Power-Set Functors and Saturated Trees

Jiří Adámek[1], Stefan Milius[1], Lawrence S. Moss[2], and Lurdes Sousa[3]

1 **Institut für Theoretische Informatik, Technische Universität Braunschweig Germany**
2 **Department of Mathematics, Indiana University Bloomington, IN, USA**
3 **Departamento de Matemática, Instituto Politécnico de Viseu, Portugal**

Abstract

We combine ideas coming from several fields, including modal logic, coalgebra, and set theory. Modally saturated trees were introduced by K. Fine in 1975. We give a new purely combinatorial formulation of modally saturated trees, and we prove that they form the limit of the final ω^{op}-chain of the finite power-set functor $\mathscr{P}_f$. From that, we derive an alternative proof of J. Worrell's description of the final coalgebra as the coalgebra of all strongly extensional, finitely branching trees. In the other direction, we represent the final coalgebra for $\mathscr{P}_f$ in terms of certain maximal consistent sets in the modal logic K. We also generalize Worrell's result to M-labeled trees for a commutative monoid M, yielding a final coalgebra for the corresponding functor $\mathscr{M}_f$ studied by P. Gumm and T. Schröder. We introduce the concept of an i-saturated tree for all ordinals i, and then prove that the i-th step in the final chain of the power set functor consists of all i-saturated trees. This leads to a new description of the final coalgebra for the restricted power-set functors $\mathscr{P}_\lambda$ (of subsets of cardinality smaller than λ).

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1 Introduction

Final coalgebras play a fundamental rôle in the theory of systems represented as coalgebras: J. Rutten [18] demonstrated that the final coalgebra describes all possible behaviors of states of systems. For Kripke structures considered as the coalgebras for the finite power-set functor $\mathscr{P}_f$ two beautiful descriptions of the final coalgebra exist: as the set of all hereditarily finite sets in the non-wellfounded set theory due to P. Aczel, see [2], and as the set of all strongly extensional, finitely branching trees[1] due to J. Worrell [21]. He used metric spaces: he described the limit $\mathscr{P}_f^\omega 1$ of the final chain of $\mathscr{P}_f$ as the set of all strongly extensional, compactly branching trees. From that he derived the above description of the final coalgebra. We give below two new descriptions that do not need topology, one combinatorial and one using modal logic. We prove that the limit $\mathscr{P}_f^\omega 1$ consists (a) of all saturated trees or (b) of all maximal consistent theories of the modal logic K. And an alternative description of the final coalgebra is: the set of all hereditarily finite (maximal consistent) theories. Related

[1] Throughout the paper trees are directed graphs with a distinguished node called the root from which every other node can be reached by a unique directed path, and they are always considered up to isomorphism. Strong extensionality for trees is recalled in Section 2 below.

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descriptions were provided by S. Abramsky [1], A. Kurz and D. Pattinson [14] and by J. Rutten [17, Theorem 7.4].

We also present a generalization in two directions: one uses finite multisets with multiplicities drawn from a given commutative monoid M, as introduced by H. P. Gumm and T. Schröder [12]. Form the functor $\mathscr{M}_f$ of all such finite multisets; its coalgebras are labeled transition systems with actions from $M \setminus \{0\}$. We prove a direct generalization for all monoids for which $\mathscr{M}_f$ preserves weak pullbacks: the final coalgebra for $\mathscr{M}_f$ consists of all finitely branching, strongly extensional M-labeled trees. For general monoids this result is not true, but we prove that the final coalgebra for $\mathscr{M}_f$ is the coalgebra of extensional, finitely branching M-labeled trees modulo an equivalence generalizing M. Barr's equivalence for $\mathscr{P}_f$, see [7].

The other direction of generalization of the final coalgebra for $\mathscr{P}_f$ is from finite subsets to subsets of cardinality less than λ, where λ is an infinite cardinal. The corresponding power-set functor $\mathscr{P}_\lambda$ has the final coalgebra of all strongly extensional λ-branching trees, as proved by D. Schwencke [19]. We present a different proof based on the description of the final chain $\mathscr{P}^i 1$ of the (full) power-set functor $\mathscr{P}$. We introduce the concept of an i-saturated tree for every ordinal i (where ω-saturated is the above concept), and we describe $\mathscr{P}^i 1$ as the set of all strongly extensional i-saturated trees.

