* Z *are* Z, *Some* Y *are* X, and *Some* Z *are* Y, and the axioms of the system: sentences of the form *All* U *are* U and J *is* J.

There are obvious notions of *submodel* and *homomorphism* of models.

Proposition 1.5 *Sentences in* $\mathcal{L}(\text{all}, \text{no}, \textit{names})$ *are preserved under submodels. Sentences in* $\mathcal{L}(\text{some}, \textit{names})$ *are preserved under homomorphisms. Sentences in* $\mathcal{L}(\text{all})$ *are preserved under surjective homomorphic images.*

2 Basic Syllogistic Fragments

2.1 *All*

These notes are organized in sections corresponding to different fragments. To begin, we present a system for $\mathcal{L}(\textit{all})$. All of our logical systems are sound by Lemma 1.1.

Theorem 2.1 *The logic of Figure 2 is complete for* $\mathcal{L}(\text{all})$.

Proof Suppose that $\Gamma \models S$. Let S be *All* X *are* Y. Let $\{*\}$ be any singleton, and define a model $\mathcal{M}$ by $M = \{*\}$, and

$$[\![Z]\!] \quad = \quad \begin{cases} M & \text{if } \Gamma \vdash \textit{All } X \textit{ are } Z \\ \emptyset & \text{otherwise} \end{cases} \tag{3}$$

It is important that in (3), X is the same variable as in the sentence S with which we began. We claim that if Γ contains *All* V *are* W, then $[\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that $[\![V]\!] \neq \emptyset$ (otherwise the result is trivial). So $[\![V]\!] = M$. Thus $\Gamma \vdash$ *All* X *are* V. So we have a proof tree over Γ as indicated by the vertical dots $\vdots$ below:

$$\dfrac{\begin{matrix}\vdots \\ \textit{All } X \textit{ are } V\end{matrix} \quad \textit{All } V \textit{ are } W}{\textit{All } X \textit{ are } W}$$

The tree overall has as leaves *All* V *are* W plus the leaves of the tree above *All* X *are* V. Overall, we see that all leaves are labeled by sentences in Γ. This tree shows that $\Gamma \vdash$ *All* X *are* W. From this we conclude that $[\![W]\!] = M$. In particular, $[\![V]\!] \subseteq [\![W]\!]$.

Now our claim implies that the model $\mathcal{M}$ we have defined makes all sentences in Γ true. So it must make the conclusion true. Therefore $[\![X]\!] \subseteq [\![Y]\!]$. And $[\![X]\!] = M$, since we have a one-point tree for *All* X *are* X. Hence $[\![Y]\!] = M$ as well. But this means that $\Gamma \vdash$ *All* X *are* Y, just as desired. $\dashv$

Remark The completeness of $\mathcal{L}(all)$ appears to be the simplest possible completeness result of any logical system! (One can also make this claim about the pure identity fragment, the one whose statements are of the form J *is* M and whose logical presentation amounts to the reflexive, symmetric, and transitive laws.) At the same time, we are not aware of any prior statement of its completeness.

2.2 The canonical model property

We introduce a property which some of the logical systems in this subject enjoy, and others have to some degree or other. First we need some preliminary points. For any set Γ of sentences, define $\leq_\Gamma$ on the set of variables by

$$U \leq_\Gamma V \quad \text{iff} \quad \Gamma \vdash \textit{All } U \textit{ are } V \tag{4}$$

Lemma 2.2 *The relation $\leq_\Gamma$ is a preorder: a reflexive and transitive relation.*

We shall often use preorders $\leq_\Gamma$ defined by (4).

Also define a preorder $\preceq_\Gamma$ on the variables by: $U \preceq_\Gamma V$ if Γ contains *All* U *are* V. Let $\preceq^*_\Gamma$ be the reflexive-transitive closure of $\preceq_\Gamma$.

Usually we suppress mention of Γ and simply write $\leq$, $\preceq$, and $\preceq^*$.

Proposition 2.3 *Let Γ be any set of sentences in this fragment, let $\preceq^*$ be defined from Γ as above. Let X and Y be any variables. Then the following are equivalent:*

1. $\Gamma \vdash$ All X are Y.

2. $\Gamma \models$ All X are Y.

3. $X \preceq^* Y$.

Proof (1)$\Longrightarrow$(2) is by soundness, and (3)$\Longrightarrow$(1) is by induction on $\preceq^*$. The most significant part is (2)$\Longrightarrow$(3). We build a model $\mathcal{M}$. As in the proof of Theorem 2.1, we take $M = \{*\}$. But we modify (3) by taking $[\![Z]\!] = M$ iff $X \preceq^* Z$. We claim that $\mathcal{M} \models \Gamma$. Consider *All* V *are* W in Γ. We may assume that $[\![V]\!] = M$, or else our claim is trivial. Then $X \preceq^* V$. But $V \preceq W$, so we have $X \preceq^* W$, as desired. This verifies that $\mathcal{M} \models \Gamma$. But $[\![X]\!] = M$, and therefore $[\![Y]\!] = M$ as well. Hence $X \preceq^* Y$, as desired. ⊣

Definition Let $\mathcal{F}$ be a fragment, let Γ be a set of sentences in $\mathcal{F}$, and consider a fixed logical system for $\mathcal{F}$. A model $\mathcal{M}$ is *canonical for* Γ if for all $S \in \mathcal{F}$, $\mathcal{M} \models S$ iff $\Gamma \vdash S$. A fragment $\mathcal{F}$ has the *canonical model property* (for the given logical system) if every set $\Gamma \subseteq \mathcal{F}$ has a canonical model.

(For example, in $\mathcal{L}(all)$, $\mathcal{M}$ is canonical for Γ provided: $X \leq Y$ iff $[\![X]\!] \subseteq [\![Y]\!]$.)

Notice, for example, that classical propositional and first-order logic do not have the canonical model property. A model of $\Gamma = \{p\}$ will have to commit to a value on a different propositional symbol q, and yet neither q nor $\neg q$ follow from Γ. These systems do have the property that every *maximal consistent* set has a canonical model. Since they also have negation, this last fact leads to completeness. As it turns out, syllogistic fragments exhibit differing behavior with respect to the canonical model property. Some have it, some do not, and some have it for certain classes of sentences.

Proposition 2.4 $\mathcal{L}$(all) *has the canonical model property with respect to our logical system for it.*

Proof Given Γ, let $\mathcal{M}$ be the model whose universe is the set of variables, and with $[\![U]\!] = \{Z : Z \leq U\}$. Consider a sentence $S \equiv$ *All* X *are* Y. Then $[\![X]\!] \subseteq [\![Y]\!]$ in $\mathcal{M}$ iff $X \leq Y$. (Both rules of the logic are used here.) ⊣

The canonical model property is stronger than completeness. To see this, let $\mathcal{M}$ be canonical for a fixed set Γ. In particular $\mathcal{M} \models \Gamma$. Hence if $\Gamma \models S$, then $\mathcal{M} \models S$; so $\Gamma \vdash S$.

2.3 A digression: *All* X *which are* Y *are* Z

At this point, we digress from our main goal of the examination of the syllogistic system *All*. Instead, we consider the logic of *All* X *which are* Y *are* Z. To save space, we abbreviate this by (X, Y, Z). We take this sentence to be true in a given model $\mathcal{M}$ if $[\![X]\!] \cap [\![Y]\!] \subseteq [\![Z]\!]$. Note that *All* X *are* Y is semantically equivalent to (X, X, Y).

First, we check that the logic is genuinely new. The result in Proposition 2.5 clearly also holds for the closure of $\mathcal{L}(all, some, no)$ under (infinitary) boolean operations.

Proposition 2.5 *Let* R *be* All X which are Y are Z. *Then* R *cannot be expressed by any set in the language* $\mathcal{L}$(all, some, no). *That is, there is no set* Γ *of sentences in* $\mathcal{L}$(all, some, no) *such that for all* $\mathcal{M}$, $\mathcal{M} \models \Gamma$ *iff* $\mathcal{M} \models R$.

Proof Consider the model $\mathcal{M}$ with universe $\{x, y, a\}$ with $[\![X]\!] = \{x, a\}$, $[\![Y]\!] = \{y, a\}$, $[\![Z]\!] =$

$$\frac{}{(X,Y,X)} \qquad \frac{}{(X,Y,Y)} \qquad \frac{(X,Y,U) \quad (X,Y,V) \quad (U,V,Z)}{(X,Y,Z)}$$

Figure 3: The logic of *All X which are Y are Z*, written here (X, Y, Z).

$\{a\}$, and also $[\![U]\!] = \emptyset$ for other variables U. Consider also a model $\mathcal{N}$ with universe $\{x, y, a, b\}$ with $[\![X]\!] = \{x, a, b\}$, $[\![Y]\!] = \{y, a, b\}$, $[\![Z]\!] = \{a\}$, and the rest of the structure the same as in $\mathcal{M}$. An easy examination shows that for all sentences $S \in \mathcal{L}(all, some, no)$, $\mathcal{M} \models S$ iff $\mathcal{N} \models S$.

Now suppose towards a contradiction that we could express R, say by the set Γ. Then since $\mathcal{M}$ and $\mathcal{N}$ agree on $\mathcal{L}(all, some, no)$, they agree on Γ. But $\mathcal{M} \models R$ and $\mathcal{N} \not\models R$, a contradiction. $\dashv$

Theorem 2.6 *The logic of* All X which are Y are Z *in Figure 3 is complete.*

Proof Suppose $\Gamma \models (X, Y, Z)$. Consider the interpretation $\mathcal{M}$ given by $M = \{*\}$, and for each variable W, $[\![W]\!] = \{*\}$ iff $\Gamma \vdash (X, Y, W)$. We claim that for $(U, V, W) \in \Gamma$, $[\![U]\!] \cap [\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that $M = [\![U]\!] \cap [\![V]\!]$. So we use the proof tree

$$\frac{\begin{array}{c}\vdots\\(X,Y,U)\end{array} \quad \begin{array}{c}\vdots\\(X,Y,V)\end{array} \quad (U,V,W)}{(X,Y,W)}$$

This shows that $[\![W]\!] = M$, as desired.

Returning to our sentence (X, Y, Z), our overall assumption that $\Gamma \models (X, Y, Z)$ tells us that $\mathcal{M} \models (X, Y, Z)$. The first two axioms show that $* \in [\![X]\!] \cap [\![Y]\!]$. Hence $* \in [\![Z]\!]$. That is, $\Gamma \vdash (X, Y, Z)$. $\dashv$

Remark Instead of the axiom (X, Y, Y), we could have taken the symmetry rule

$$\frac{(Y,X,Z)}{(X,Y,Z)}$$

The two systems are equivalent.

Remark The fragment with (X, X, Y) is a conservative extension of the fragment with *All*, via the translation of *All X are Y* as (X, X, Y).

Exercise 4 Find a complete set of axioms for *All X which are Y are Z which are W* on top of the fragment of this section. The semantics should be $[\![X]\!] \cap [\![Y]\!] \subseteq [\![Z]\!] \cap [\![W]\!]$.

2.4 *All* and *Some*

We enrich our language with sentences *Some X are Y* and our rules with those of Figure 4. The symmetry rule for *Some* may be dropped if one 'twists' the transitivity rule to read

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } Z \textit{ are } X}$$

Then symmetry is derivable. We will use the twisted form in later work, but for now we want the three rules of Figure 4 because the first two alone are used in Theorem 2.8 below.

Example 2.7 Perhaps the first non-trivial derivation in the logic is the following one:

$$\dfrac{\textit{All } Z \textit{ are } Y \qquad \dfrac{\dfrac{\textit{All } Z \textit{ are } X \quad \textit{Some } Z \textit{ are } Z}{\textit{Some } Z \textit{ are } X}}{\textit{Some } X \textit{ are } Z}}{\textit{Some } X \textit{ are } Y}$$

That is, if there is a Z, and if all Zs are Xs and also Ys, then some X is a Y.

In working with *Some* sentences, we adopt some notation parallel to (4): for *All*

$$U \uparrow_\Gamma V \quad \text{iff} \quad \Gamma \vdash \textit{Some } U \textit{ are } V \tag{5}$$

Usually we drop the subscript Γ. Using the symmetry rule, $\uparrow$ is symmetric.

The next result is essentially due to van Benthem [2], Theorem 3.3.5.

Theorem 2.8 *The first two rules in Figure 4 give a logical system with the canonical model property for* $\mathcal{L}(\text{some})$. *Hence the system is complete.*

Proof Let $\Gamma \subseteq \mathcal{L}(\textit{some})$. Let $\mathcal{M} = \mathcal{M}(\Gamma)$ be the set of unordered pairs (i.e., sets with one or two elements) of variables. Let

$$[\![U]\!] \;=\; \{\{U, V\} : U \uparrow V\}.$$

Observe that the elements of $[\![U]\!]$ are unordered pairs with one element being U. If $U \uparrow V$, then $\{U, V\} \in [\![U]\!] \cap [\![V]\!]$. Assume first $X \neq Y$ and that Γ contains $S = \textit{Some } X \textit{ are } Y$. Then $\{X, Y\} \in [\![X]\!] \cap [\![Y]\!]$, so $\mathcal{M} \models S$. Conversely, if $\{U, V\} \in [\![X]\!] \cap [\![Y]\!]$, then by what we have said above $\{U, V\} = \{X, Y\}$. In particular, $\{X, Y\} \in M$. So $X \uparrow Y$. Second, we consider the situation when $X = Y$. If Γ contains $S = \textit{Some } X \textit{ are } X$, then $\{X\} \in [\![X]\!]$. So $\mathcal{M} \models S$. Conversely, if $\{U, V\} \in [\![X]\!]$, then (without loss of generality) $U = X$, and $X \uparrow V$. Using our second rule of *Some*, we see that $X \uparrow X$. ⊣

The rest of this section is devoted to the combination of *All* and *Some*.

Lemma 2.9 *Let* $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some})$. *Then there is a model* $\mathcal{M}$ *with the following properties:*

1. *If* $X \leq Y$, *then* $[\![X]\!] \subseteq [\![Y]\!]$.
2. $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ *iff* $X \uparrow Y$.

In particular, $\mathcal{M} \models \Gamma$.

Proof Let $N = |\Gamma_{\textit{some}}|$. We think of N as the ordinal number $\{0, 1, \ldots, N-1\}$. For $i \in N$,

$\dfrac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X}$	$\dfrac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}$	$\dfrac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}$

Figure 4: The logic of *Some* and *All*, in addition to the logic of *All*.

let U_i and V_i be such that

$$\Gamma_{some} \quad = \quad \{\textit{Some } V_i \textit{ are } W_i : i \in I\} \tag{6}$$

Note that for $i \neq j$, we might well have $V_i = V_j$ or $W_i = W_j$. For the universe of $\mathcal{M}$ we take the set N. For each variable Z, we define

$$[\![Z]\!] \quad = \quad \{i \in N : \text{either } V_i \leq Z \text{ or } W_i \leq Z\}. \tag{7}$$

(As in (4), the relation $\leq$ is: $X \leq Y$ iff $\Gamma \vdash \textit{All } X \textit{ are } Y$.) This defines the model $\mathcal{M}$.

For the first point, suppose that $X \leq Y$. It follows from (7) and Lemma 2.2 that $[\![X]\!] \subseteq [\![Y]\!]$.

Second, take a sentence *Some* V_i *are* W_i on our list in (6) above. Then i itself belongs to $[\![V_i]\!] \cap [\![W_i]\!]$, so this intersection is not empty. At this point we know that $\mathcal{M} \models \Gamma$, and so by soundness, we then get half of the second point in this lemma.

For the left-to-right direction of the second point, assume that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. Let $i \in [\![X]\!] \cap [\![Y]\!]$. We have four cases, depending on whether $V_i \leq X$ or $V_i \leq Y$, and whether $W_i \leq X$ or $W_i \leq Y$. In each case, we use the logic to see that $X \uparrow Y$. The formal proofs are all similar to what we saw in Example 2.7 above. $\dashv$

Theorem 2.10 *The logic of Figures 2 and 4 is complete for* $\mathcal{L}(\text{all}, \text{some})$.

Proof Suppose that $\Gamma \models S$. There are two cases, depending on whether S is of the form *All* X *are* Y or of the form *Some* X *are* Y. In the first case, we claim that $\Gamma_{all} \models S$. To see this, let $\mathcal{M} \models \Gamma_{all}$. We get a new model $\mathcal{M}' = \mathcal{M} \cup \{*\}$ via $[\![X]\!]' = [\![X]\!] \cup \{*\}$. The model $\mathcal{M}'$ so obtained satisfies Γ_{all} and all *Some* sentences whatsoever in the fragment. Hence $\mathcal{M}' \models \Gamma$. So $\mathcal{M}' \models S$. And since S is a universal sentence, $\mathcal{M} \models S$ as well. This proves our claim that $\Gamma_{all} \models S$. By Theorem 2.1, $\Gamma_{all} \vdash S$. Hence $\Gamma \vdash S$.