2 Extensional and saturated trees

For an endofunctor H of **Set** recall that a *coalgebra* is a set A together with a morphism $a\colon A \to HA$. A *coalgebra homomorphism* into $b\colon B \to HB$ is a morphism $f\colon A \to B$ with $b{\cdot}f = Hf{\cdot}a$. The final coalgebra, if it exists, is denoted by νH; by Lambek's Lemma [15] its coalgebra structure is an isomorphism $\nu H \xrightarrow{\sim} H(\nu H)$. For example Kripke structures (W, R, l) where $R \subseteq W \times W$ and $l\colon W \to 2^{\mathsf{AP}}$ are precisely the coalgebras for $HX = \mathscr{P}X \times 2^{\mathsf{AP}}$ where AP is a fixed set of atomic propositions and $\mathscr{P}$ is the power-set functor. In the present paper we restrict ourselves to the case $\mathsf{AP} = \emptyset$. Then Kripke structures are simply graphs, or coalgebras for $\mathscr{P}$. And the finitely branching graphs are coalgebras for the finite power-set functor $\mathscr{P}_f$.

In this and the next section we describe the final coalgebra for $\mathscr{P}_f$. Lambek's Lemma implies that $\mathscr{P}$ does not have a final coalgebra, but we describe the final chain of $\mathscr{P}$ in Section 5.

Recall from [7], dualizing the initial chain of [4], the *final chain* of H which is the chain $W\colon \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set}$ determined (uniquely up-to natural isomorphism) by its objects W_i, $i \in \mathbf{Ord}$, and connecting morphisms $w_{i,j}\colon W_i \to W_j$ $(i \geq j)$ as follows $W_0 = 1$, $W_{i+1} = HW_i$, and $W_i = \lim_{j<i} W_j$ for limit ordinals i and $w_{i+1,j+1} = Hw_{i,j}$, whereas $(w_{i,j})_{j<i}$ is a limit cone for limit ordinals i. If this chain *converges* at some ordinal i, i.e., the connecting map $HW_i \to W_i$ is an isomorphism, then its inverse yields the final coalgebra for H. The finite steps of the final chain of H are called the *final* ω^{op}*-chain of* H.

▶ **Remark 2.1.** Recall that coalgebras for $\mathscr{P}$ are simply graphs. When speaking about morphisms between graphs (in particular trees) we always mean coalgebra homomorphisms $f : A \to B$. That is, f preserves edges, and for every edge from $f(a)$ to b in B there exists an edge from a to a' in A with $b = f(a')$. Quotients of graphs are, as usual, represented by epimorphisms, that is, surjective morphisms.

▶ **Definition 2.2.** A tree is *extensional* if distinct children of any node define non-isomorphic subtrees.

The *extensional modification* of a tree t is the smallest quotient of t which is extensional. It is obtained from t by recursively identifying isomorphic subtrees whose roots have a joint parent.

▶ **Example 2.3.** The extensional modification of the complete binary tree is the single path.

▶ **Notation 2.4.** For every tree t denote by $\partial_n t$ the extensional tree obtained by cutting t at level n (i.e. deleting all nodes of depth $> n$) and forming the extensional modification. For all trees t and u, we write $t \sim_n u$ to mean that $\partial_n t = \partial_n u$ (remember that we identify isomorphic trees).

▶ **Remark 2.5.** The final ω^{op}-chain of $\mathscr{P}_f$ can be described as follows:

$$\mathscr{P}_f^n 1 = \text{all extensional trees of depth} \le n \text{ with the connecting maps } \partial_n \colon \mathscr{P}_f^{n+1} 1 \to \mathscr{P}_f^n 1.$$

Indeed, the unique element of 1 can be taken to be the root-only tree. Given a set $M \subseteq \mathscr{P}_f^n 1$, we identify it with the tree tupling of its elements and obtain a tree in $\mathscr{P}_f^{n+1} 1$. The first connecting map from $\mathscr{P}_f 1$ to 1 is obviously ∂_0, and given that the n-th connecting map is $\partial_n \colon \mathscr{P}_f^{n+1} 1 \to \mathscr{P}_f^n 1$, it follows that the next connecting map, $\mathscr{P}_f \partial_n$, is (with the above tree tupling identification) precisely ∂_{n+1}.

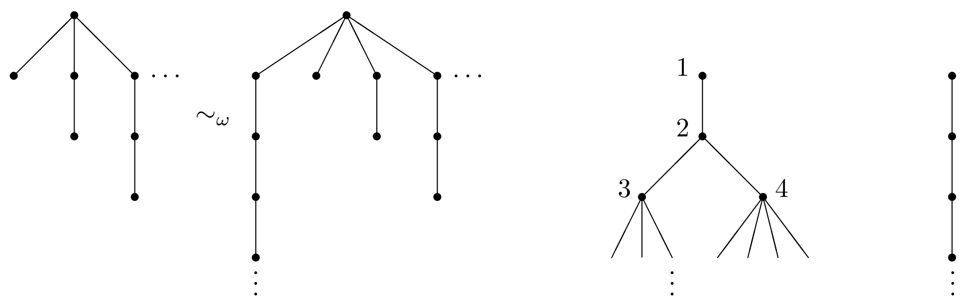
▶ **Definition 2.6.** Two trees t and u are called *Barr equivalent*, notation $t \sim_\omega u$, provided that $t \sim_n u$ holds for all $n < \omega$.

▶ **Remark 2.7.** The set B of all finitely branching extensional trees is a coalgebra for $\mathscr{P}_f$: the coalgebra map is the inverse of tree tupling. This coalgebra is weakly final, and a final coalgebra can be described as its quotient:

▶ **Theorem 2.8** (see [7])**.** *The final coalgebra for $\mathscr{P}_f$ can be described as the quotient $B/{\sim_\omega}$ of the coalgebra of all finitely branching, extensional trees modulo Barr-equivalence.*

▶ **Definition 2.9** (see [21])**.** For trees t and s a *tree bisimulation* is a relation $R \subseteq t \times s$ such that the roots are related, two related child nodes always have related parents, and R is a bisimulation w.r.t. $\mathscr{P}$; i.e., given related nodes $a\ R\ b$ then for every child a' of a in t there exists a child b' in s with $a'\ R\ b'$, and vice versa.

▶ **Example 2.10.** The first two trees in the picture below are Barr-equivalent trees. The third and fourth trees are two extensional trees (the third tree has n children of the n-th node) that are bisimilar. In fact, every tree without leaves is bisimilar to the infinite path.



▶ **Definition 2.11** (see [21])**.** A tree t is called *strongly extensional* if distinct children of any node are not bisimilar. Equivalently, every tree bisimulation $R \subseteq t \times t$ satisfies $R \subseteq \Delta_t$, where Δ_t is the diagonal relation on t.

▶ **Example 2.12.** (1) Every finite extensional tree is strongly extensional.

(2) The infinite path is a strongly extensional tree. This is the only strongly extensional tree without leaves: for every tree t without leaves the relation

$$x \; R \; y \quad \text{iff } x \text{ and } y \text{ have the same depth}$$

is a tree bisimulation. Thus, the third tree in Example 2.10 shows an extensional tree which is not strongly extensional.

▶ **Definition 2.13.** Given a tree t, the subtree of t rooted at the node x is denoted by t_x. A tree t is called *saturated* provided that for all nodes x of t and all trees s, if for all n, there are children x_n of x with $s \sim_n t_{x_n}$ $(n < \omega)$, then there is some fixed child y of x with $s \sim_\omega t_y$.

▶ **Example 2.14.** (1) Every finite tree is saturated.

(2) More generally: all finitely branching trees are saturated. Indeed, given x_n as above, there exists $k < \omega$ with $x_n = x_k$ for infinitely many n, and then $s \sim_n t_{x_k}$ for infinitely many n, proving $s \sim_\omega t_{x_k}$.

(3) The left-hand tree of Example 2.10 is not saturated. We obtain a saturated tree by adding a new child whose subtree is the infinite path.

(4) For every set $A \subseteq \omega$ the following tree r_A is saturated and strongly extensional: take an infinite path and add a leaf at depth n iff $n+1$ lies in A. We know from item (2) that r_A is saturated, and strong extensionality is obvious.