The second case, where S is of the form *Some* X *are* Y, is an immediate application of Lemma 2.9. $\dashv$

Remark Let $\Gamma \subseteq \mathcal{L}(\textit{all}, \textit{some})$, and let $S \in \mathcal{L}(\textit{some})$. As we know from Lemma 2.9, if $\Gamma \not\vdash S$, there is a $\mathcal{M} \models \Gamma$ which makes S false. The proof gets a model $\mathcal{M}$ whose size is $|\Gamma_{some}|$. We can get a countermodel of size at most 2. To see this, let $\mathcal{M}$ be as in Lemma 2.9, and let S be *Some* X *are* Y. If either $[\![X]\!]$ or $[\![Y]\!]$ is empty, we can coalesce all the points in $\mathcal{M}$ to a single point $*$, and then take $[\![U]\!]' = \{*\}$ iff $[\![U]\!] \neq \emptyset$. So we assume that $[\![X]\!]$ and $[\![Y]\!]$ are non-empty. Let $\mathcal{N}$ be the two-point model $\{1, 2\}$. Define $f : \mathcal{M} \to \mathcal{M}$ by $f(x) = 1$ iff $x \in [\![X]\!]$. The structure of $\mathcal{N}$ is that $[\![U]\!]_{\mathcal{N}} = f[[\![U]\!]_{\mathcal{N}}]$. This makes f a surjective homomorphism. By Proposition 1.5, $\mathcal{N} \models \Gamma$. And the construction insures that in $\mathcal{N}$, $[\![X]\!] \cap [\![Y]\!] = \emptyset$.

Note that 2 is the smallest we can get, since on models of size 1,

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \models \textit{Some } X \textit{ are } Z.$$

$$\frac{}{J \textit{ is } J} \qquad \frac{J \textit{ is } M \quad M \textit{ is } F}{F \textit{ is } J} \qquad \frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y}$$

$$\frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \qquad \frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X}$$

Figure 5: The logic of names, on top of the logic of *All* and *Some*.

Remark $\mathcal{L}(\textit{all}, \textit{some})$ does not have the canonical model property with respect to any logical system. To see this, let Γ be the set $\{\textit{All } X \textit{ are } Y\}$. Let $\mathcal{M} \models \Gamma$. Then either $\mathcal{M} \models \textit{All } Y \textit{ are } X$, or $\mathcal{M} \models \textit{Some } Y \textit{ are } Y$. But neither of these sentences follows from Γ. We cannot hope to avoid the split in the proof of Theorem 2.10 due to the syntax of S.

Remark Suppose that one wants to say that *All* X *are* Y is true when $[\![X]\!] \subseteq [\![Y]\!]$ *and also* $[\![X]\!] \neq \emptyset$. Then the following rule becomes sound:

$$\frac{\textit{All } X \textit{ are } Y}{\textit{Some } X \textit{ are } Y} \tag{8}$$

On the other hand, is is no longer sound to take *All* X *are* X to be an axiom. So we drop that rule in favor of (8). In this way, we get a complete system for the modified semantics. Here is how one sees this. Given Γ, let $\overline{\Gamma}$ be Γ with all sentences *Some* X *are* Y such that *All* X *are* Y belongs to Γ. An easy induction on proofs shows that $\Gamma \vdash S$ in the modified system iff $\overline{\Gamma} \vdash S$ in the old system.

2.5 Adding Proper Names

In this section we obtain completeness for sentences in $\mathcal{L}(\textit{all}, \textit{some}, \text{names})$. The proof system adds rules in Figure 5 to what we already have seen in Figures 2 and 4.

Fix a set $\Gamma \subseteq \mathcal{L}(\textit{all}, \textit{some}, \text{names})$. Let $\equiv$ and $\in$ be the relations defined from Γ by

$$\begin{array}{lll} J \equiv M & \text{iff} & \Gamma \vdash J \textit{ is } M \\ J \in X & \text{iff} & \Gamma \vdash J \textit{ is an } X \end{array}$$

Lemma 2.11 *$\equiv$ is an equivalence relation. And if $J \equiv M \in X \leq Y$, then $J \in Y$.*

Lemma 2.12 *Let $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some}, \textit{names})$. Then there is a model $\mathcal{N}$ with the following properties:*

1. *If $X \leq Y$, then $[\![X]\!] \subseteq [\![Y]\!]$.*
2. *$[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ iff $X \uparrow Y$.*
3. *$[\![J]\!] = [\![M]\!]$ iff $J \equiv M$.*
4. *$[\![J]\!] \in [\![X]\!]$ iff $J \in X$.*

Proof Let $\mathcal{M}$ be any model satisfying the conclusion of Lemma 2.9 for $\Gamma_{all} \cup \Gamma_{some}$. Let $\mathcal{N}$

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{No } Z \textit{ are } Y}{\textit{No } Y \textit{ are } X} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{No } X \textit{ are } Y} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{All } X \textit{ are } Y}$$

Figure 6: The logic of *No* X *are* Y on top of *All* X *are* Y.

be defined by

$$\begin{array}{rcl} N & = & M + \{[J] : J \text{ a name}\} \\ [\![X]\!] & = & [\![X]\!]_M + \{[J] : \Gamma \vdash J \textit{ is an } X\} \end{array} \tag{9}$$

The + here denotes a disjoint union. It is easy to check that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences in *All*, that the *Some* sentences true in $\mathcal{M}$ are still true in $\mathcal{N}$, and that points (3) and (4) in our lemma hold. So what remains is to check that if $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ in $\mathcal{N}$, then $X \uparrow Y$. The only interesting case is when $J \in [\![X]\!] \cap [\![Y]\!]$ for some name J. So $J \in X$ and $J \in Y$. Using the one rule of the logic which has both names and *Some*, we see that $X \uparrow Y$. ⊣

Theorem 2.13 *The logic of Figures 2, 4, and 5 is complete for* $\mathcal{L}(\text{all}, \text{some}, \textit{names})$.

Proof The proof is nearly the same as that of Theorem 2.10. In the part of the proof dealing with *All* sentences, we had a construction taking a model $\mathcal{M}$ to a one-point extension $\mathcal{M}'$. To interpret names in $\mathcal{M}'$, we let $[\![J]\!] = *$ for all names J. Then all sentences involving names are automatically true in $\mathcal{M}'$. ⊣

2.6 *All* and *No*

In this section, we consider $\mathcal{L}(\textit{all}, \textit{no})$. Note that *No* X *are* X just says that there are no Xs. In addition to the rules of Figure 2, we take the rules in Figure 6. As in (4) and (5), we write

$$U \perp_\Gamma V \quad \text{iff} \quad \Gamma \vdash \textit{No } U \textit{ are } V \tag{10}$$

This relation is symmetric.

Lemma 2.14 $\mathcal{L}(\text{all}, \text{no})$ *has the canonical model property with respect to our logic.*

Proof Let Γ be any set of sentences in *All* and *No*. Let

$$\begin{array}{rcl} M & = & \{\{U, V\} : U \not\perp V\} \\ [\![W]\!] & = & \{\{U, V\} \in M : U \leq W \text{ or } V \leq W\} \end{array} \tag{11}$$

The semantics is monotone, and so if $X \leq Y$, then $[\![X]\!] \subseteq [\![Y]\!]$. Conversely, suppose that $[\![X]\!] \subseteq [\![Y]\!]$. If $[\![X]\!] = \emptyset$, then $X \perp X$, for otherwise $\{X\} \in [\![X]\!]$. From the last rule in Figure 6, we see that $X \leq Y$, as desired. In the other case, $[\![X]\!] \neq \emptyset$, Fix $\{V, W\} \in [\![X]\!]$ so that $V \not\perp W$, and either $V \leq X$ or $W \leq X$. Without loss of generality, $V \leq X$. We cannot have $X \perp X$, or else $V \perp V$ and then $V \perp W$. So $\{X\} \in [\![X]\!] \subseteq [\![Y]\!]$. Thus $X \leq Y$.

We have shown $X \leq Y$ iff $[\![X]\!] \subseteq [\![Y]\!]$. This is half of the canonical model property, the other half being $X \perp Y$ iff $[\![X]\!] \cap [\![Y]\!] = \emptyset$. Suppose first that $[\![X]\!] \cap [\![Y]\!] = \emptyset$. Then $\{X, Y\} \notin M$, lest it belong to both $[\![X]\!]$ and $[\![Y]\!]$. So $X \perp Y$. Conversely, suppose that $X \perp Y$. Suppose towards a contradiction that $\{V, W\} \in [\![X]\!] \cap [\![Y]\!]$. There are four cases, and two representative ones are (i) $V \leq X$ and $W \leq Y$, and (ii) $V \leq X$ and $V \leq Y$. In (i), we have the following tree over Γ:

$$\dfrac{\begin{matrix}\vdots\\ \textit{All } W \textit{ are } Y\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \textit{All } V \textit{ are } X\end{matrix} \quad \begin{matrix}\vdots\\ \textit{No } X \textit{ are } Y\end{matrix}}{\textit{No } Y \textit{ are } V}}{\textit{No } V \textit{ are } W}$$

This contradicts $\{V, W\} \in M$. In (ii), we replace W by V in the tree above, so that the root is *No* V *are* V. Then we use one of the rules to conclude that *No* V *are* W, again contradicting $\{V, W\} \in M$. ⊣

Since the canonical model property is stronger than completeness, we have shown the following result:

Theorem 2.15 *The logic of Figures 2 and 6 is complete for* All *and* No.

2.7 $\mathcal{L}$(*all*, *some*, *no*, names)

At this point, we put together our work on the previous systems by proving a completeness result for $\mathcal{L}$(*all*, *some*, *no*, names). For the logic, we take all the rules in Figure 7. This includes the all rules from Figures 2, 4, 5, and 6. But we also must add a principle relating *Some* and *No*. For the first time, we face the problem of potential inconsistency: there are no models of *Some* X *are* Y and *No* X *are* Y. Hence any sentence S whatsoever follows from these two. This explains the last rule, a new one, in Figure 7.

Definition A set Γ is *inconsistent* if $\Gamma \vdash S$ for all S. Otherwise, Γ is *consistent*.

Before we turn to the completeness result in Theorem 2.17 below, we need a result specifically for $\mathcal{L}$(*all*, *no*, names).

Lemma 2.16 *Let* $\Gamma \subseteq \mathcal{L}(\text{all}, \text{no}, \textit{names})$ *be a consistent set. Then there is a model* $\mathcal{N}$ *such that*

1. $[\![X]\!] \subseteq [\![Y]\!]$ *iff* $X \leq Y$.
2. $[\![X]\!] \cap [\![Y]\!] = \emptyset$ *iff* $X \perp Y$.
3. $[\![J]\!] = [\![M]\!]$ *iff* $J \equiv M$.
4. $[\![J]\!] \in [\![X]\!]$ *iff* $J \in X$.

Proof Let $\mathcal{M}$ be from Lemma 2.14 for $\Gamma_{all} \cup \Gamma_{no}$. Let $\mathcal{N}$ come from $\mathcal{M}$ by the definitions in (9) in Lemma 2.12. (That is, we add the equivalence classes of the names in the natural way.) It is easy to check all of the parts above except perhaps for the second. If $[\![X]\!] \cap [\![Y]\!] = \emptyset$ in $\mathcal{N}$, then the same holds in its submodel $\mathcal{M}$. And so $X \perp Y$. In the other direction, assume that $X \perp Y$ but towards a contradiction that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. There are no points in the intersection in $M \subseteq N$. So let J be such that $[J] \in [\![X]\!] \cap [\![Y]\!]$. Then by our last point, $J \in X$ and $J \in Y$. Using the one rule of the logic which has both names and *Some*, we see that $\Gamma \vdash$ *Some* X *are* Y. Since $X \perp Y$, we see that Γ is inconsistent. ⊣

Theorem 2.17 *The logic in Figure 7 is complete for* $\mathcal{L}(\text{all}, \text{some}, \text{no}, \textit{names})$.

Proof Suppose that $\Gamma \models S$. We show that $\Gamma \vdash S$. We may assume that Γ is consistent, or

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } Z \textit{ are } X}$$

$$\frac{}{J \textit{ is } J} \qquad \frac{J \textit{ is } M \quad M \textit{ is } F}{F \textit{ is } J}$$

$$\frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y} \qquad \frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y}$$

$$\frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{No } Z \textit{ are } Y}{\textit{No } Y \textit{ are } X}$$

$$\frac{\textit{No } X \textit{ are } X}{\textit{No } X \textit{ are } Y} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } X \textit{ are } Y \quad \textit{No } X \textit{ are } Y}{S}$$

Figure 7: A complete set of rules for $\mathcal{L}(\textit{all}, \textit{some}, \textit{no}, \text{names})$.

else our result is trivial. There are a number of cases, depending on S.

First, suppose that $S \in \mathcal{L}(\textit{some}, \text{names})$. Let $\mathcal{N}$ be from Lemma 2.12 for $\Gamma_{all} \cup \Gamma_{some} \cup \Gamma_{names}$. There are two cases. If $\mathcal{N} \models \Gamma_{no}$, then by hypothesis, $\mathcal{N} \models S$. Lemma 2.12 then shows that $\Gamma \vdash S$, as desired. Alternatively, there may be some *No A are B* in Γ_{no} such that $[\![A]\!] \cap [\![B]\!] \neq \emptyset$. And again, Lemma 2.12 shows that $\Gamma_{all} \cup \Gamma_{some} \cup \Gamma_{names} \vdash$ *Some A are B*. So Γ is inconsistent.

Second, suppose that $S \in \mathcal{L}(\textit{all}, \textit{no})$. Let $\mathcal{N}$ come from Lemma 2.16 for $\mathcal{N} \models \Gamma_{all} \cup \Gamma_{names}$. If $\mathcal{N} \models \Gamma_{some}$, then by hypothesis $\mathcal{N} \models S$. By Lemma 2.16, $\Gamma \vdash S$. Otherwise, there is some sentence *Some A are B* in Γ_{some} such that $[\![A]\!] \cap [\![B]\!] = \emptyset$. And then $\mathcal{N} \models$ *No A are B*. By Lemma 2.16, $\Gamma \vdash$ *No A are B*. Again, Γ is inconsistent. $\dashv$

3 Adding Boolean Sentential Operations

The classical syllogisms include sentences *Some X is not a Y*. In our setting, it makes sense also to add other sentences with negative verb phrases: *J is not an X*, and *J is not M*. It is possible to consider the logical system that is obtained by adding just these sentences. But it is also possible to simply add the boolean operations on top of the language which we have already considered. So we have atomic sentences of the kinds we have already seen (the sentences in $\mathcal{L}(\textit{all}, \textit{some}, \textit{no}, \text{names})$), and then we have arbitrary conjunctions, disjunctions, and negations of sentences. We present a Hilbert-style axiomatization of this logic in Figure 8 below. The completeness of it appears in Łukasiewicz [6] (in work with Słupecki; they also showed decidability), and also by Westerståhl [18], and axioms 1–6 are essentially the system SYLL. We include Theorem 3.2 in these notes because it is a natural next step, because the

1. All substitution instances of propositional tautologies.
2. $All\ X\ are\ X$
3. $(All\ X\ are\ Z) \wedge (All\ Z\ are\ Y) \to All\ X\ are\ Y$
4. $(All\ Y\ are\ Z) \wedge (Some\ X\ are\ Y) \to Some\ Z\ are\ X$
5. $Some\ X\ are\ Y \to Some\ X\ are\ X$
6. $No\ X\ are\ X \to All\ X\ are\ Y$
7. $No\ X\ are\ Y \leftrightarrow \neg(Some\ X\ are\ Y)$
8. $J\ is\ J$
9. $(J\ is\ M) \wedge (M\ is\ F) \to F\ is\ J$
10. $(J\ is\ an\ X) \wedge (J\ is\ a\ Y) \to Some\ X\ are\ Y$
11. $(All\ X\ are\ Y) \wedge (J\ is\ an\ X) \to J\ is\ a\ Y$
12. $(M\ is\ an\ X) \wedge (J\ is\ M) \to J\ is\ an\ X$

Figure 8: Axioms for boolean combinations of sentences in $\mathcal{L}(all, some, no, \text{names})$.

techniques build on what we have already seen, and because we shall generalize the result in Section 7.3.

It should be noted that the axioms in Figure 8 are not simply transcriptions of the rules from our earlier system in Figure 7. The biconditional (7) relating *Some* and *No* is new, and using it, one can dispense with two of the transcribed versions of the *No* rules from earlier. Similarly, we should emphasize that the pure syllogistic logic is computationally much more tractable than the boolean system, being in polynomial time.

3.1 Propositional Logic

Before we turn to the boolean syllogistic system, it makes sense to review propositional logic. We do this partly because propositional logic is itself a 'natural logic' system, and partly because the particular algebraic treatment that we have in mind will re-appear in Section 4.2 below.

the rest of this section is missing

3.2 Boolean Syllogistic Logic

As with any Hilbert-style system, the only rule of the system in this section is modus ponens. (We think of the other systems in these notes as having many rules.) We define $\vdash \varphi$ in the usual way, and then we say that. $\Gamma \vdash \varphi$ if there are $\psi_1, \ldots, \psi_n$ from Γ such that $\vdash (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$.

The soundness of this system is routine.

Proposition 3.1 *If* $\Gamma_0 \cup \{\chi\} \subseteq \mathcal{L}(\text{all}, \text{some}, \text{no}, names)$*, and if* $\Gamma_0 \vdash \chi$ *using the system of Figure 7, then* $\Gamma_0 \vdash \chi$ *in the system of Figure 8.*

The proof is by induction on proof trees in the previous system. We shall use this result above frequently in what follows, without special mention.