▶ **Lemma 2.15.** *Given saturated, strongly extensional trees t and u with $t \sim_\omega u$, we have $t = u$. Therefore there exist precisely $2^{\aleph_0}$ saturated, strongly extensional trees and they have branching at most $2^{\aleph_0}$.*

Proof. (a) For every child z of the root of u there exists a child y of the root of t with $t_y \sim_\omega u_z$. Indeed, since $t \sim_n u$ for every n there exist children x_n with $t_{x_n} \sim_n u_z$ $(n < \omega)$ and then y exists since t is saturated. Conversely, for every y there exists z with $t_y \sim_\omega u_z$. By continuing to lower nodes we conclude that the relation $R \subseteq t \times u$ defined recursively by

$$y \; R \; z \qquad \text{iff} \qquad \begin{cases} \text{either } y \text{ and } z \text{ are the roots} \\ \text{or } y \text{ and } z \text{ have } R\text{-related parents and } t_y \sim_\omega u_z \end{cases}$$

is a tree bisimulation. Clearly, the opposite relation R^{op} is a tree bisimulation, too, and, as $\mathscr{P}_f$ preserves weak pullbacks, so are the composite relations $R^{\mathrm{op}} \circ R \subseteq t \times t$ and $R \circ R^{\mathrm{op}} \subseteq u \times u$. Since t and u are strongly extensional, we conclude $R^{\mathrm{op}} \circ R \subseteq \Delta_t$ and $R \circ R^{\mathrm{op}} \subseteq \Delta_u$. Finally, since R and R^{op} are total relations, the last two inequalities are equalities, and this implies that R is the graph of an isomorphism from t to u, i.e., $t = u$.

(b) The number of saturated, strongly extensional trees is at least $2^{\aleph_0}$ by Example 2.14(4). It is at most $2^{\aleph_0}$ because every saturated strongly extensional tree t is determined by the set $M = \{t_x; x \text{ a child of the root of } t\}$, and M is determined, due to (a), by the sequence of sets $M_n = \{\partial_n s; s \in M\}$ for $n < \omega$. Since M_n is finite, the number of these sequences is at most $2^{\aleph_0}$.

The last statement follows since every subtree of a saturated tree is saturated. ◀

Recall Worrell's description of the limit $\mathscr{P}_f^\omega 1 = \lim\limits_{n<\omega} \mathscr{P}_f^n 1$ of the final chain of $\mathscr{P}_f$ as the set of all compactly branching trees [21]. Here is a new combinatorial description:

▶ **Theorem 2.16.** *The limit $\mathscr{P}_f^\omega 1$ of the final ω^{op}-chain of $\mathscr{P}_f$ can be described as the set of all saturated, strongly extensional trees. The limit cone is $(\partial_n)_{n<\omega}$.*

Proof. Let S be the set of all saturated, strongly extensional trees. We prove that $\partial_n : S \to \mathscr{P}_f^n 1$ (see Remark 2.5) is a limit cone. For definiteness, we denote the connecting morphism of the final ω^{op}-chain by $\partial'_n : \mathscr{P}_f^{n+1} 1 \to \mathscr{P}_f^n 1$. It is obvious that $\partial_n = \partial'_n \cdot \partial_{n+1}$, thus (∂_n) is a cone on the final ω^{op}-chain.

(a) The cone ∂_n is collectively monic by Lemma 2.15.

(b) For every compatible family $r^n \in \mathscr{P}_f^n 1$ we prove that there exists $t \in S$ with $\partial_n t = r^n$ for every n. Compatibility means $r^n = \partial'_n(r^{n+1})$ for $n < \omega$. Let $\hat{r}^{n+1}$ be the tree obtained by cutting r^{n+1} at depth n. Since r^n is extensional, the above equation tells us that r^n is a quotient of $\hat{r}^{n+1}$. Let $e_n : \hat{r}^{n+1} \to r^n$ be the corresponding epimorphism. Define a tree t to have as nodes of depths $k = 0, 1, 2 \dots$ all sequences $\bar{x} = (\bar{x}^k, \bar{x}^{k+1}, \bar{x}^{k+2} \dots)$ of nodes $\bar{x}^n \in r^n$ of depth k with $e_n(\bar{x}^{n+1}) = \bar{x}^n$ for all $n \geq k$. Thus the sequence of roots of $r^0, r^1, r^2 \dots$ is the root of t. And edges are defined componentwise: there is an edge from $(\bar{x}^k, \bar{x}^{k+1}, \bar{x}^{k+2} \dots)$ to $(\bar{y}^{k+1}, \bar{y}^{k+2}, \bar{y}^{k+3}, \dots)$ iff $(\bar{x}^n, \bar{y}^n)$ is an edge of r^n for all $n \geq k+1$. It is easy to verify that t is a well-defined tree.

(b1) We prove $\partial_n t = r^n$. To this end it suffices to establish that there is an epimorphism of graphs from the cutting of t at level n to r^n (the desired equality then follows since r^n is extensional). Consider the n-th projection. This is surjective:

For every node $z \in r^n$ there exists a node $\bar{x} \in t$ with $z = \bar{x}^n$. Indeed, put $\bar{x}^n = z$, and since e_n is an epimorphism, choose $\bar{x}^{n+1} \in e_n^{-1}(\bar{x}^n)$, etc. Then $\bar{x} = (\bar{x}^n, \bar{x}^{n+1}, \bar{x}^{n+2} \dots)$ has the required property.

It is clear that the projection is a graph morphism: it preserves edges by the definition of t. And analogously to the argument of surjectivity above, for every $\bar{x}$ in t and every edge from $z = \bar{x}^n$ to z' in r^n there exists an edge from $\bar{x}$ to x' in t with $z' = (\bar{x}')^n$.

(b2) t is strongly extensional. Indeed, given a tree bisimulation $R \subseteq t \times t$, we prove that $\bar{x} \mathrel{R} \bar{y}$ implies $\bar{x} = \bar{y}$. Let k be the depth of $\bar{x}$ and $\bar{y}$. From (b1) it follows that $r^n_{\bar{x}^n} = \partial_n(t_{\bar{x}})$ and $r^n_{\bar{y}^n} = \partial_n(t_{\bar{y}})$ for all $n \geq k$. But $\bar{x} \mathrel{R} \bar{y}$ implies that R restricts to a tree bisimulation between $t_{\bar{x}}$ and $t_{\bar{y}}$, thus, $t_{\bar{x}} \sim_n t_{\bar{y}}$. Consequently, $r^n_{\bar{x}^n} = r^n_{\bar{y}^n}$. This implies $\bar{x}^n = \bar{y}^n$. (This is clear from extensionality of r^n in case $k = 1$. This finishes the proof of $\bar{x} = \bar{y}$ if $k = 1$. For $k = 2$, we conclude that $\bar{x}$ and $\bar{y}$ have the same parent, $\bar{z}$, and apply the above to $t_{\bar{z}}$ in lieu of t, etc.)

(b3) The tree t is saturated. Indeed, let s be a tree for which the condition of Definition 2.13 holds, taking x to be the root of t. (The proof for all other nodes x of t is completely analogous.) That is, we have children $\bar{x}_n$ of the root of t with $s \sim_n t_{\bar{x}_n}$ for $n < \omega$. We prove that the node $\bar{y}$ of t with components $\bar{y}^n = \big(\bar{x}_n\big)^n$ for all $n \geq 1$ fulfils $s \sim_\omega t_y$.

Firstly, we need to verify that $\bar{y}$ is a node: $e_n(\bar{y}^{n+1}) = \bar{y}^n$. Both of these nodes are children of the root of r^n, thus, by extensionality we only need to prove that they define the same subtree of r^n. Let $\hat{t}_{\bar{x}_n}$ be the cutting of $t_{\bar{x}_n}$ at level $n-1$, then from (b1) we know that $r^n_{\bar{y}^n}$ is the image of $\hat{t}_{\bar{x}_n}$ under the n-th projection $t \to r^n$. Since $r^n_{\bar{y}_n}$ is extensional, this proves $\partial_n t_{\bar{x}_n} = r^n_{\bar{y}_n}$ and analogously for $n+1$. Moreover, from $t_{\bar{x}_n} \sim_n s \sim_{n+1} t_{\bar{x}_{n+1}}$ we deduce $t_{\bar{x}_n} \sim_n t_{\bar{x}_{n+1}}$. Consequently,

$$r^n_{\bar{y}^n} = \partial_n t_{\bar{x}_{n+1}} \qquad \text{and} \qquad r^{n+1}_{\bar{y}^{n+1}} = \partial_{n+1} t_{\bar{x}_{n+1}}.$$

We also have $\partial'_n r^{n+1}_{\bar{y}^{n+1}} = r^n_{e_n(\bar{y}^{n+1})}$ because the right-hand tree is extensional, and it is the image of $\hat{r}^{n+1}_{\bar{y}^{n+1}}$ under e_n. Consequently, $r^n_{e_n(\bar{y}^{n+1})} = \partial'_n \partial_{n+1} t_{\bar{x}_{n+1}} = \partial_n t_{\bar{x}_{n+1}}$. This proves $r^n_{e_n(\bar{y}^{n+1})} = r^n_{\bar{y}_n}$, thus, $e_n(\bar{y}^{n+1}) = \bar{y}^n$ by extensionality of r^n.