Theorem 3.2 *The logic of Figure 8 is complete for assertions* $\Delta \models \varphi$ *in the language of boolean combinations from* $\mathcal{L}(\text{all}, \text{some}, \text{no}, \textit{names})$*.*

The rest of this section is devoted to proof of Theorem 3.2. As usual, the presence of negation in the language allows us to prove completeness by showing that every consistent Δ in the language of this section has a model. We may as well assume that Δ is *maximal* consistent.

Definition The *basic* sentences are those of the form *All X are Y*, *Some X and Y*, *J is M*, and *J is an X* or their negations. Let

$$\Gamma \quad = \quad \{S : \Delta \models S \text{ and } S \text{ is basic}\}.$$

Note that Γ might contain sentences $\neg$(*All X are Y*) which do not belong to the syllogistic language $\mathcal{L}(\textit{all}, \textit{some}, \textit{no}, \text{names})$.

Claim 3.3 $\Gamma \models \Delta$*. That is, every model of* Γ *is a model of* Δ*.*

To see this, let $\mathcal{M} \models \Gamma$ and let $\varphi \in \Delta$. We may assume that φ is in disjunctive normal form. It is sufficient to show that some disjunct of φ holds in $\mathcal{M}$. By maximal consistency, let ψ be a disjunct of φ which also belongs to Δ. Each conjunct of ψ belongs to Γ and so holds in $\mathcal{M}$.

The construction of a model of Γ is similar to what we saw in Theorem 2.13. Define $\leq$ to be the relation on variables given by $X \leq Y$ if the sentence *All X are Y* belongs to Γ. We claim that $\leq$ is reflexive and transitive. We'll just check the transitivity. Suppose that *All X are Y* and *All Y are Z* belong to Γ. Then they belong to Δ. Using Proposition 3.1, we see that $\Delta \vdash$ *All X are Z*. Since Δ is maximal consistent, it must contain *All X are Z*; thus so must Γ.

Define the relation $\equiv$ on names by $J \equiv M$ iff the sentence J is M belongs to Γ. Then $\equiv$ is an equivalence relation, just as we saw above for $\leq$. Let the set of equivalence classes of $\equiv$ be $\{[J_1], \ldots, [J_m]\}$. (Incidentally, this result does not need Γ to be finite, and we are only pretending that it is finite to simplify the notation a bit.)

Let the set of *Some X are Y* sentences in Γ be $S_1, \ldots, S_n$, and for $1 \leq i \leq n$, let U_i and V_i be such that S_i is *Some* U_i *are* V_i. So

$$\Gamma_{some} \quad = \quad \{\textit{Some } U_i \textit{ are } V_i : i = 1, \ldots, n\} \tag{12}$$

Let the set of $\neg$(*All X are Y*) sentences in Γ be $T_1, \ldots, T_p$. For $1 \leq i \leq p$, let W_i and X_i be such that T_i is $\neg$(*All* W_i *are* X_i). So this time we are concerned with

$$\{\neg(\textit{All } W_i \textit{ are } X_i) : i = 1, \ldots, p\} \tag{13}$$

Note that for $i \neq j$, we might well have $U_i = U_j$ or $U_i = W_j$, or some other such equation. (This is the part of the structure that goes beyond what we saw in Theorem 2.13.)

We take $\mathcal{M}$ to be a model with M the following set

$$\{(a, 1), \ldots, (a, m)\} \cup \{(b, 1), \ldots, (b, n)\} \cup \{(c, 1), \ldots, (c, p)\}.$$

Here m, n, and p are the numbers we saw in the past few paragraphs. The purpose of a, b, and c is to make a disjoint union. Let $[\![J]\!] = (a, i)$, where i is the unique number between 1 and m such that $J \equiv J_i$. And for a variable Z we set

$$
\begin{array}{rcll}
[\![Z]\!] & = & & \{(a,i) : 1 \leq i \leq n \text{ and } J_i \textit{ is a } Z \text{ belongs to } \Gamma\} \\
 & & \cup & \{(b,i) : 1 \leq i \leq m \text{ and either } U_i \leq Z \text{ or } V_i \leq Z\} \\
 & & \cup & \{(c,i) : 1 \leq i \leq p \text{ and } W_i \leq Z\}
\end{array}
\tag{14}
$$

This completes the specification of $\mathcal{M}$. The rest of our work is devoted to showing that all sentences in Γ are true in $\mathcal{M}$. We must argue case-by-case, and so we only give the parts of the arguments that differ from what we have seen in Theorem 2.13.

Consider the sentence T_i, that is $\neg(\textit{All } W_i \textit{ are } X_i)$. We want to make sure that $[\![W_i]\!] \setminus [\![X_i]\!] \neq \emptyset$. For this, consider (c, i). This belongs to $[\![W_i]\!]$ by the last clause in (14). We want to be sure that $(c, i) \notin [\![X_i]\!]$. For if $(c, i) \in [\![X_i]\!]$, then Γ would contain *All* W_i *are* X_i. And then Γ would be inconsistent in our previous system, so our original Δ would be inconsistent in our Hilbert-style system.

Continuing, consider a sentence $\neg(\textit{Some } P \textit{ are } Q)$ in Γ. We have to make sure that $[\![P]\!] \cap [\![Q]\!] = \emptyset$. We argue by contradiction. There are three cases, depending on the first coordinate of a putative element of the intersection. Perhaps the most interesting case is when $(c, i) \in [\![P]\!] \cap [\![Q]\!]$ for $1 \leq i \leq p$. Then Γ contains both *All* W_i *are* P and *All* W_i *are* Q. Now the fact that Γ contains $\neg(\textit{All } W_i \textit{ are } X_i)$ implies that it must contain *Some* W_i *are* W_i. For if not, then it would contain *No* W_i *are* W_i and hence *All* W_i *are* X_i; as always, this would contradict the consistency of Δ. Thus Γ contains *All* W_i *are* P, *All* W_i *are* Q and *Some* W_i *are* W_i. Using our previous system, we see that Γ contains *Some* P *are* Q (see Example 2.7). This contradiction shows that $[\![P]\!] \cap [\![Q]\!]$ cannot contain any element of the form (c, i). The other two cases are similar, and we conclude that the intersection is indeed empty.

This concludes our outline of the proof of Theorem 3.2.

4 Adding a Complement Operation

This section adds an *explicit noun-level complement operator* to the syntax. So we now can say, for example, *All* X' *are* Y, or *Some non-X are* Y.

We again start with the syntax and semantics of a language. This time we call it $\mathcal{L}(\textit{all}, \textit{some}, ')$. Let $\mathcal{V}$ be an arbitrary set whose members will be called *variables*. We use X, Y, ..., for variables. The idea is that they represent plural common nouns. We also assume that there is a *complementation operation* $' : \mathcal{V} \to \mathcal{V}$ on the variables such that $X'' = X$ for all X. This *involutive* property implies that complementation is a bijection on $\mathcal{V}$. In addition, to avoid some uninteresting technicalities, we shall always assume that $X \neq X'$. Then we consider sentences *All* X *are* Y and *Some* X *are* Y. Here X and Y are any variables, including the case when they are the same. We call this language $\mathcal{L}(\textit{all}, \textit{some}, ')$. We shall use letters like S to denote sentences.

Semantics One starts with a set M and a subset $[\![X]\!] \subseteq M$ for each variable X, subject to the requirement that $[\![X']\!] = M \setminus [\![X]\!]$ for all X. This gives a *model* $\mathcal{M} = (M, [\![\ \ ]\!])$. We then

$$\frac{}{\textit{All } X \textit{ are } X}\ \textit{Axiom} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}\ \textit{Some}_1 \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X}\ \textit{Some}_2$$

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}\ \textit{Barbara} \qquad \frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}\ \textit{Darii}$$

$$\frac{\textit{All } Y \textit{ are } Y'}{\textit{All } Y \textit{ are } X}\ \textit{Zero} \qquad \frac{\textit{All } Y' \textit{ are } Y}{\textit{All } X \textit{ are } Y}\ \textit{One}$$

$$\frac{\textit{All } Y \textit{ are } X'}{\textit{All } X \textit{ are } Y'}\ \textit{Antitone} \qquad \frac{\textit{All } X \textit{ are } Y \quad \textit{Some } X \textit{ are } Y'}{S}\ \textit{Contrad}$$

Figure 9: Syllogistic logic with complement.

define, just as before:

$$\begin{array}{lll} \mathcal{M} \models \textit{All } X \textit{ are } Y & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} \models \textit{Some } X \textit{ are } Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \end{array}$$

Example 4.1 We claim that $\Gamma \models \textit{All } A \textit{ are } C$, where

$$\Gamma \quad = \quad \{\textit{All } B' \textit{ are } X, \textit{All } X \textit{ are } Y, \textit{All } Y \textit{ are } B, \textit{All } B \textit{ are } X, \textit{All } Y \textit{ are } C\}.$$

Here is an informal explanation. Since all B and all B' are X, everything whatsoever is an X. And since all X are Y, and all Y are B, we see that everything is a B. In particular, all A are B. But the last two premises and the fact that all X are Y also imply that all B are C. So all A are C.

No In previous work, we took *No* X *are* Y as a basic sentence in the syntax. There is no need to do this here: we may regard *No* X *are* Y as a variant notation for *All* X *are* Y'. So the semantics would be

$$\mathcal{M} \models \textit{No } X \textit{ are } Y \quad \text{iff} \quad [\![X]\!] \cap [\![Y]\!] = \emptyset$$

In other words, if one wants to add *No* as a basic sentence forming-operation, on a par with *Some* and *All*, it would be easy to do so.

Proof trees We have discussed the meager syntax of $\mathcal{L}(\textit{all}, \textit{some}, ')$ and its semantics. We next turn to the proof theory. A *proof tree over* Γ is a finite tree $\mathcal{T}$ whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules for the fragment listed in Figure 9. $\Gamma \vdash S$ means that there is a proof tree $\mathcal{T}$ for over Γ whose root is labeled S.

We attached names to the rules in Figure 9 so that we can refer to them later. We usually do not display the names of rules in our proof trees except when to emphasize some point or other. The only purpose of the axioms *All* X *are* X is to derive these sentences from all sets;

otherwise, the axioms are invisible. The names "Barbara" and "Darii" are traditional from Aristotelian syllogisms. But the (*Antitone*) rule is not part of traditional syllogistic reasoning. It is possible to drop (Some_2) if one changes the conclusion of (*Darii*) to $\mathit{Some}\ Z\ \mathit{are}\ X$. But at one point it will be convenient to have (Some_2), and so this guides the formulation. The rules (*Zero*) and (*One*) are concerned with what is often called vacuous universal quantification. That is, if $Y' \subseteq Y$, then Y is the whole universe and Y' is empty; so Y is a superset of every set and Y' a subset. It would also be possible to use binary rules instead; in the case of (*Zero*), for example, we would infer $\mathit{All}\ X\ \mathit{are}\ Z$ from $\mathit{All}\ X\ \mathit{are}\ Y$ and $\mathit{All}\ X\ \mathit{are}\ Y'$. The (*Contrad*) rule is *ex falso quodlibet*; it permits inference of any sentence S whatsoever from a contradiction.

Example 4.2 Returning to Example 4.1, here is a proof tree showing $\Gamma \vdash \mathit{All}\ A\ \mathit{are}\ C$:

$$\dfrac{\dfrac{\dfrac{\mathit{All}\ B'\ \mathit{are}\ X \quad \dfrac{\mathit{All}\ X\ \mathit{are}\ Y \quad \mathit{All}\ Y\ \mathit{are}\ B}{\mathit{All}\ X\ \mathit{are}\ B}}{\mathit{All}\ B'\ \mathit{are}\ B}}{\mathit{All}\ A\ \mathit{are}\ B} \qquad \dfrac{\mathit{All}\ B\ \mathit{are}\ X \quad \dfrac{\mathit{All}\ X\ \mathit{are}\ Y \quad \mathit{All}\ Y\ \mathit{are}\ C}{\mathit{All}\ X\ \mathit{are}\ C}}{\mathit{All}\ B\ \mathit{are}\ C}}{\mathit{All}\ A\ \mathit{are}\ C}$$

Exercise 5 Show that

$$\{\mathit{All}\ B\ \mathit{are}\ X, \mathit{All}\ B'\ \mathit{are}\ X, \mathit{All}\ Y\ \mathit{are}\ C, \mathit{Some}\ A\ \mathit{are}\ C'\} \vdash \mathit{Some}\ X\ \mathit{are}\ Y'.$$

Lemma 4.3 *The following are derivable:*

1. Some X are $X' \vdash S$ *(a contradiction fact)*
2. All X are Z, No Z are $Y \vdash$ No Y are X *(Celarent)*
3. No X are $Y \vdash$ No Y are X *(E-conversion)*
4. Some X are Y, $\mathit{No}\ Y\ \mathit{are}\ Z \vdash$ Some X are Z' *(Ferio)*
5. All Y are Z, All Y are $Z' \vdash$ No Y are Y *(complement inconsistency)*

Proof For the assertion on contradictions,

$$\dfrac{\dfrac{}{\mathit{All}\ X\ \mathit{are}\ X}\ \mathit{Axiom} \quad \mathit{Some}\ X\ \mathit{are}\ X'}{S}\ \mathit{Contrad}$$

(*Celarent*) in this formulation is just a re-phrasing of (*Barbara*), using complements:

$$\dfrac{\mathit{All}\ X\ \mathit{are}\ Z \quad \mathit{All}\ Z\ \mathit{are}\ Y'}{\mathit{All}\ Y\ \mathit{are}\ Z'}\ \mathit{Barbara}$$

(*E-conversion*) is similarly related to (*Antitone*), and (*Ferio*) to (*Darii*). For complement inconsistency, use (*Antitone*) and (*Barbara*). ⊣

The logic is easily seen to be sound: if $\Gamma \vdash S$, then $\Gamma \models S$. The main contribution of this paper is the completeness of this system.

Some syntactic abbreviations The language lacks boolean connectives, but it is convenient to use an informal notation for it. It is also worthwhile specifying an operation of *duals*.

$$\begin{array}{lcl|lcl} \neg(\textit{All } X \textit{ are } Y) & = & \textit{Some } X \textit{ are } Y' & (\textit{All } X \textit{ are } Y)^d & = & \textit{All } Y' \textit{ are } X' \\ \neg(\textit{Some } X \textit{ are } Y) & = & \textit{All } X \textit{ are } Y' & (\textit{Some } X \textit{ are } Y)^d & = & \textit{Some } Y \textit{ are } X \end{array}$$

Here are some uses of this notation. We say that Γ is *inconsistent* if for some S, $\Gamma \vdash S$ and $\Gamma \vdash \neg S$. The first part of Lemma 4.3 tells us that if $\Gamma \vdash \textit{Some } X \textit{ is } X'$, then Γ is inconsistent. Also, we have the following result:

Proposition 4.4 *If* $S \vdash T$*, then* $\neg T \vdash \neg S$*.*

This fact is not needed below, but we recommend thinking about it as a way of getting familiar with the rules.

4.1 The indirect calculus

Frequently the logic of syllogisms is set up as an *indirect* system, where one in effect takes reductio ad absurdum to be part of the system. We formulate a notion $\Gamma \vdash_{raa} S$ of indirect proof in this section, and then later we show that $\Gamma \vdash_{raa} S$ iff $\Gamma \vdash S$.

We define $\Gamma \vdash_{raa} S$ as follows:

1. If $S \in \Gamma$ or S is *All* X *are* X, then $\Gamma \vdash_{raa} S$
2. For all rules in Figure 9 except the contradiction rule, if S_1 and S_2 are the premises of some instance of the rule, and T the conclusion, if $\Gamma \vdash_{raa} S_1$ and $\Gamma \vdash_{raa} S_2$, then also $\Gamma \vdash_{raa} T$.
3. If $\Gamma \cup \{S\} \vdash_{raa} T$ and $\Gamma \cup \{S\} \vdash_{raa} \neg T$, then $\Gamma \vdash_{raa} \neg S$.

In effect, one is adding hypothetical reasoning in the manner of the sequent calculus.

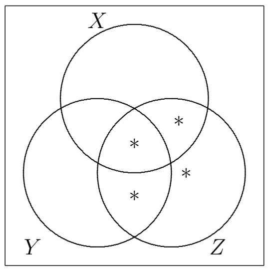
Proposition 4.5 *If* $\Gamma \vdash S$*, then* $\Gamma \vdash_{raa} S$*.*

Proof By induction on the heights of proof trees for $\vdash$. The only interesting step is when $\Gamma \vdash S$ via application of the contradiction rule. So for some T, $\Gamma \vdash T$ and $\Gamma \vdash \neg T$. Using the induction hypothesis, $\Gamma \vdash_{raa} T$ and $\Gamma \vdash_{raa} \neg T$. Clearly we also have $\Gamma \cup \{\neg S\} \vdash_{raa} T$ and $\Gamma \cup \{\neg S\} \vdash_{raa} \neg T$. Hence $\Gamma \vdash_{raa} S$. ⊣

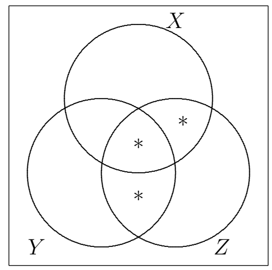
It is natural to ask whether the converse holds. This takes a little more work, and we won't present it in these notes.