Next, we need to verify $s \sim_n t_{\bar{y}}$ for every $n < \omega$. Indeed, we have $s \sim_n t_{\bar{x}_n}$, and to prove $t_{\bar{x}_n} \sim_n t_{\bar{y}}$ observe that the n-th projection $t \to r^n$ maps the cutting of $t_{\bar{y}}$ onto $r^n_{\bar{y}^n}$, thus, $r^n_{\bar{y}^n} = \partial t_{\bar{y}}$. We already observed that $r^n_{\bar{y}^n} = \partial_n t_{\bar{x}_n}$, thus, $\partial_n t_{\bar{x}_n} = \partial_n t_{\bar{y}}$. ◀

▶ **Corollary 2.17** (J. Worrell). *The final chain of $\mathscr{P}_f$ converges in $\omega + \omega$ steps with the step $\omega + n$ given by the set $\mathscr{P}_f^{\omega+n}1$ of all saturated, strongly extensional trees finitely branching up to level $n - 1$. Moreover, the final coalgebra for $\mathscr{P}_f$ is given by*

$$\mathscr{P}_f^{\omega+\omega}1 = \textit{all finitely branching, strongly extensional trees.}$$

Indeed, for $n = 1$ we have $\mathscr{P}_f^{\omega+1} = \mathscr{P}_f(\mathscr{P}_f^{\omega}1)$ and we identify, again, every finite set $M \subseteq \mathscr{P}_f^{\omega}1$ of saturated trees with its tree-tupling. This is, by Example 2.14(2), a saturated, strongly extensional tree which is finitely branching at the root—and conversely, every such tree is a tree tupling of a finite subset of $\mathscr{P}_f^{\omega}1$. Analogously for $n = 2$: we have $\mathscr{P}_f^{\omega+2} = \mathscr{P}_f(\mathscr{P}_f^{\omega+1}1)$ and the resulting trees are precisely those trees in $\mathscr{P}_f^{\omega}1$ that are finitely branching at levels 0 and 1, etc. The connecting maps are the inclusion maps. The limit $\mathscr{P}_f^{\omega+\omega}1 = \lim_{n<\omega} \mathscr{P}_f^{\omega+n}1$ is the intersection of these subsets of $\mathscr{P}_f^{\omega}1$ which consists of all finitely branching, strongly extensional trees: they are saturated, see Example 2.14(2).

3 Modally saturated trees

K. Fine [10] introduced the concept of modal saturatedness for Kripke structures in modal logic. In this section, we review all of the needed definitions, and we prove that modally saturated trees are the same as saturated trees.

(a) We work with modal logic formulated *without* atomic propositions. The sentences φ of modal logic are then given by

$$\varphi ::= \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

We use the usual abbreviations:

$$\bot = \neg\top \qquad \varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi) \qquad \varphi \to \psi = \neg\varphi \vee \psi \qquad \Diamond\varphi = \neg\Box\neg\varphi.$$

A sentence has *depth n* if n is the maximum of nested $\Box$ in it.

(b) We interpret modal logic on Kripke structures. Since we have no atomic sentences, our Kripke structures are just graphs $G = (G, \to)$, where $\to$ is a binary relation on the set G. The main semantic relation is the *satisfaction* relation $\models$ between the node set of a given graph and the sentences of the logic. This is defined as follows:

$$\begin{array}{lll} a \models \top & & \text{always} \\ a \models \neg\varphi & \text{iff} & \text{it is not the case that } a \models \varphi \\ a \models \varphi \wedge \psi & \text{iff} & a \models \varphi \text{ and } a \models \psi \\ a \models \Box\varphi & \text{iff} & \text{for all neighbors } b \text{ of } a,\ b \models \varphi \end{array}$$

Given a tree t we write $t \vDash \varphi$ if the root satisfies φ.

(c) A *theory* is a set S of sentences. We write $a \vDash S$ if $a \vDash \varphi$ for all $\varphi \in S$ and call a a *model* of S.