Comparison with previous work The proof system in this section, the one presented by the rules in Figure 9, appears to be new. However, the indirect system appears to be close to the earlier work of Corcoran [4] and Martin [7]. Thus, the fact that the systems turn out to be equivalent is of some interest. In any case, these papers are mostly concerned with modern reconstruction of Aristotleian syllogisms, as is the pioneering work in this area, Łukasiewicz's book [6]. We are not so concerned with this project, but rather our interest lies in logical completeness results for fragments of natural language. The fragment in this paper is obviously quite small, but we believe that the techniques used in studying it may help with larger fragments. This is the main reason for this work.

The language $\mathcal{L}(all, some, ')$ of this section is more expressive than $\mathcal{L}(all, some, no)$, in the following precise sense: Consider the two models $\mathcal{M}$ and $\mathcal{N}$ shown below:



$\mathcal{M}$



$\mathcal{N}$

They satisfy the same sentences in $\mathcal{L}(all, some, no)$. (They also satisfy the same sentences of the form *Some A are B'*.) But let S be *Some X' are Y'* so that $\neg S$ is *All X' are Y*. $\mathcal{M} \models S$ but $\mathcal{N} \models \neg S$. We conclude from this example is that a logical system for the language with complements cannot simply be a translation into the smaller language.

4.2 Completeness via representation of orthoposets

An important step in our work is to develop an *algebraic semantics* for $\mathcal{L}(all, some, ')$. There are several definitions, and then a representation theorem. As with other uses of algebra in logic, the point is that the representation theorem is also a *model construction technique.*

An *orthoposet* is a tuple $(P, \leq, 0, ')$ such that

1. $(P, \leq)$ is a partial order: $\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.

2. 0 is a minimum element: $0 \leq p$ for all $p \in P$.

3. $x \mapsto x'$ is an antitone map in both directions: $x \leq y$ iff $y' \leq x'$.

4. $x \mapsto x'$ is involutive: $x'' = x$.

5. complement inconsistency: If $x \leq y$ and $x \leq y'$, then $x = 0$.

The notion of an orthoposet mainly appears in papers on quantum logic. (In fact, the stronger notion of an *orthomodular poset* appears to be more central there. However, I do not see any application of this notion to logics of the type considered in this paper.)

Example 4.6 For example, for all sets X we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $\subseteq$ is the inclusion relation, $\emptyset$ is the empty set, and $a' = X \setminus a$ for all subsets a of X.

Example 4.7 Let Γ be any set of sentences in $\mathcal{L}(all, some, ')$. We define a relation $\leq_\Gamma$ on the set $\mathcal{V}$ of variables of our logical system by

$$X \leq_\Gamma Y \quad \text{iff} \quad \Gamma \vdash \textit{All } X \textit{ are } Y.$$

We always drop the subscript Γ because it will be clear from the context which set Γ is used. We have an induced equivalence relation $\equiv$, and we take $\mathcal{V}_\Gamma$ to be the quotient $\mathcal{V}/\!\equiv$. It is a partial order under the induced relation. If there is some X such that $X \leq X'$, then for all Y we have $[X] \leq [Y]$ in $\mathcal{V}/\!\equiv$. In this case, set 0 to be $[X]$ for any such X. (If such X exists, its equivalence class is unique.) We finally define $[X]' = [X']$. If there is no X such that $X \leq X'$, we add fresh elements 0 and 1 to $\mathcal{V}/\!\equiv$. We then stipulate that $0' = 1$, and that for all $x \in \mathcal{V}_\Gamma$, $0 \leq x \leq 1$.

It is not hard to check that we have an orthoposet $\mathcal{V}_\Gamma = (\mathcal{V}_\Gamma, \leq, 0, ')$. The antitone property comes from the axiom with the same name, and the complement inconsistency is verified using the similarly-named part of Lemma 4.3.

A *morphism of orthoposets* is a map m preserving the order (if $x \leq y$, then $mx \leq my$), the complement $m(x') = (mx)'$, and minimum elements ($m0 = 0$). We say m is *strict* if the following extra condition holds: $x \leq y$ iff $mx \leq my$.

A *point* of a orthoposet $P = (P, \leq, 0, ')$ is a subset $S \subseteq P$ with the following properties:

1. If $p \in S$ and $p \leq q$, then $q \in S$ (S is *up-closed*).
2. For all p, either $p \in S$ or $p' \in S$ (S is *complete*), but not both (S is *consistent*).

Example 4.8 Let $X = \{1, 2, 3\}$, and let $\mathcal{P}(X)$ be the power set orthoposet from Example 4.6. Then S is a point, where

$$S \quad = \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

(More generally, if X is any finite set, then the collection of subsets of X containing more than half of the elements of X is a point of $\mathcal{P}(X)$.) Also, it is easy to check that the points on this $\mathcal{P}(X)$ are exactly S as above and the three principal ultrafilters. S shows that a point of a boolean algebra need not be an ultrafilter or even a filter. Also, the lemma just below shows that for $\mathcal{P}(X)$, a collection of elements is included in a point iff every pair of elements has a non-empty intersection.

Lemma 4.9 *For a subset S_0 of an orthoposet $P = (P, \leq, ')$, the following are equivalent:*

1. *S_0 is a subset of a point S in P.*
2. *For all $x, y \in S_0$, $x \not\leq y'$.*

Proof Clearly (1) $\implies$ (2). For the more important direction, use Zorn's Lemma to get a

$\subseteq$-maximal superset S_1 of S_0 with the consistency property. Let $S = \{q : (\exists p \in S_1) q \geq p\}$. So S is up-closed. We check that consistency is not lost: suppose that $r, r' \in S$. Then there are $q_1, q_2 \in S_1$ such that $r \geq q_1$ and $r' \geq q_2$. But then $q_2' \geq r \geq q_1$. Since $q_1 \in S_1$, so too $q_2' \in S_1$. Thus we see that S_1 is not consistent, and this is a contradiction. To conclude, we only need to see that for all $r \in P$, either r or r' belongs to S. If $r \notin S$, then $r \notin S_1$. By maximality, there is $q \in S_1$ such that $q_1 \leq r'$. (For otherwise, $S_1 \cup \{r\}$ would be a consistent proper superset of S_1.) And as $r' \notin S$, there is $q_2 \in S_1$ such that $q_2 \leq r$. Then as above $q_1 \leq q_2'$, leading to the same contradiction. $\dashv$

We now present a representation theorem that implies the completeness of the logic. It is due to Calude, Hertling, and Svozil [3]. We also state an additional technical point.

Theorem 4.10 ([3]; see also [5, 19]) *Let $P = (P, \leq, ')$ be an orthoposet. There is a set* points(P) *and a strict morphism of orthoposets $m : P \to \mathcal{P}(\text{points}(P))$.*

Moreover, if $S \cup \{p\} \subseteq P$ has the following two properties, then $m(p) \setminus \bigcup_{q \in S} m(q)$ is non-empty:

1. *For all $q \in S$, $p \not\leq q$.*
2. *For all $q, r \in S$, $q \not\geq r'$.*

Proof Let *points*(P) be the collection of points of P. The map m is defined by $m(p) = \{S : p \in S\}$. The preservation of complement comes from the completeness and consistency requirement on points, and the preservation of order from the up-closedness. Clearly $m0 = \emptyset$. We must check that if $q \not\geq p$, then there is some point S such that $p \in S$ and $q \notin S$. For this, take $S = \{q\}$ in the "moreover" part. And for that, let $T = \{p\} \cup \{q' : q \in S\}$. Lemma 4.9 applies, and so there is some point $U \supseteq T$. Such U belongs to $m(p)$. But if $q \in S$, then $q' \in T \subseteq U$; so U does not belong to $m(q)$. $\dashv$

Completeness of the indirect system The algebraic machinery that we have just seen gives an easy completeness theorem for the indirect system.

Lemma 4.11 *Let Γ be consistent in $\mathcal{L}(\text{all}, \text{some}, ')$. There is a model $\mathcal{M} = (M, [\![\ ]\!])$ such that*

1. *$\mathcal{M} \models \Gamma$.*
2. *If T is a sentence in* All *and $\mathcal{M} \models T$, then $\Gamma \vdash T$.*

Proof Let $\mathcal{V}_\Gamma$ be the orthoposet from Example 4.7 for Γ. Let n be the natural map of $\mathcal{V}$ into $\mathcal{V}_\Gamma$, taking a variable X to its equivalence class $[X]$. If $X \leq Y$, then $[X] \leq [Y]$ by definition of the structure. In addition, n preserves the order in both directions. We also apply Theorem 4.10, to obtain a strict morphism of orthoposets m as shown below:

$$\mathcal{V} \xrightarrow{\ n\ } \mathcal{V}_\Gamma \xrightarrow{\ m\ } \textit{points}(\mathcal{V}_\Gamma)$$

Let $M = \textit{points}(\mathcal{V}_\Gamma)$, and let $[\![\ ]\!] : \mathcal{V} \to \mathcal{P}(M)$ be the composition $n \circ m$. We thus have a model $\mathcal{M} = (\textit{points}(\mathcal{V}_\Gamma), [\![\ ]\!])$.

We check that $\mathcal{M} \models \Gamma$. Note that n and m are strict monotone functions. So the semantics has the property that the *All* sentences holding in $\mathcal{M}$ are exactly the consequences of Γ. We turn to a *Some* sentence in Γ such as *Some U are V*. Then by consistency of Γ, $U \not\leq V'$. Thus $[\![U]\!] \not\subseteq ([\![V]\!])'$. That is, $[\![U]\!] \cap [\![V]\!] \neq \emptyset$. ⊣

(Unfortunately, the last step in this proof is not reversible, in the following precise sense. $U \not\leq W'$ does not imply that $\Gamma \vdash$ *Some U are W*. (For example, if Γ is the empty set we have $U \not\leq W'$, and indeed $\mathcal{M}(\Gamma) \models$ *Some U are W*. But Γ only derives valid sentences.)

Theorem 4.12 *The indirect system is complete for* $\mathcal{L}(\text{all}, \text{some}, ')$*:* $\Gamma \vdash_{raa} S$ *iff* $\Gamma \models S$.

Proof Suppose that $\Gamma \not\vdash_{raa} S$ in $\mathcal{L}(all, some, ')$. We shall construct a model of Γ where S fails. Γ must be consistent in the logic. If S is a sentence in *All*, $\mathcal{M}(\Gamma)$ from Lemma 4.11 works. Otherwise, let S be *Some X are Y*. The indirect calculus is set up so that we immediately infer the consistency of $\Gamma \cup \{\neg S\}$. So in the notation of Lemma 4.11 again, $\mathcal{M}(\Gamma \cup \{\neg S\})$ does what we want. ⊣

The direct system is also complete: see [10] and Pratt-Hartmann [**?**].

5 Verbs I: An Explicitly Scoped Fragment with Verbs

We now move on to the addition of verbs to syllogistic fragments. A natural question to ask would be: what about quantifier scope ambiguity? One way to address this issue would be to add explicit scope information to sentences. This is only needed in sentences with both universal and existential quantifiers. We can add explicit information either to the whole sentence or just to the V; the choice at this level is immaterial.

Our system is given in Figure 10 It is inspired by system in Nishihara, Morita, and Iwata [12]. However, we made an important change: we use an explicit scoping. The original treatment uses unambiguous sentences with a convention that existential quantifiers have wide scope. And then to get the inverse scope readings, one uses negation. So their system requires more boolean operations in the first place, and also the syntax does not look to us to be as natural as the one below.

We shall indicate the scoping on the outside of a sentence. For example, we shall write

$$(\textit{All X love some Y})_{\mathsf{ows}} \tag{15}$$

to mean that we intend the object wide scope reading. so this is what the original NMI system writes as *All X love some Y*, and which standard logic writes as

$$(\exists y)(Y(y) \wedge (\forall x)(X(x) \rightarrow L(x, y)),$$

or even as $(\exists y \in Y)(\forall x \in X)L(x, y)$. For the subject wide scope reading, we use sws. We also adopt a convention that the scope markings are only used in cases with two different quantifiers. In sentences with two universal or existential sentences, the different scope readings are logically equivalent, so there is no point of using the extra notation.

Since we have the two scope readings explicitly present, there is no need to also include a sentential negation operation. This simplifies the system quite a bit. However, there are

$$\frac{(\textit{All } X \ V \ NP)_{\mathsf{sws}} \quad \textit{All } Y \ \textit{are } X}{(\textit{All } Y \ V \ NP)_{\mathsf{sws}}} \qquad \frac{(\textit{All } X \ V \ NP)_{\mathsf{ows}} \quad \textit{All } Y \ \textit{are } X}{(\textit{All } Y \ V \ NP)_{\mathsf{ows}}}$$

$$\frac{(NP \ V \ \textit{all } X)_{\mathsf{sws}} \quad \textit{All } Y \ \textit{are } X}{(NP \ V \ \textit{all } Y)_{\mathsf{sws}}} \qquad \frac{(NP \ V \ \textit{all } X)_{\mathsf{ows}} \quad \textit{All } Y \ \textit{are } X}{(NP \ V \ \textit{all } Y)_{\mathsf{ows}}}$$

$$\frac{(\textit{Some } X \ V \ NP)_{\mathsf{sws}} \quad \textit{All } X \ \textit{are } Y}{(\textit{Some } Y \ V \ NP)_{\mathsf{sws}}} \qquad \frac{(\textit{Some } X \ V \ NP)_{\mathsf{ows}} \quad \textit{All } X \ \textit{are } Y}{(\textit{Some } Y \ V \ NP)_{\mathsf{ows}}}$$

$$\frac{(NP \ V \ \textit{some } X)_{\mathsf{sws}} \quad \textit{All } X \ \textit{are } Y}{(NP \ V \ \textit{some } Y)_{\mathsf{sws}}} \qquad \frac{(NP \ V \ \textit{some } X)_{\mathsf{ows}} \quad \textit{All } X \ \textit{are } Y}{(NP \ V \ \textit{some } Y)_{\mathsf{ows}}}$$

$$\frac{(\textit{All } X \ V \ NP)_{\mathsf{sws}} \quad \textit{Some } X \ \textit{is a } Y}{(\textit{Some } Y \ V \ NP)_{\mathsf{sws}}} \qquad \frac{(NP \ V \ \textit{all } X)_{\mathsf{ows}} \quad \textit{Some } X \ \textit{are } Y}{(NP \ V \ \textit{some } Y)_{\mathsf{ows}}}$$

$$\frac{(\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{sws}}}{(\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{ows}}} \qquad \frac{(\textit{All } X \ V \ \textit{some } Y)_{\mathsf{ows}}}{(\textit{All } X \ V \ \textit{some } Y)_{\mathsf{sws}}}$$

$$\frac{(\textit{Some } X \ V \ NP)_{\mathsf{sws}}}{\textit{Some } X \ \textit{is an } X} \qquad \frac{(NP \ V \ \textit{some } X)_{\mathsf{ows}}}{\textit{Some } X \ \textit{is an } X}$$

$$\frac{\textit{All } X \ V \ \textit{all } X}{(\textit{All } X \ V \ \textit{some } X)_{\mathsf{sws}}}$$

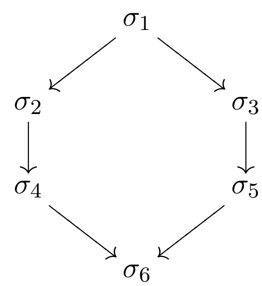
$$\frac{(\textit{All } X \ V \ \textit{some } X)_{\mathsf{sws}}}{(\textit{Some } X \ V \ \textit{all } X)_{\mathsf{ows}}} \qquad \frac{(\textit{Some } X \ V \ \textit{all } X)_{\mathsf{sws}}}{(\textit{All } X \ V \ \textit{some } X)_{\mathsf{ows}}}$$

Figure 10: Rules for a scoped formulation in the spirit of Nishihara, Morita, and Iwata [12]. We omitted he rules we have seen for *All* and *Some*. The scope markings are only needed for sentences which are ambiguous. In the unambiguous cases, the rules are simpler. The double-lined rules at the bottom go both ways, so each is really two rules.

still eight different types of syntactic expressions, and so proofs about the system must involve lengthy case-by-case work.

Our proof system for this language is given in Figure 10. It should be clear that these rules are schematic in several ways. First, V can be any verb. It is fine to have more than one verb in the fragment, but the rules don't allow for any conclusions to be drawn that change verbs in a sentence. (In more linguistic terms, since we have no relative clauses, there is no way to make an argument with multiple verbs.) In addition, the NP's in the rules can be any NP's in the language, but in each rule one must use the same NP in the premise and the conclusion, of course.