(d) Turning to the proof system, the modal logic K extends the propositional logic (Hilbert's style) by one axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, called K, and one deduction rule: if $\varphi \in \mathsf{K}$ then $\Box\varphi \in \mathsf{K}$. We write $\vdash \varphi$ if φ can be derived in this logic.

This logic is sound and complete. That is, $\vdash \varphi$ holds iff for every node a of any graph, $a \vDash \varphi$.

(e) A theory S is *inconsistent* if for some finite $\{\varphi_1, \dots, \varphi_n\} \subseteq S$, $\vdash \neg \bigwedge \varphi_i$. S is *consistent* if S is not inconsistent. Or, equivalently, S has a model. If, moreover, $S \cup \{\varphi\}$ is inconsistent for every sentence $\varphi \notin S$, then S is *maximal consistent*.

(f) $\Box S$ denotes the theory $\{\Box\varphi : \varphi \in S\}$, and $\Box^k S = \Box(\Box^{k-1} S)$ for $k \geq 2$.

▶ **Definition 3.1.** We define *canonical sentences* χ *of depth* n by recursion on n, as follows:
(a) $\top$ is the only canonical sentence of depth 0, and
(b) canonical sentences of depth $n+1$ are precisely the sentences

$$\nabla S = (\bigwedge \Diamond S) \wedge \Box \bigvee S$$

where S is a set of canonical sentences of depth n. We use the conventions that $\bigwedge \emptyset = \top$, $\bigvee \emptyset = \bot$, and we often identify sentences φ and ψ when $\vdash \varphi \leftrightarrow \psi$ in K.

▶ **Example 3.2.** We have two canonical sentences of depth 1.

$$\nabla\emptyset = \top \wedge \Box \bot = \Box\bot \qquad \text{and} \qquad \nabla\{\top\} = \Diamond\top \wedge \Box\top = \Diamond\top$$

distinguishing whether the given node has a neighbor or not.

▶ **Theorem 3.3** (K. Fine [10] and L. Moss [16]). *For every node a of a graph and every $n \in \mathbb{N}$ there exists a unique canonical sentence χ of depth n satisfied by a. Moreover, for every canonical sentence χ of depth n and every sentence ψ of depth at most n, either $\vdash \chi \to \psi$ or $\vdash \chi \to \neg\psi$.*

▶ **Corollary 3.4.** *The sentences of depth at most n form a finite set (up to logical equivalence in K).*

Proof. Observe first that there are only finitely many canonical sentences of depth n. Let ψ be any sentence of depth n. Let A be the set of all canonical sentences χ of depth n with $\vdash \chi \to \psi$ and let B be the canonical sentences χ of depth n with $\vdash \chi \to \neg\psi$. So we have $\vdash \bigvee A \to \psi$ and $\vdash \bigvee \to \neg\psi$. By Theorem 3.3, we have

$$\vdash \bigvee A \vee \bigvee B.$$

So by propositional logic we have $\vdash \bigvee A \leftrightarrow \psi$. Thus, every sentence of depth n is equivalent to a disjunction of canonical sentences of depth n from which the desired result follows. ◀

▶ **Notation 3.5.** (a) For every tree t we denote by $\chi_n(t)$ the unique canonical sentence of depth n satisfied in the root. It is easy to prove that

$$\chi_{n+1}(t) = \nabla\big\{\chi_n(t_x) : x \text{ child of the root of } t\big\}.$$

(b) For any graph G, and any $a \in G$, we denote by S_a the set of all sentences φ with $a \vDash \varphi$ in G. For a tree t, we similarly denote by S_t the set of sentences satisfied by the root of t.

(c) Recall from [8] that the *canonical model of* K is the graph C whose nodes are the maximal consistent theories, and with $S \to S'$ iff $\Diamond S' \subseteq S$ (equivalently, $\Box S \subseteq S'$). The *Truth Lemma* (see [8], Lemma 4.21) is the statement that for all $S \in C$,

$$\{\varphi : S \models \varphi \text{ in } C\} = S.$$

This lemma is easy to check by induction on φ.