σ_1	*All U V all W*	$\forall u \forall w$
σ_2	$(\textit{Some } U\ V\ \textit{all } W)_{\mathsf{sws}}$	$\exists u \forall w$
σ_3	$(\textit{All } U\ V\ \textit{some } W)_{\mathsf{ows}}$	$\exists w \forall u$
σ_4	$(\textit{Some } U\ V\ \textit{all } W)_{\mathsf{ows}}$	$\forall w \exists u$
σ_5	$(\textit{All } U\ V\ \textit{some } W)_{\mathsf{sws}}$	$\forall u \exists w$
σ_6	*Some U V some W*	$\exists u \exists w$



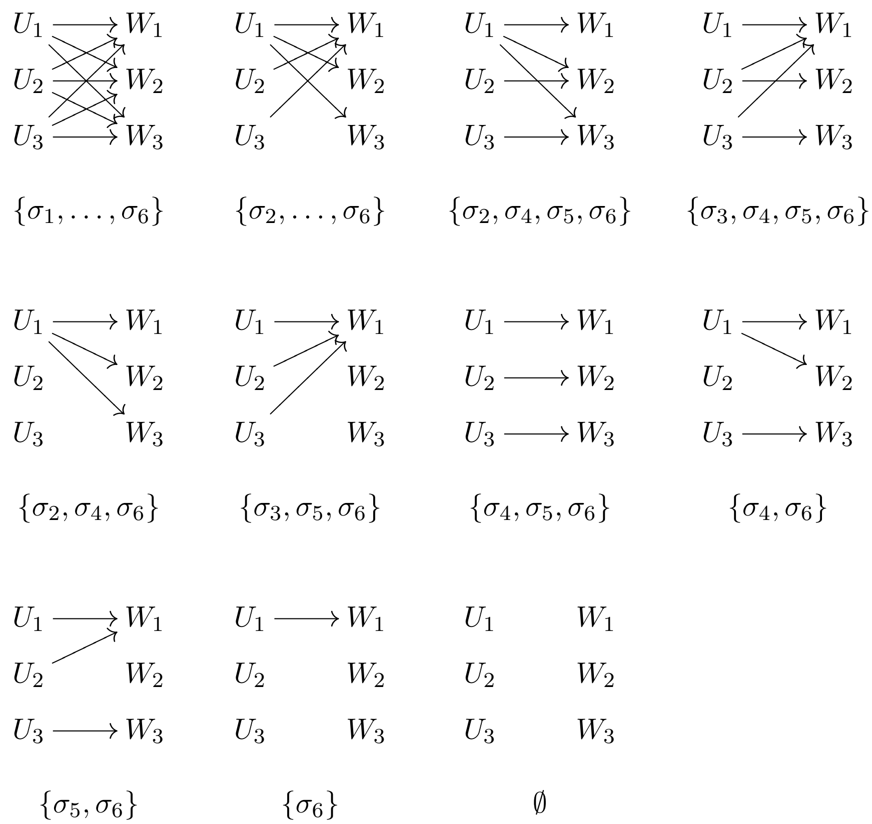


Figure 11: At the top are the six sentences in fixed variables U and W, and some mnemonics for their first-order translations. (These are only useful when U and W are distinct.) The hexagon shows the semantic implication relations among these sentences. Below these are the closed sets and the definition of special relations used in the completeness proof, again, only when U and W are distinct. When $U = W$, the closed sets are $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$, $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$, $\{\sigma_4, \sigma_5, \sigma_6\}$, $\{\sigma_6\}$, and $\emptyset$. We use the same relations in these cases.

5.1 Completeness

To simplify our notation, we abbreviate our sentences as σ_1, …, σ_6 as defined in Figure 11. Actually, we should write these as $\sigma_{i,U,W}$; we only do this when we need to. We also note the diagram of implications above. Each arrow is provable in our logic, and assuming that $U \neq W$, no other arrows are sound. We continue to assume that $U \neq W$ for the time being; later we will consider the opposite case. We consider subsets of $\{\sigma_1, \ldots, \sigma_6\}$. There are eleven subsets which are implication-closed; in pictures, these are the downward closed subsets of the hexagonal diagram. We call them *closed* sets in the sequel. For example, for each U and W, we have a closed set

$$Th_{\Gamma,V}(U,W) \quad = \quad \{\sigma_{i,U,W} : \Gamma \vdash \sigma_{i,U,W}\}. \tag{16}$$

(As the notation indicates, Γ and V are silent partners.) We list all closed sets s in the figure, along with particular relations $R_{U,W,s}$ that will be used in our work. These relations will be subsets of the fixed set

$$A_{U,W} \quad = \quad \{U_1, U_2, U_3\} \times \{W_1, W_2, W_3\}$$

In case $U = W$, $\sigma_{2,U,U}$ and $\sigma_{3,U,U}$ are inter-derivable, as are $\sigma_{4,U,U}$ and $\sigma_{5,U,U}$. So in this case we have only five closed sets. We drop the subscripts U and list the sets s in the bottom of Figure 11, along with our definitions of the subsets $R_{U,U,s} \subseteq A_{U,U}$.

Proposition 5.1 *Let U and W be variables, either the same or different.*

1. *The closed subsets of $\{\sigma_{1,U,W}, \ldots, \sigma_{6,U,W}\}$ are exactly the sets listed in Figure 11.*
2. *For all closed s, and all $1 \leq i \leq 6$,*

$$(A_{U,W}, R_{U,W,s}) \models \sigma_i \quad \textit{iff} \quad \sigma_i \in s.$$

3. *If $(A_{U,W}, R_{U,W,s}) \models \sigma_2$, then for all j, $(U_1, W_j) \in R_{U,W,s}$.*
4. *If $(A_{U,W}, R_{U,W,s}) \models \sigma_3$, then for all i, $(U_i, W_1) \in R_{U,W,s}$.*
5. *If $(U_3, W_2) \in R_{U,W,s}$ or $(U_2, W_3) \in R_{U,W,s}$, then $(A_{U,W}, R_{U,W,s}) \models \sigma_1$.*
6. *If $(U_2, W_j) \in R_{U,W,s}$ for some j, then $(A_{U,W}, R_{U,W,s}) \models \sigma_5$.*
7. *If $(U_i, W_2) \in R_{U,W,s}$ for some i, then $(A_{U,W}, R_{U,W,s}) \models \sigma_4$.*

Proof All of the parts are verified by direct inspection. ⊣

Theorem 5.2 *The logic consisting of the* All *and* Some *rules from Figure* **??** *together with the rules in Figure 10 is complete for* $\mathcal{L}(\text{all}, \text{some}, \textit{verbs})$.

Proof Suppose $\Gamma \models S$. We show that $\Gamma \vdash S$. We first deal with the case that $S \in \mathcal{L}(\textit{All}, \textit{Some})$.

Let

$$\Delta \quad = \quad \{T \in \mathcal{L}(All, Some) : \Gamma \vdash T\}.$$

We show that $\Delta \models S$ in $\mathcal{L}(All, Some)$. For this, let $\mathcal{M} \models \Delta$. We may assume that if $[\![Z]\!] \neq \emptyset$, then $\Gamma \vdash \exists Z$. (Otherwise, re-set $[\![Z]\!]$ to be empty, and check that this does not change the truth values in $\mathcal{L}(All, Some)$.) Turn $\mathcal{M}$ into a structure $\mathcal{M}^+$ for $\mathcal{L}(all, some, \text{verbs})$ by $[\![V]\!] = M \times M$; i.e., by relating every point to every point under all verbs. We shall check that $\mathcal{M}^+ \models \Gamma$. It then follows that $\mathcal{M}^+ \models S$. And since the set variables are interpreted the same way on the two models, we see that $\mathcal{M} \models S$. Now to check that $\mathcal{M}^+ \models \Gamma$, we argue by cases. For example, suppose that Γ contains the sentence in (15). One of our axioms implies directly that $\Gamma \vdash T$, where T is *Some Y is a Y*. Thus $T \in \Delta$. So $[\![Y]\!] \neq \emptyset$ in $\mathcal{M}$. Then the structure of $\mathcal{M}^+$ tells us that this model indeed satisfies (15). Similar arguments apply to sentences of forms different than that of (15). We'll look at one more case, the subject wide scope version of (15). We may assume that $[\![X]\!] \neq \emptyset$. Recall from above that we may assume that $\Gamma \vdash$ *Some X is an X*. And now we have the following Γ-deduction:

$$\dfrac{\dfrac{(All\ X\ love\ some\ Y)_{\mathsf{sws}} \qquad \begin{matrix}\vdots \\ Some\ X\ is\ an\ X\end{matrix}}{Some\ X\ loves\ some\ Y}}{Some\ Y\ is\ a\ Y}$$

The rest of the argument is similar. We now know that $\Delta \models S$. We use Theorem 2.10 to see that $\Delta \vdash S$. A fortiori, $\Gamma \vdash S$.

The main work We assume that S is a scoped sentence $(Q_1X)\ V\ (Q_2Y)$. Let

$$\Delta \quad = \quad \Gamma \cup \{\exists X : Q_1 = \text{all}\} \cup \{\exists Y : Q_2 = \text{all}\}$$

So we add existential statements to Γ. The point is that we aim to construct a model $\mathcal{M} = \mathcal{M}(\Gamma, S)$ in which X and Y are interpreted by *non-empty* sets. Continuing, we let $\mathcal{M}$ be the following model:

$$M \quad = \quad \{U_1, U_2, U_3 : \Delta \vdash \exists U\} \cup \{\{A, B\} : A \neq B \text{ and } \Gamma \vdash Some\ A\ are\ B\}$$

So we have three copies of the variables whose existence follows from Δ together with some other unordered pairs. The only use of subscripts in the rest of this proof is to refer to these copied points. The purpose of $\{A, B\}$ will be to insure that *Some A are B* will hold in our $\mathcal{M}$. (Incidentally, if Γ has no *Some*-sentences, then the whole argument is much easier. We encourage the reader to either work out a proof of that special case first, or to read what follows first in the simpler case.) The structure of $\mathcal{M}$ is given by

$W_i \in [\![U]\!]$	iff $\Gamma \vdash$ *All W are U*
$\{A, B\} \in [\![U]\!]$	iff $\Gamma \vdash$ *All A are U*, or $\Gamma \vdash$ *All B are U*
$U_i [\![V]\!] W_j$	iff $(U_i, W_j) \in R_{U,W,Th(U,W)}$
$\{U, Z\} [\![V]\!] W_2$	iff $\Gamma \vdash \sigma_{1,U,W}$ or $\Gamma \vdash \sigma_{1,Z,W}$
$\{U, Z\} [\![V]\!] W_1$	iff $\{U, Z\} [\![V]\!] W_2$, or $\Gamma \vdash \sigma_{3,U,W}$, or $\Gamma \vdash \sigma_{3,Z,W}$
$\{U, Z\} [\![V]\!] W_3$	iff $\{U, Z\} [\![V]\!] W_2$, or $\Gamma \vdash \sigma_{5,U,W}$ or $\Gamma \vdash \sigma_{5,Z,W}$
$U_2 [\![V]\!] \{W, Z\}$	iff $\Gamma \vdash \sigma_{1,U,W}$ or $\Gamma \vdash \sigma_{1,U,Z}$
$U_1 [\![V]\!] \{W, Z\}$	iff $U_2 [\![V]\!] \{W, Z\}$, or $\Gamma \vdash \sigma_{3,U,W}$, or $\Gamma \vdash \sigma_{2,U,Z}$
$U_3 [\![V]\!] \{W, Z\}$	iff $U_2 [\![V]\!] \{W, Z\}$, or $\Gamma \vdash \sigma_{4,U,W}$, or $\Gamma \vdash \sigma_{4,Z,W}$
$\{A, B\} [\![V]\!] \{C, D\}$	iff $\Gamma \vdash \sigma_{1,A,C}$, or $\Gamma \vdash \sigma_{1,A,D}$, or $\Gamma \vdash \sigma_{1,B,C}$, or $\Gamma \vdash \sigma_{1,B,D}$

(See (16) for $Th(U, W)$ and Figure 11 for the relation $R_{U,W,s}$.)

We first claim that $\mathcal{M} \models \Gamma$. It is easy to check this for sentences in $\mathcal{L}$(all, some): for *All* sentences, this is a routine monotonicity point, and for *Some* sentences this comes from the elements $\{A, B\}$.

Moving on, consider a sentence in Γ such as $\sigma_{3,U,W}$ from above, $(\textit{All } U \ V \ \textit{some } W)_{\mathsf{ows}}$. Hence $\Gamma \vdash \exists W$. We consider $W_1 \in M$, and we claim that for all $Z_i \in [\![U]\!]$, $Z_i [\![V]\!] W_1$; also, for all $\{A, B\} \in [\![U]\!]$, $\{A, B\} [\![V]\!] W_1$. Here is the reason for the first assertion. Let $Z_i \in [\![U]\!]$, so $\Gamma \vdash$ *All Z are U*. Using our logic, we see that $\Gamma \vdash \sigma_{3,Z,W}$. Then $[\![V]\!]_{Z,W} = R_{U,Z,s}$ for some set s containing the sentence $\sigma_{3,Z,W}$. So Proposition 5.1 implies that $(Z_i, W_1) \in [\![V]\!]$. Next, we turn to the second part of our claim. Let $\{A, B\} \in [\![U]\!]$. We may assume that $\Gamma \vdash$ *All A are U*. As before, we have $\Gamma \vdash \sigma_{3,A,W}$. And then we see that $\{A, B\} [\![V]\!] W_1$.

We omit the rest of the similar verifications showing that $\mathcal{M} \models \Gamma$. We conclude that $\mathcal{M} \models S$, where S is the statement in our theorem. The point now is to use this information to read off that $\Gamma \vdash S$. We argue by cases on S. Perhaps the most interesting case is when X is $\sigma_{4,X,Y} = (\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{ows}}$.

The easiest case is when S is $\sigma_{1,X,Y}$, *All X V all Y*. Being a universal sentence, S is preserved under submodels. $(A_{X,Y}, R_{X,Y,Th(X,Y)})$ is a submodel of $\mathcal{M}$, and so it satisfies $\sigma_{1,X,Y}$. By Proposition 5.1, part 2, $\sigma_{1,X,Y} \in Th(X, Y)$. This is what we want.

More interesting is the case that S is $\sigma_{2,X,Y} = (\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{sws}}$. If the witness to the wide-scope existential quantifier belongs to $\{X_1, X_2, X_3\}$, we are easily done. So we only worry about the case that the witness is of the form $\{A, B\}$. Since we have $\{A, B\} [\![V]\!] Y_2$, we either have $\sigma_{1,A,Y}$ from Γ, or $\sigma_{1,B,Y}$. Let us assume that it is $\sigma_{1,A,Y}$, *All A V all Y*. We also have *All A are X*, and *Some A is an A*, so we have *Some X is an A*. But then we easily get from this and $\sigma_{1,A,Y}$ that $(\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{sws}}$.

The case of $\sigma_{3,X,Y}$ is similar to that of $\sigma_{2,X,Y}$. Furthermore, $\sigma_{5,X,Y}$ is similar to $\sigma_{4,X,Y}$.

Consider next $\sigma_{4,X,Y} = (\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{ows}}$. Looking at Y_2, suppose that there is some $\{A, B\} \in [\![X]\!]$ such that $\{A, B\} [\![V]\!] Y_2$. (We shall later consider the other case, when there is some $Z_i \in [\![X]\!]$ such that $Z_i [\![V]\!] Y_2$.) Recall the semantics of V in our model. We may assume that $\Gamma \vdash \sigma_{1,A,Y}$. Working under Γ, we have *Some A is an X*. We next see that $(\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{sws}}$ and hence $(\textit{Some } X \ V \ \textit{all } Y)_{\mathsf{ows}}$. This is our goal in this paragraph. So we assume that there are no $\{A, B\} \in [\![X]\!]$ such that $\{A, B\} [\![V]\!] Y_2$. That is, the element of $[\![X]\!]$ related by $[\![V]\!]$ to Y_2 is one of $\{W_1, W_2, W_3\}$ for some W such that $\Gamma \vdash$ *All W are X*. For some i, $(W_i, Y_2) \in R_{W,Y,Th(W,Y)}$. By Proposition 5.1, parts 7 and 2, $\sigma_{4,W,Y} \in Th(W,Y)$. That is, $\Gamma \vdash \sigma_{4,W,Y}$. By monotonicity, $\Gamma \vdash \sigma_{4,X,Y}$.

We conclude with the case that S is $\sigma_{6,X,Y}$, *Some X V some Y*. If $[\![V]\!]$ contains any pair (U_i, W_j) whatsoever, then we are done easily by Proposition 5.1. We also are done easily in case $[\![V]\!]$ contains a pair such as $(\{A,B\},\{C,D\})$. There would be four subcases here, and we go into details on only one of them: suppose $\{A,B\}[\![V]\!]\{C,D\}$. Without loss of generality, $\Gamma \vdash \sigma_{1,A,C}$. Since Γ also derives *Some A exists*, *All A are X*, *Some B exists*, *All B are Y*, we easily get the desired $\sigma_{6,X,Y}$. The reasoning is similar when $[\![V]\!]$ contains a pair such as $(\{A,B\},W_j)$ or one such as $(U_i,\{A,B\})$.

This concludes the proof of Theorem 5.2. $\dashv$

6 Verbs II: Fragments of the McAllester-Givan Type

McAllester and Givan in [8] study a fragment which we'll call $\mathcal{L}_{MG}(\mathit{all}, \mathit{some})$. It begins with variables X, Y, etc., and also *verbs* V, W. The fragment then has *class expressions* c, d, etc., of the following forms:

1. X, Y, Z, ...
2. V *all* c
3. V *some* c

Note that we have recursion, so we have class expressions like

$$R\ \mathit{all}(S\ \mathit{some}(T\ \mathit{all}\ c))$$

We might use this in the symbolization of a predicate like

recognizes everyone who sees someone who trades all umbrellas.

$\mathcal{L}_{MG}(\mathit{all}, \mathit{some})$ is our first infinite fragment. We also have formulas of the form *all c d* and *some c d*. In these, c and d are class expressions. The original paper also uses boolean combinations and proper names; we shall not do so in this section.

We write $\exists c$ for *some c c*. As a matter of fact, we shall be interested not only in $\mathcal{L}_{MG}(\mathit{all}, \mathit{some})$ but also in the smaller fragment $\mathcal{L}_{MG}(\mathit{all}, \exists)$ in which all of the *some* sentences are in fact $\exists$ sentences.

The semantics interprets variables by subsets of an underlying model $\mathcal{M}$, just as we have been doing. It also interprets a verb like V by a binary relation $[\![V]\!] \subseteq M^2$. Then

$$[\![V \text{ all } c]\!] \quad = \quad \{x \in M : \text{for all } y \in [\![c]\!],\ [\![V]\!](x,y).\}$$

We sometimes write relations in infix notation, writing $x[\![V]\!]y$ for $[\![V]\!](x,y)$.

The main technical result in [8] is that the satisfiability problem for $\mathcal{L}_{MG}(\mathit{all}, \mathit{some})$ is NP-complete. We are not so concerned in this paper with complexity results but rather with logical completeness results. However, some of the steps are the same, and our treatment owes a lot to McAllester and Givan [8].

$$\frac{all\ c\ d}{all\ (V\ \ all\ d)\ (V\ \ all\ c)}$$

Figure 12: The rule for $\mathcal{L}_{MG}(all)$, in addition to the two rules for *all* from Figure 2.

6.1 Completeness for $\mathcal{L}_{MG}(\boldsymbol{all})$

In this section, we study the fragment $\mathcal{L}_{MG}(all)$ obtained from the variables and verbs by the constructions $V\ \ all\ c$ for the class expressions, and all $c\ d$ for the sentences. Our logical system uses the two *All*-rules and the one extra rule in Figure 12.

A model construction for $\mathcal{L}_{MG}(\boldsymbol{all})$ Let $\mathcal{S}$ be a set of class expressions closed under sub-class-expressions. We make a model $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$ as follows.

$$\begin{array}{lcl} M & = & \mathcal{S} \\ [\![X]\!] & = & \{c \in M : \Gamma \vdash all\ c\ X\} \\ [\![V]\!] & = & \{(d,c) \in M \times M : \Gamma \vdash all\ d\ (V\ \ all\ c)\} \end{array}$$

Lemma 6.1 *For all $c \in \mathcal{S}$, $[\![c]\!] = \{d \in M : \Gamma \vdash all\ d\ c\}$.*

Proof By induction on c. The base case being immediate, we assume our lemma for c and then consider a class expression in $\mathcal{S}$ of the form $V\ \ all\ c$. Since $\mathcal{S}$ is closed under sub-class-expressions, $c \in \mathcal{S}$ and the statement of our lemma applies to it.

Let $d \in [\![V \text{ all } c]\!]$. The induction hypothesis and reflexivity imply that $c \in [\![d]\!]$. So we have $d[\![V]\!]c$, and thus $\Gamma \vdash all\ d\ (V\ \ all\ c)$.

Conversely, suppose that $\Gamma \vdash all\ d\ (V\ \ all\ c)$. We claim that $d \in [\![V \text{ all } c]\!]$. For this, let $c' \in [\![c]\!]$ so that $\Gamma \vdash all\ c'\ c$. We have the following derivation from Γ:

$$\frac{\begin{array}{c}\vdots\\ all\ d\ (\text{V}\ \ all\ c)\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ all\ c'\ c\end{array}}{all\ (V\ \text{all}\ c)\ (V\ \text{all}\ c')}}{all\ d\ (\text{V}\ \ all\ c')}$$

We see that $d[\![V]\!]c'$. This for all $c' \in [\![c]\!]$ shows that $d \in [\![V \text{ all } c]\!]$. ⊣

Lemma 6.2 *Let $\mathcal{S}$ include the sub-class-expressions of a sentence $\varphi \in \mathcal{L}_{MG}(\text{all})$. Then $\mathcal{M}(\Gamma, \mathcal{S}) \models \varphi$ iff $\Gamma \vdash \varphi$.*

Proof Suppose that $\Gamma \vdash all\ c\ d$. Consider $\mathcal{M}(\Gamma, \mathcal{S})$. By transitivity and Lemma 6.1, $[\![c]\!] \subseteq [\![d]\!]$. So $\mathcal{M} \models all\ c\ d$. For the converse, assume that $[\![c]\!] \subseteq [\![d]\!]$. Then $c \in [\![c]\!] \subseteq [\![d]\!]$. So by Lemma 6.4, $\Gamma \vdash all\ c\ d$, just as desired. ⊣

Theorem 6.3 *The logic containing the* all *rules in Figure 2 and the rules in Figure 12 is complete for $\mathcal{L}_{MG}(\text{all})$: $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.*

Proof Let $\mathcal{S}$ be the set of sub-class-expressions of all sentences in $\Gamma \cup \{\varphi\}$. Consider $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$. By Lemma 6.2, $\mathcal{M} \models \Gamma$. So $\mathcal{M} \models \varphi$. And thus by Lemma 6.2 again, $\Gamma \vdash \varphi$. ⊣

$$\frac{\textit{all } c\ d}{\textit{all } (V\ \textit{all } d)\ (V\ \textit{all } c)} \qquad \frac{\textit{all } c\ d}{\textit{all } (V\ \textit{some } c)\ (V\ \textit{some } d)}$$

$$\frac{\textit{some } c\ d}{\textit{all } (V\ \textit{all } c)\ (V\ \textit{some } d)} \qquad \frac{\exists (V\ \textit{some } c)}{\exists c}$$

$$\frac{\Gamma \cup \{\exists c\} \vdash \varphi \quad \Gamma \cup \{\textit{all } c\ d : d \in \mathcal{L}\} \cup \{\textit{all } d\ (V \text{ all } c) : d, V \in \mathcal{L}\} \vdash \varphi}{\Gamma \vdash \varphi}$$

Figure 13: Rules for $\mathcal{L}_{MG}(\textit{all}, \textit{some})$, our version of the McAllester-Givan fragment. We also use rules for *all* and *some*. The inference rule at the bottom is called the Cases Rule. We write $\Gamma \vdash \varphi$ if $\Gamma \vdash \varphi$ without the Cases Rule, and $\Gamma \vdash^* \varphi$ if the Cases Rule is used. Finally, we are also interested in the fragment $\mathcal{L}_{MG}(\textit{all}, \exists)$ which uses just *all* and $\exists$, and for this one of the rules needs to be modified.

6.2 The Cases Rule

Our logical system for $\mathcal{L}_{MG}(\textit{all}, \textit{some})$ is presented in Figure 13. The system is a sequent calculus. One should read the first four rules as saying, for example,

$$\frac{\Gamma \vdash \exists c}{\Gamma \vdash \textit{all } (V \text{ all } c)\ (V \text{ some } c)}$$

The soundness of these first four rules is easy. Then last rule allows for a case-by-case analysis on whether the interpretation of a class expression is empty or not.

As an example of how this rule is used, we show that

$$\exists d, \textit{all } (V\ \textit{all } c)\ c \vdash \exists c$$

Before we give the formal derivation, here is the informal semantic argument. Take any model $\mathcal{M}$ of the hypotheses. If $[\![c]\!]$ is non-empty, we are done. Otherwise, let $x \in [\![d]\!]$ by the first hypothesis. Then (vacuously) we have $[\![V]\!](x, y)$ for all $y \in [\![c]\!]$. So by our second hypothesis, $x \in [\![c]\!]$.

We also have the following consequence:

If $\Gamma \vdash$ *every (R every s) t*, and also $\Gamma +$ *some c t* $\vdash \exists s$, then $\Gamma + \exists c \vdash \exists s$.

6.3 Completeness for $\mathcal{L}_{MG}(\textbf{\textit{all}}, \exists)$

Before we obtain completeness for the full fragment $\mathcal{L}_{MG}(\textit{all}, \textit{some})$, but we find it easier to study the smaller fragment $\mathcal{L}_{MG}(\textit{all}, \exists)$.

The logical system contains the *All* rules that we have seen, the existence rule

$$\frac{\exists\ Y \quad \textit{All } Y \textit{ are } X}{\exists\ X}$$

and the rules of Figure 13, except that we change

$$\frac{\textit{some } c\ d}{\textit{all } (V\ \textit{all } c)\ (V\ \textit{some } d)} \qquad \frac{\exists c}{\textit{all } (V\ \textit{all } c)\ (V\ \textit{some } c)}$$

the rule on the left above to the one on the right by weakening both the hypothesis and the conclusion. The new rule is sound. We make the change because the hypothesis of the rule on the left goes beyond $\mathcal{L}_{MG}(all, \exists)$ when $c \neq d$.

Definition Let $\mathcal{S}$ be a set of class expressions closed under subexpressions. We say that Γ *determines existentials in* $\mathcal{S}$ if for all $c \in \mathcal{S}$, either (1) or (2) below holds:

1. $\Gamma \vdash \exists c$.

2. For all $d \in \mathcal{S}$ and all V, $\Gamma \vdash$ *All d* $(V\ \ all\ c)$; and also $\Gamma \vdash$ *all c d*.

For example, the set of sentences true in any model determines existentials in the set of all class expressions.

A model construction for $\mathcal{L}_{MG}(\textbf{\textit{all}}, \exists)$ Let $\mathcal{S}$ be a set of class expressions closed under sub-class-expressions, and let $\Gamma \subseteq \mathcal{L}_{MG}(all, \exists)$ determine existentials in $\mathcal{S}$. We make a model $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$ as follows.

$$\begin{array}{lcl} M & = & \{c \in \mathcal{S} : \Gamma \vdash \exists c\} \times \{\forall, \exists\} \\ [\![X]\!] & = & \{(c, Q) \in M : \Gamma \vdash all\ c\ X\} \\ [\![V]\!] & = & \{((c, Q), (d, Q')) \in M \times M : \Gamma \vdash all\ c\ (V\ \ Q'\ \ d)\} \end{array}$$

Lemma 6.4 *For all $c \in \mathcal{S}$, $[\![c]\!] = \{(d, Q) \in M : \Gamma \vdash all\ d\ c\}$.*

Proof By induction on c. The base case being immediate, we assume our lemma for c and then consider class expressions in $\mathcal{S}$ of the form $V\ \ all\ c$ and $V\ \ some\ c$. The induction hypothesis implies that provided $\Gamma \vdash \exists c$, both $(c, \forall)$ and $(c, \exists)$ belong to $[\![c]\!]$.

Let $(d, Q) \in [\![V \text{ all } c]\!]$. There are two cases, depending on whether $\Gamma \vdash \exists c$ or not. In the first case $(c, \forall) \in [\![c]\!]$. So $(d, Q)[\![V]\!](c, \forall)$. And we see that $\Gamma \vdash all\ d\ (V\ \ all\ c)$, as desired. In the second case, we trivially have the same conclusion $\Gamma \vdash all\ d\ (V\ \ all\ c)$. (This is where the condition that Γ determines existentials enters into this lemma; it also is used in Lemma 6.5 below.) And so we are done then also.

Conversely, suppose that $\Gamma \vdash all\ d\ (V\ \ all\ c)$. We claim that $(d, Q) \in [\![V \text{ all } c]\!]$ for both Q. For this, let $(c', Q') \in [\![c]\!]$ so that $\Gamma \vdash all\ c'\ c$. We'll only work out the case of $Q' = \exists$. Using our various assumptions, we have the following derivation from Γ:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ all\ d\ (V\ \ all\ c)\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ all\ c'\ c\end{array}}{all\ (V\ \text{all}\ c)\ (V\ \text{all}\ c')}}{all\ d\ (V\ \ all\ c')} \quad \dfrac{\begin{array}{c}\vdots\\ \exists c'\end{array}}{all\ (V\ \text{all}\ c')\ (V\ \text{some}\ c')}}{all\ d\ (V\ \ some\ c')}$$

And this means that $(d, Q)[\![V]\!](c', \exists)$. This for all elements of $[\![c]\!]$ shows that $(d, Q) \in [\![V \text{ all } c]\!]$.

The steps for $V\ \ some\ c$ are similar: Let $(d, Q) \in [\![V \text{ some } c]\!]$. Let $(e, Q') \in [\![c]\!]$ be such that $(d, Q)[\![V]\!](e, Q')$. We may assume that $Q' = \forall$, so $\Gamma \vdash all\ d\ (V\ \ all\ e)$. Also, $\Gamma \vdash \exists e$. And by induction hypothesis, $\Gamma \vdash$ *all e c*. We have

$$\dfrac{\begin{array}{c}\vdots\\ \textit{all } d\ (V\ \textit{all } e)\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists e\end{array}}{\textit{all }(V\ \text{all } e)\ (V\ \text{some } e)} \quad \dfrac{\begin{array}{c}\vdots\\ \textit{all } e\ c\end{array}}{\textit{all }(V\ \text{some } e)\ (V\ \text{some } c)}}{\textit{all }(V\ \text{all } e)\ (V\ \text{some } c)}}{\textit{all } d\ (V\ \textit{some } c)}$$

Conversely, suppose that $(d, Q) \in M$ and $\Gamma \vdash$ *all d (V some c)*. Then $\Gamma \vdash \exists d$, and we have $\Gamma \vdash \exists c$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists d\end{array} \quad \begin{array}{c}\vdots\\ \textit{all } d\ (V\ \textit{some } c)\end{array}}{\exists(V\ \textit{some } c)}}{\exists c}$$

This goes to show that $(c, \exists) \in M$. Then $(d, Q)[\![V]\!](c, \exists)$. So $(d, Q) \in [\![V \text{ some } c]\!]$. $\dashv$

Lemma 6.5 *Let $\mathcal{S}$ include the set of sub-class-expressions of a sentence $\varphi \in \mathcal{L}_{MG}(\text{all}, \exists)$. Suppose that Γ determines existentials in $\mathcal{S}$. Then $\mathcal{M}(\Gamma, \mathcal{S}) \models \varphi$ iff $\Gamma \vdash \varphi$.*

Proof First let φ be $\exists c$. Consider $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$. If $\Gamma \vdash \varphi$, then $(c, \exists) \in M$. Indeed, $(c, \exists) \in [\![c]\!]$, by Lemma 6.4. (We note that the lemma applies, since $\mathcal{S}$ is closed under sub-class-expressions.) And so we see that $\mathcal{M} \models \exists c$. Conversely, suppose that $\mathcal{M} \models \exists c$. Let $(Q, d) \in [\![c]\!]$. Then by Lemma 6.4, $\Gamma \vdash$ *all d c*. And as $\Gamma \vdash \exists d$, we also see that $\Gamma \vdash \exists c$, as desired.

Next, let φ be *all c d*. Suppose that $\Gamma \vdash \varphi$. By a routine monotonicity calculation and Lemma 6.4, $[\![c]\!] \subseteq [\![d]\!]$. For the converse, assume $[\![c]\!] \subseteq [\![d]\!]$. We have two cases, depending on whether $\Gamma \vdash \exists c$, or not. In the first case, $(c, \forall) \in [\![c]\!] \subseteq [\![d]\!]$. So by Lemma 6.4, $\Gamma \vdash$ *all c d*, just as desired. In the second case, the condition that Γ determines existentials in $\mathcal{S}$ tells us directly that $\Gamma \vdash$ *all c d*. $\dashv$

Theorem 6.6 *The logic described above is complete for $\mathcal{L}_{MG}(\text{all}, \exists)$: $\Gamma \models \varphi$ iff $\Gamma \vdash^* \varphi$.*

Proof Suppose that $\Gamma \models \varphi$. Since the fragment is a sublanguage of first-order logic, if is compact. So we may assume that Γ is finite. Let $\mathcal{S}$ be the set of class expressions which occur in either Γ or φ. These will be fixed throughout this proof.

For any set $\Delta \subseteq \mathcal{L}_{MG}(\textit{all}, \exists)$, Let $n(\Delta, \mathcal{S})$ be the number of class expressions $c \in \mathcal{S}$ such that both of the following hold:

1. $\Delta \not\vdash \exists c$,

2. For some $d \in \mathcal{S}$ and V, $\Delta \not\vdash$ *All d* $(V$ *all* $c)$; or else for some $d \in \mathcal{S}$, $\Delta \not\vdash$ *all c d*.

This number $n(\Delta, \mathcal{S})$ measures how far Δ is from determining existentials in $\mathcal{S}$. We show by induction on the number k that for all $\Delta \supseteq \Gamma$ such that $n(\Delta, \mathcal{S}) = k$, $\Delta \vdash \varphi$. Applying this to the original Γ with with $k = n(\Gamma, \mathcal{S})$, we see that $\Gamma \vdash \varphi$, as required.

If $k = 0$, then $n(\Delta, \mathcal{S}) = 0$ and so Δ determines existentials in all sub-class-expressions of all $\psi \in \Gamma$. By Lemma 6.5, $\mathcal{M}(\Delta, \mathcal{S}) \models \Gamma$. So since we are assuming that $\Gamma \models \varphi$, we see that $\mathcal{M}(\Delta, \mathcal{S}) \models \varphi$. And then by Lemma 6.5 again, we have $\Delta \vdash \varphi$.

Now assume our result for k, and suppose that $n(\Delta, \mathcal{S}) = k+1$. Fix a class expression c with either (1) or (2) above. Consider

$$\begin{array}{lcl} \Delta_1 & = & \Delta \cup \{\exists c\} \\ \Delta_2 & = & \Delta \cup \{All\ d\ (V\ \ all\ c) : d, V\} \cup \{All\ c\ d : d\} \end{array}$$

A fortiori, $\Delta_1 \models \varphi$ and also $\Delta_2 \models \varphi$. Further, $n(\Delta_1) \leq k$, and similarly for Δ_2. By induction hypothesis, $\Delta_1 \vdash^* \varphi$, and similarly for Δ_2. So using the Cases Rule, $\Delta \vdash^* \varphi$. $\dashv$

6.4 Completeness for $\mathcal{L}_{MG}(\boldsymbol{all}, \boldsymbol{some})$

In this section we prove the completeness of our system for $\mathcal{L}_{MG}(all, some)$. Our proof is a modification of the work we saw in Section 6.3 for $\mathcal{L}_{MG}(all, \exists)$. We use the same definition of "determines existentials in $\mathcal{S}$", except that we obviously read $\vdash$ as "proves in the full logic". We employ a different model construction. Lemma 6.4 therefore changes; our new argument is a generalization of the previous one, and so we shall not bother to exhibit proof trees in Lemma 6.7 below. Lemma 6.5 finds an extra step in Lemma 6.8, and our new version of Lemma 6.6 gives the full result as before.

A model construction for $\mathcal{L}_{MG}(\boldsymbol{all}, \boldsymbol{some})$ Let $\mathcal{S}$ be a set of class expressions closed under sub-class-expressions, and let $\Gamma \subseteq \mathcal{L}_{MG}(all, some)$ determine existentials in $\mathcal{S}$. We make a model $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$ as follows.

$$\begin{array}{lcl} M & = & \{(c_1, c_2, Q) \in \mathcal{S} \times \mathcal{S} \times \{\forall, \exists\} : \Gamma \vdash some\ c_1\ c_2\} \\ [\![X]\!] & = & \{(c_1, c_2, Q) \in M : \Gamma \vdash all\ c_1\ X \text{ or } \Gamma \vdash all\ c_2\ X\} \\ (c_1, c_2, Q)[\![V]\!](d_1, d_2, \forall) & \text{iff} & \text{for some } i \text{ and } j,\ \Gamma \vdash all\ c_i\ (V\ \ all\ d_j) \\ (c_1, c_2, Q)[\![V]\!](d_1, d_2, \exists) & \text{iff} & \text{either } (c_1, c_2, Q)[\![V]\!](d_1, d_2, \forall) \\ & & \text{or else for some } i \text{ and } j, \\ & & \Gamma \vdash all\ c_i\ (V\ \ some\ d_j) \text{ and also } \Gamma \vdash all\ d_j\ d_{3-j} \end{array}$$

Lemma 6.7 *For all $c \in \mathcal{S}$, $[\![c]\!] = \{(d_1, d_2, Q) \in M :$ either $\Gamma \vdash all\ d_1\ c$, or $\Gamma \vdash all\ d_2\ c\}$.*

Proof By induction on c. The base case being immediate, we assume our lemma for c and then consider class expressions in $\mathcal{S}$ of the form $V\ \ all\ c$ and $V\ \ some\ c$. The induction hypothesis implies that provided $\Gamma \vdash \exists c$, both $(c, c, \forall)$ and $(c, c, \exists)$ belong to $[\![c]\!]$.

Let $(d_1, d_2, Q) \in [\![V \text{ all } c]\!]$. If $\Gamma \vdash \exists c$, then $(c, c, \forall) \in [\![c]\!]$. By the overall semantics of $\mathcal{L}_{MG}(all, some)$, $(d_1, d_2, Q)[\![V]\!](c, c, \forall)$. And we see that $\Gamma \vdash all\ d_i\ (V\ \ all\ c)$ for some i, as desired. If $\Gamma \not\vdash \exists c$, we trivially have the same conclusion $\Gamma \vdash all\ d_i\ (V\ \ all\ c)$, this time for both i. And so we are done then also.

Conversely, fix i and suppose that $\Gamma \vdash all\ d_i\ (V\ \ all\ c)$. Fix Q; we claim that $(d_1, d_2, Q) \in [\![V \text{ all } c]\!]$. For this, let $(e_1, e_2, Q') \in [\![c]\!]$ so that $\Gamma \vdash all\ e_j\ c$ for some j. Then $\Gamma \vdash all\ d_i\ (V\ \ all\ e_j)$. We conclude that $(d_1, d_2, Q)[\![V]\!](e_1, e_2, \forall)$, and also $(d_1, d_2, Q)[\![V]\!](e_1, e_2, \exists)$. This for all elements of $[\![c]\!]$ shows that $(d_1, d_2, Q) \in [\![V \text{ all } c]\!]$.

Here is the induction step for $V\ some\ c$. Let $(d_1, d_2, Q) \in [\![V \text{ some } c]\!]$. Thus we have $(d_1, d_2, Q)[\![V]\!](e_1, e_2, Q')$ for some $(e_1, e_2, Q') \in [\![c]\!]$. We first consider the case that $Q' = \forall$.

Here there are four subcases. One representative subcase is $\Gamma \vdash$ *all* d_1 (V *all* e_1). We have $\Gamma \vdash$ *some* e_1 e_2, since $(e_1, e_2, Q') \in M$. By induction hypothesis, either $\Gamma \vdash$ *all* e_1 c or else $\Gamma \vdash$ *all* e_2 c. Either way, $\Gamma \vdash$ *all* d_1 (V *some* c).

This concludes the work when $Q' = \forall$. In the other case, $Q' = \exists$. We again have a number of subcases; one is that $\Gamma \vdash$ *all* d_1 (V *some* e_1) and $\Gamma \vdash$ *all* e_1 e_2. And by induction hypothesis, $\Gamma \vdash$ *all* e_1 c or else $\Gamma \vdash$ *all* e_2 c. Either way, we get $\Gamma \vdash$ *all* e_1 c. And further we get the desired conclusion, $\Gamma \vdash$ *all* d_1 (V *some* c). This concludes half of the induction step for V *some* c.

For the other half, let $(d_1, d_2, Q) \in M$ and fix i such that $\Gamma \vdash$ *all* d_i (V *some* c). Then $\Gamma \vdash \exists d_i$, and in a few steps we also have $\Gamma \vdash \exists c$. Therefore $(c, c, \exists) \in M$. By induction hypothesis, $(c, c, \exists) \in [\![c]\!]$. We have $(d_1, d_2, Q)[\![V]\!](c, c, \exists)$ because $\Gamma \vdash$ *all* c. So $(d_1, d_2, Q) \in [\![V \text{ some } c]\!]$. ⊣

Lemma 6.8 *Let $\mathcal{S}$ include the set of sub-class-expressions of a sentence $\varphi \in \mathcal{L}_{MG}(\text{all}, \text{some})$. Suppose that Γ determines existentials in $\mathcal{S}$. Then $\mathcal{M}(\Gamma, \mathcal{S}) \models \varphi$ iff $\Gamma \vdash \varphi$.*

Proof As in Lemma 6.5. The only thing that changes is that φ might be *some* c d. Consider $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$. If $\Gamma \vdash \varphi$, then $(c, d, \exists) \in M$. Indeed, $(c, d, \exists) \in [\![c]\!] \cap [\![d]\!]$, by Lemma 6.7. Conversely, suppose that $\mathcal{M} \models \varphi$. Let $(c', d', Q) \in [\![c]\!] \cap [\![d]\!]$. We use Lemma 6.7 again and reduce to four cases; one of them is $\Gamma \vdash$ *all* c' c, and $\Gamma \vdash$ *all* d' d. And as we also have $\Gamma \vdash$ *some* c' d', we also have $\Gamma \vdash \varphi$ (see Example 2.7). ⊣

Theorem 6.9 *The* All *and* Some *rules in Figure 2 and Figure 4, together with the rules in Figure 13, give a complete logic for $\mathcal{L}_{MG}(\text{all}, \text{some})$: $\Gamma \models \varphi$ iff $\Gamma \vdash^* \varphi$.*

Proof As in Theorem 6.6. ⊣

7 *There are at least as many X as Y*

In our next section, we show that it is possible to have complete syllogistic systems for logics which go are not first-order. We regard this as a proof-of-concept; it would be of interest to get complete systems for richer fragments, such the ones in Pratt-Hartmann [15].

We write $\exists^{\geq}(X, Y)$ for *There are at least as many X as Y*, and we are interested in adding these sentences to our fragments. We are usually interested in sentences in this fragment on *finite* models. We write $|S|$ for the cardinality of the set S. The semantics is that $\mathcal{M} \models \exists^{\geq}(X, Y)$ iff $|[\![X]\!]| \geq |[\![Y]\!]|$ in $\mathcal{M}$.

$\mathcal{L}(\mathit{all}, \exists^{\geq})$ does not have the canonical model property of Section 2.2. We show this via establishing that the semantics is not compact. Consider

$$\Gamma \quad = \quad \{\exists^{\geq}(X_1, X_2), \exists^{\geq}(X_2, X_3), \ldots, \exists^{\geq}(X_n, X_{n+1}), \ldots\}$$

Suppose towards a contradiction that $\mathcal{M}$ were a canonical model for Γ. In particular, $\mathcal{M} \models \Gamma$. Then $|[\![X_1]\!]| \geq |[\![X_2]\!]| \geq \ldots$. For some n, we have $|[\![X_n]\!]| = |[\![X_{n+1}]\!]|$. Thus $\mathcal{M} \models \exists^{\geq}(X_{n+1}, X_n)$. However, this sentence does not follow from Γ.

Remark In the remainder of this section, Γ denotes a *finite* set of sentences.

$$\frac{\textit{All } Y \textit{ are } X}{\exists^{\geq}(X,Y)} \qquad \frac{\exists^{\geq}(X,Y) \quad \exists^{\geq}(Y,Z)}{\exists^{\geq}(X,Z)} \qquad \frac{\textit{All } Y \textit{ are } X \quad \exists^{\geq}(Y,X)}{\textit{All } X \textit{ are } Y}$$

Figure 14: Rules for $\exists^{\geq}(X,Y)$ and *All*.

In this section, we consider $\mathcal{L}(all, \exists^{\geq})$. For proof rules, we take the rules in Figure 14 together with the rules for *All* in Figure 2. The system is sound. The last rule is perhaps the most interesting, and it uses the assumption that our models are finite. That is, if all Y are X, and there are at least as many elements in the bigger set Y as in X, then the sets have to be the same.

We need a little notation at this point. Let Γ be a (finite) set of sentences. We write $X \leq_c Y$ for $\Gamma \vdash \exists^{\geq}(Y,X)$. We also write $X \equiv_c Y$ for $X \leq_c Y \leq_c X$, and $X <_c Y$ for $X \leq_c Y$ but $X \not\equiv_c Y$. We continue to write $X \leq Y$ for $\Gamma \vdash$ *All* X *are* Y. And we write $X \equiv Y$ for $X \leq Y \leq X$.

Proposition 7.1 *Let* $\Gamma \subseteq \mathcal{L}(\text{all}, \exists^{\geq})$ *be a (finite) set. Let* $\mathcal{V}$ *be the set of variables in* Γ.

1. *If* $X \leq Y$, *then* $X \leq_c Y$.
2. $(\mathcal{V}, \leq_c)$ *is a preorder: a reflexive and transitive relation.*
3. *If* $X \leq_c Y \leq X$, *then* $X \leq Y$.
4. *If* $X \leq_c Y$, $X \equiv X'$, *and* $Y \equiv Y'$, *then* $X' \leq_c Y'$.
5. $(\mathcal{V}, \leq_c)$ *is pre-wellfounded: a preorder with no descending sequences in its strict part.*

Proof Part (1) uses the first rule in Figure 14. In part (2), the reflexivity of $\leq_c$ comes from that of $\leq$ and part (1); the transitivity is by the second rule of $\exists^{\geq}$. Part (3) is by the last rule of $\exists^{\geq}$. Part 4 uses part (1) and transitivity. Part 5 is just a summary of the previous parts. $\dashv$

Theorem 7.2 *The logic of Figures 2 and 14 is complete for* $\mathcal{L}(\text{all}, \exists^{\geq})$.

Proof Suppose that $\Gamma \models \exists^{\geq}(Y,X)$. Let $\{*\}$ be any singleton, and define a model $\mathcal{M}$ by taking M to be a singleton $\{*\}$, and

$$[\![Z]\!] \quad = \quad \begin{cases} M & \text{if } \Gamma \vdash \exists^{\geq}(Z,X) \\ \emptyset & \text{otherwise} \end{cases} \tag{17}$$

We claim that if Γ contains $\exists^{\geq}(W,V)$ or *All* V *are* W, then $[\![V]\!] \subseteq [\![W]\!]$. We only verify the second assertion. For this, we may assume that $[\![V]\!] \neq \emptyset$ (otherwise the result is trivial). So $[\![V]\!] = M$. Thus $\Gamma \vdash \exists^{\geq}(V,X)$. So we see that $\Gamma \vdash \exists^{\geq}(W,X)$. From this we conclude that $[\![W]\!] = M$. In particular, $[\![V]\!] \subseteq [\![W]\!]$.

Now our claim implies that $\mathcal{M} \models \Gamma$. Therefore $|[\![X]\!]| \leq |[\![Y]\!]|$. And $[\![X]\!] = M$, since $\Gamma \vdash \exists^{\geq}(X,X)$. Hence $[\![Y]\!] = M$ as well. But this means that $\Gamma \vdash \exists^{\geq}(Y,X)$, just as desired.

We have shown one case of the general completeness theorem that we are after. In the other case, we have $\Gamma \models$ *All* X *are* Y. We construct a model $\mathcal{M} = \mathcal{M}_{\Gamma}$ such that for all A and B,

(α) $[\![A]\!] \subseteq [\![B]\!]$ iff $A \leq B$.

(β) If $A \leq_c B$, then $|[\![A]\!]| \leq |[\![B]\!]|$.

Let $\mathcal{V}/\equiv_c$ be the (finite) set of equivalences classes of variables in Γ under $\equiv_c$. This set is then well-founded by the natural relation induced on it by $\leq_c$. It is then standard that we may list the elements of $\mathcal{V}/\equiv_c$ in some order

$$[U_0], [U_2], \ldots, [U_k]$$

with the property that if $U_i <_c U_j$, then $i < j$. (But if $i < j$, then it might be the case that $U_i \equiv_c U_j$.)

We define by recursion on $i \leq k$ the interpretation $[\![V]\!]$ of all $V \in [U_i]$. Suppose we have $[\![W]\!]$ for all $j < i$ and all $W \equiv_c U_j$. Let

$$\mathcal{X}_i \quad = \quad \bigcup_{j<i, W \equiv_c U_j} [\![W]\!],$$

and note that this is the set of all points used in the semantics of any variable so far. Let $n = 1 + |\mathcal{X}_i|$. For all $V \equiv_c U_i$, we shall arrange that $[\![V]\!]$ be a set of size n.

Now $[U_i]$ is the equivalence class of U_i under $\equiv_c$. It splits into equivalence classes of the finer relation $\equiv$. For a moment, consider one of those finer classes, say $[A]_\equiv$. We must interpret each variable in this class by the same set. For this A, let

$$\mathcal{Y}_A \quad = \quad \bigcup \{[\![B]\!] : (\exists j < i) V_j \equiv_c B \leq A\}.$$

Note that $\mathcal{Y}_A \subseteq \mathcal{X}_i$ so that $|\mathcal{Y}_A| < n$. We set $[\![A]\!]$ to be $\mathcal{Y}_A$ together with $n - |\mathcal{Y}_A|$ fresh elements. Moving on to the other $\equiv$-classes which partition the $\equiv_c$-class of U_i, we do the same thing. We must insure that for $A \not\equiv A'$, the fresh elements added into $[\![A']\!]$ are disjoint from the fresh elements added into $[\![A]\!]$.

This completes the definition of $\mathcal{M}$. We check that so that conditions (α) and (β) are satisfied. It is easy to first check that for $i < j$, $|[\![U_i]\!]| < |[\![U_j]\!]|$. It might also be worth noting that $[\![U_0]\!] \neq \emptyset$, so no $[\![A]\!]$ is empty.

For (β), let $A \leq_c B$. Let i and j be such that $A \equiv_c U_i$ and $B \equiv_c U_j$. If $A \equiv_c B$, then $i = j$ and the construction arranged that $[\![A]\!]$ and $[\![B]\!]$ be sets of the same cardinality. If $A <_c B$, then $i < j$ by the way we enumerated the U's, and so $|[\![A]\!]| = |[\![U_i]\!]| < |[\![U_j]\!]| = |[\![B]\!]|$.

Turning to (α), we argue the two directions separately. Suppose first that $A \leq B$. Then $A \leq_c B$. If $A <_c B$, then $[\![A]\!] \subseteq \mathcal{Y}_B \subseteq [\![B]\!]$. If $A \equiv_c B$, then we also have $A \equiv B$. The construction has then arranged that $[\![A]\!] = [\![B]\!]$. In the other direction, assume that $[\![A]\!] \subseteq [\![B]\!]$, and let i and j be such that $A \equiv_c U_i$ and $B \equiv_c U_j$. On cardinality grounds, $i \leq j$. If $i < j$, then the construction shows that $A \leq B$. (For if not, $[\![A]\!]$ would be a non-empty set disjoint from $[\![B]\!]$, and this contradicts $[\![A]\!] \subseteq [\![B]\!]$.) Finally (for perhaps the most interesting point), if $i = j$, then we must have $A \equiv B$: otherwise, the construction arranged that both A and B have at least one point that is not in the other, due to the "1+" in the definition of n.

Since (α) and (β), we know that $\mathcal{M} \models \Gamma$. Recall that we are assuming that $X \leq Y$ holds semantically from Γ; we need to show that this assertion is derivable in the logic. But $[\![X]\!] \subseteq [\![Y]\!]$ in the model, and so by (α), we indeed have $\Gamma \vdash X \leq Y$. ⊣

7.1 Larger syllogistic fragments

Having worked through $\mathcal{L}(all, \exists^{\geq})$, it is natural to go on to further syllogistic fragments. We are not going to do this in detail. Instead, we simply state the results and move ahead in our final section to the largest system in the notes, the one that adds $\exists^{\geq}$ to the system from Section 3. We would need two rules:

$$\frac{\textit{Some } Y \textit{ are } Y \quad \exists^{\geq}(X,Y)}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{No } Y \textit{ are } Y}{\exists^{\geq}(X,Y)}$$

The rule on the left should be added to our existing system for $\mathcal{L}(all, some)$ (and adding the rules in Figure 14), and the resulting system would be complete for $\mathcal{L}(all, some, \exists^{\geq})$. Similarly, the rule on the right can be added to the system for $\mathcal{L}(all, no)$ to get a completeness result. Finally, adding both rules to $\mathcal{L}(all, some, no)$ would again result in a complete system.

7.2 Digression: *Most*

The semantics of *Most* is that *Most X are Y* are that this is true iff $|[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![X]\!]|$. So if $[\![X]\!]$ is empty, then *Most X are Y* is false.

As an example of what is going on, consider the following. Assume that *All X are Z*, *All Y are Z*, *Most Z are Y*, and *Most Y are X*. Does it follow that *Most X are Y*? As it happens, the conclusion does not follow. One can take $X = \{a,b,c,d,e,f,g\}$, $Y = \{e,f,g,h,i\}$, and $Z = \{a,b,c,d,e,f,g,h,i\}$. Then $|X| = 7$, $|Y| = 5$, $|Z| = 9$, $|Y \cap Z| = 5 > 9/5$, $|X \cap Y| = 3 > 5/2$, but $|X \cap Y| = 3 < 7/2$. (Another countermodel: let $X = \{1,2,4,5\}$, $Y = \{1,2,3\}$, and $Z = \{1,2,3,4,5\}$. Then $|Y \cap Z| = 3 > 5/2$, $|Y \cap X| = 2 > 3/2$, but $|X \cap Y| = 2 \not> 4/2$.)

On the other hand, the following is a sound rule:

$$\frac{\textit{All } U \textit{ are } X \quad \textit{Most } X \textit{ are } V \quad \textit{All } V \textit{ are } Y \quad \textit{Most } Y \textit{ are } U}{\textit{Some } U \textit{ are } V}$$

Here is the reason for this. Assume our hypotheses and also that towards a contradiction that U and V were disjoint. We obviously have $|V| \geq |X \cap V|$, and the second hypothesis, together with the disjointness assumption, tells us that $|X \cap V| > |X \cap U|$. By the first hypothesis, we have $|X \cap U| = |U|$. So at this point we have $|V| > |U|$. But the last two hypotheses similarly give us the opposite inequality $|U| > |V|$. This is a contradiction.

At the time of this writing, I do not have a completeness result for $\mathcal{L}(all, some, most)$. The best that is known is for $\mathcal{L}(some, most)$. The rules are are shown in Figure 15. We study these on top of the rules in Figure 4.

Proposition 7.3 *The following two axioms are complete for* Most*.*

$$\frac{\text{Most } X \text{ are } Y}{\text{Most } X \text{ are } X} \qquad \frac{\text{Most } X \text{ are } Y}{\text{Most } Y \text{ are } Y}$$

Moreover, if $\Gamma \subseteq \mathcal{L}(\text{most})$*,* $X \neq Y$*, and* $\Gamma \not\models$ Most X are Y*, then there is a model* $\mathcal{M}$ *of* Γ *which falsifies* Most X are Y *in which all sets of the form* $[\![\text{U}]\!] \cap [\![\text{V}]\!]$ *are nonempty, and* $|M| \leq 5$*.*

Proof Suppose that $\Gamma \not\vdash$ *Most X are Y*. We construct a model $\mathcal{M}$ which satisfies all sentences

$$\frac{\textit{Most } X \textit{ are } Y}{\textit{Some } X \textit{ are } Y} \qquad \frac{\textit{Some } X \textit{ are } X}{\textit{Most } X \textit{ are } X} \qquad \frac{\textit{Most } X \textit{ are } Y \quad \textit{Most } X \textit{ are } Z}{\textit{Some } Y \textit{ are } Z}$$

Figure 15: Rules of *Most* to be used in conjunction with *Some*.

in Γ, but which falsifies *Most* X *are* X. There are two cases. If $X = Y$, then X does not occur in any sentence in Γ. We let $\mathcal{M} = \{*\}$, $[\![X]\!] = \emptyset$, and $[\![Y]\!] = \{*\}$ for $Y \neq X$.

The other case is when $X \neq Y$. Let $\mathcal{M} = \{1,2,3,4,5\}$, $[\![X]\!] = \{1,2,4,5\}$, $[\![Y]\!] = \{1,2,3\}$, and for $Z \neq X, Y$, $[\![Z]\!] = \{1,2,3,4,5\}$. Then the only statement in *Most* which fails in the model $\mathcal{M}$ is *Most* X *are* Y. But this sentence does not belong to Γ. Thus $\mathcal{M} \models \Gamma$. ⊣

Theorem 7.4 *The rules in Figure 15 together with the first two rules in Figure 4 are complete for* $\mathcal{L}(\text{some}, \text{most})$. *Moreover, if* $\Gamma \not\models S$*, then there is a model* $\mathcal{M} \models \Gamma$ *with* $\mathcal{M} \not\models S$*, and* $|M| \leq 6$.

Proof Suppose $\Gamma \nvdash S$, where S is *Some* X *are* Y. If $X = Y$, then Γ contains no sentence involving X. So we may satisfy Γ and falsify S in a one-point model, by setting $[\![X]\!] = \emptyset$ and $[\![Z]\!] = \{*\}$ for $Z \neq X$.

We next consider the case when $X \neq Y$. Then Γ does not contain S, *Some* Y *are* X, *Most* X *are* Y, or *Most* Y *are* X. And for all Z, Γ does not contain both *Most* Z *are* X and *Most* Z *are* Y. Let $M = \{1,2,3,4,5,6\}$, and consider the subsets $a = \{1,2,3\}$, $b = \{1,2,3,4,5\}$, $c = \{2,3,4,5,6\}$, and $d = \{4,5,6\}$. Let $[\![X]\!] = a$ and $[\![Y]\!] = d$, so that $\mathcal{M} \not\models S$. For Z different from X and Y, if Γ does not contain *Most* Z *are* X, let $[\![Z]\!] = c$. Otherwise, Γ does not contain *Most* Z *are* Y, and so we let $[\![Z]\!] = b$. For all these Z, $\mathcal{M}$ satisfies whichever of the sentences *Most* Z *are* X and *Most* Z *are* Y (if either) which belong to Γ. $\mathcal{M}$ also satisfies all sentences *Most* X *are* Z and *Most* Y *are* Z, whether or not these belong to Γ. It also satisfies *Most* U *are* U for all U. Also, for Z, Z' each different from both X and Y, $\mathcal{M} \models$ *Most* Z *are* Z'. Finally, $\mathcal{M}$ satisfies all sentences *Some* U *are* V except for $U = X$ and $Y = V$ (or vice-versa). But those two sentences do not belong to Γ. The upshot is that $\mathcal{M} \models \Gamma$ but $\mathcal{M} \not\models S$.

Up until now in this proof, we have considered the case when S is *Some* X *are* Y. We turn our attention to the case when S is *Most* X *are* Y. Suppose $\Gamma \nvdash S$. If $X = Y$, then the second rule of Figure 15 shows that $\Gamma \nvdash$ *Some* X *are* X. So we take $M = \{*\}$ and take $[\![X]\!] = \emptyset$ and for $Y \neq X$, $[\![Y]\!] = M$. It is easy to check that $\mathcal{M} \models \Gamma$.

Finally, if $X \neq Y$, we clearly have $\Gamma_{most} \nvdash S$. Proposition 7.3 shows that there is a model $\mathcal{M} \models \Gamma_{most}$ which falsifies S in which all sets of the form $[\![U]\!] \cap [\![V]\!]$ are nonempty. So all *Some* sentences hold in $\mathcal{M}$. Hence $\mathcal{M} \models \Gamma$. ⊣

7.3 Adding $\exists^{\geq}$ to the boolean syllogistic fragment

We now put aside *Most* and return to the study of $\exists^{\geq}$ from earlier. We move on to the addition of $\exists^{\geq}$ to the fragment of Section 3.

Our logical system extends the axioms of Figure 8 by those in Figure 16. Note that the last new axiom expresses *cardinal comparison*. Axiom 4 in Figure 16 is just a transcription of the rule for *No* that we saw in Section 7.1. We do not need to also add the axiom

$$(\textit{Some } Y \textit{ are } Y) \wedge \exists^{\geq}(X, Y) \rightarrow \textit{Some } X \textit{ are } X$$

1. *All X are Y* $\to \exists^{\geq}(Y, X)$
2. $\exists^{\geq}(X, Y) \wedge \exists^{\geq}(Y, Z) \to \exists^{\geq}(X, Z)$
3. *All Y are X* $\wedge\, \exists^{\geq}(Y, X) \to$ *All X are Y*
4. *No X are X* $\to \exists^{\geq}(Y, X)$
5. $\exists^{\geq}(X, Y) \vee \exists^{\geq}(Y, X)$

Figure 16: Additions to the system in Figure 8 for $\exists^{\geq}$ sentences.

because it is derivable. Here is a sketch, in English. Assume that there are some Ys, and there are at least as many Xs as Ys, but (towards a contradiction) that there are no Xs. Then all X's are Ys. From our logic, all Ys are Xs as well. And since there are Y's, there are also X's: a contradiction.

Notice also that in the current fragment we can express *There are more X than Y*. It would be possible to add this directly to our previous systems.

Theorem 7.5 *The logic of Figures 8 and 16 is complete for assertions* $\Delta \models \varphi$ *in the language of boolean combinations of sentences in* $\mathcal{L}(\text{all}, \text{some}, \text{no}, \exists^{\geq})$.

Proof We need only build a model for a maximal consistent set Δ in the language of this section. We take the *basic* sentences to be those of the form *All X are Y*, *Some X and Y*, *J is M*, *J is an X*, $\exists^{\geq}(X, Y)$, or their negations. Let

$$\Gamma \quad = \quad \{S : \Delta \models S \text{ and } S \text{ is basic}\}.$$

As in Claim 3.3, we need only build a model $\mathcal{M} \models \Gamma$. We construct $\mathcal{M}$ such that for all A and B,

(α) $[\![A]\!] \subseteq [\![B]\!]$ iff $A \leq B$.

(β) $A \leq_c B$ iff $|[\![A]\!]| \leq |[\![B]\!]|$.

(γ) For $A \leq_c B$, $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$.

Let $\mathcal{V}$ be the set of variables in Γ. Let $\leq_c$ and $\equiv_c$ be as in Section 7. Proposition 7.1 again holds, and now the quotient $\mathcal{V}/\equiv_c$ is a linear order due to the last axiom in Figure 16. We write it as

$$[U_0] \quad <_c \quad [U_2] \quad <_c \quad \cdots \quad <_c \quad [U_k]$$

We define by recursion on $i \leq k$ the interpretation $[\![V]\!]$ of all $V \in [U_i]$. The case of $i = 0$ is special. If $\Gamma \models$ *No* U_0 *is a* U_0, then the same holds for all $W \equiv_c U_0$. In this case, we set $[\![W]\!] = \emptyset$ for all these W. Note that by our fourth axiom in Figure 16, all of the other variables W are such that $\Gamma \vdash \exists W$. In any case, we must interpret the variables in $[U_0]$ even when $\Gamma \vdash (\exists U_0)$. In this case, we may take each $[\![W]\!]$ to be a singleton, with the added condition that $V \equiv W$ iff $[\![V]\!] = [\![W]\!]$.

Suppose we have $[\![W]\!]$ for all $j \leq i$ and all $W \equiv_c U_j$. Let

$$\mathcal{X}_{i+1} \quad = \quad \bigcup_{j \leq i, W \equiv_c U_j} [\![W]\!],$$

and note that this is the set of all points used in the semantics of any variable so far. Let $m = |\Gamma_{some}|$, and let

$$n \quad = \quad 1 + m + |\mathcal{X}_{i+1}| \tag{18}$$

For all $V \equiv_c U_{i+1}$, we shall arrange that $[\![V]\!]$ be a set of size n.

Now $[U_{i+1}]$ splits into equivalence classes of the finer relation $\equiv$. For a moment, consider one of those finer classes, say $[A]_\equiv$. We must interpret each variable in this class by the same set. For this A, let

$$\mathcal{Y}_A \quad = \quad \bigcup \{[\![B]\!] : (\exists j \leq i)\ V_j \equiv_c B \leq A\}.$$

Note that $\mathcal{Y}_A \subseteq \mathcal{X}_{i+1}$ so that $|\mathcal{Y}_A| \leq |\mathcal{X}_{i+1}|$ for all $A \equiv_c U_{i+1}$. We shall set $[\![A]\!]$ to be $\mathcal{Y}_A$ plus other points. Let $\mathcal{Z}_A$ be the set of pairs $\{A, B\}$ with $B \equiv_c U_{i+1}$ and $A \uparrow_\Gamma B$. (This is the same as saying that *Some A are B* in Γ_{some}.) Notice that if both A and B are $\equiv_c U_{i+1}$ and $A \uparrow_\Gamma B$, then $\{A, B\} \in \mathcal{Z}_A \cap \mathcal{Z}_B$.) We shall set $[\![A]\!]$ to be $\mathcal{Y}_A \cup \mathcal{Z}_A$ plus one last group of points. If $C <_c U_{i+1}$ and $A \uparrow_\Gamma C$, then we must pick some element of $[\![C]\!]$ and put it into $[\![A]\!]$. Note that the number of points selected like this plus $|\mathcal{Z}_A|$ is still $\leq |\Gamma_{some}|$. So the number of points so far in $[\![A]\!]$ is $\leq |\Gamma_{some}| + m$. We finally add fresh elements to $[\![A]\!]$ so that the total is n.

We do all of this for all of the other $\equiv$-classes which partition the $\equiv_c$-class of U_{i+1}. We must insure that for $A \not\equiv A'$, the fresh elements added into $[\![A']\!]$ are disjoint from the fresh elements added into $[\![A]\!]$. This is needed to arrange that neither $[\![A]\!]$ nor $[\![A']\!]$ will be a subset of the other.

This completes the definition of the model. We say a few words about why requirements (α)–(γ) are met. First, and easy induction on i shows that if $j < i$, then $|[\![U_j]\!]| < |[\![U_i]\!]|$. The point is that $|[\![U_j]\!]| \leq |\mathcal{X}_i| < |[\![U_i]\!]|$. The argument for (β) is the same as in the proof of Theorem 7.2. For that matter, the proof of (α) is also essentially the same. The point is that when $A \equiv_c B$ and $A \not\equiv B$, then $[\![A]\!]$ and $[\![B]\!]$ each contain a point not in the other.

For (γ), suppose that $A \leq_c B$. Let $i \leq j$ be such that $A \equiv_c U_i$ and $B \equiv U_j$. The construction arranged that $[\![A]\!]$ and $[\![B]\!]$ be disjoint except for the case that $A \uparrow B$.

So this verifies that (α)–(γ) hold. We would like to conclude that $\mathcal{M} \models \Gamma$, but there is one last point: (γ) appears to be a touch too weak. We need to know that $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$ (without assuming $A \leq_c B$). But either $A \leq_c B$ or $B \leq_c A$ by our last axiom. So we see that indeed $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$. ⊣

The next step in this direction would be to consider *At least as many X as Y are Z*.

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