▶ **Corollary 3.6.** *For two trees t and s we have $t \sim_n s$ iff $t \vDash \chi_n(s)$. Consequently, $t \sim_\omega s$ iff $S_t = S_s$.*

▶ **Proposition 3.7.** *The limit $\mathscr{P}_f^\omega 1$ can be described as the set C of all maximal consistent theories in K.*

Proof. We have described $\mathscr{P}_f^\omega 1$ as the set of all saturated, strongly extensional trees. We prove that $t \mapsto S_t$ is a bijection between this set and C. This finishes the proof. (a) For every $t \in \mathscr{P}_f^\omega 1$ the theory S_t is maximal consistent. Indeed, it is is obviously consistent. Given $\varphi \notin S$ of depth n, we have $t \nvDash \varphi$ and $t \vDash \chi_n(t)$, thus, $\nvdash \chi_n(t) \to \varphi$. By Theorem 3.3 $\vdash \chi_n(t) \to \neg\varphi$. Therefore, $S_t \cup \{\varphi\}$ is inconsistent. (b) By the Truth Lemma, every maximal consistent theory S is of the form S_t for some t: let t be the expansion of the canonical graph C at S. Moreover, t can be taken as saturated and strongly extensional, since the saturation operation on trees preserves modal theories (see Corollary 3.6). ◀

▶ **Definition 3.8.** A theory S is called *hereditarily finite* if it is maximal consistent and for every $k \in \mathbb{N}$ there exist only finitely many maximal consistent theories S' with $\diamond^k S' \subseteq S$.

▶ **Theorem 3.9.** *The set of all hereditarily finite theories is a final coalgebra for $\mathscr{P}_f$ via the coalgebra map $S \mapsto \{S' : \diamond S' \subseteq S\}$.*

Proof. We prove that the bijection $t \mapsto S_t$ of Proposition 3.7 has the property that for $t \in \mathscr{P}_f^\omega 1$ we have that t is finitely branching iff S_t hereditarily finite. From that our theorem follows, since the coalgebra map above corresponds to the coalgebra map of $\nu\mathscr{P}_f$. Indeed:

(a) If S_t is hereditarily finite, then t is finitely branching. It is sufficient to verify that t is finitely branching at the root. Given a node x of depth k, we then apply this to t_x: the theory of this subtree is also hereditarily finite, since $\diamond^k S_{t_x} \subseteq S_t$ (indeed: if $t_x \vDash \varphi$ then $t \vDash \diamond^k \varphi$).

Every child a of the root of t fulfils $\diamond S_{t_a} \subseteq S_t$. Thus, there are only finitely many such theories S_{t_a}. Now let a and b be children of the root of t with $S_{t_a} = S_{t_b}$, whence $t_a \sim_\omega t_b$ by Corollary 3.6. So since t_a and t_b are saturated and strongly extensional, we have $t_a = t_b$ by Lemma 2.15. Therefore, the root has only finitely many children.

(b) If t is finitely branching, then S_t is hereditarily finite. Indeed, for every maximal consistent theory S' with $\diamond^k S' \subseteq S_t$ let s be a tree with $S' = S_s$ (see Proposition 3.7). Then for every $n \in \mathbb{N}$ we have $t \vDash \diamond^k \chi_n(s)$, i.e., some node of t of depth k satisfies $\chi_n(s)$. Since we have only finitely many such nodes, one of them, say a, satisfies $\chi_n(s)$ for all n. That is, $t_a \sim_n s$ for $n \in \mathbb{N}$, hence, $S_{t_a} = S'$, see Corollary 3.6. Since we have only finitely many nodes a of depth k, we see that S_t is hereditarily finite. ◀

▶ **Definition 3.10** (see [10]). A graph is called *modally saturated* if for every node a, given a theory S such that

$$a \vDash \diamond \bigwedge S_0 \qquad \text{for every finite } S_0 \subseteq S \tag{3.1}$$

there exists a neighbor b of a satisfying S.

▶ **Theorem 3.11.** *A tree is saturated iff it is modally saturated.*

Proof. (a) Let t be modally saturated. Let a be a node in t, and let s be a tree with the property that there exist children x_n of a with $s \sim_n t_{x_n}$ $(n < \omega)$. We prove $s \sim_\omega t_b$ for some child b. The theory S_s fulfils (3.1): given $S_0 \subseteq S_s$ finite, let n be the maximum of the depths of all $\psi \in S_0$; then $\vdash \chi_n(s) \to \psi$ for all $\psi \in S_0$ (see Theorem 3.3). By Corollary 3.6, $s \sim_n t_{x_n}$ iff $x_n \models \chi_n(s)$, and this implies $x_n \vDash \psi$ for all $\psi \in S_0$. Thus, $a \vDash \diamond \bigwedge S_0$. Let b be a neighbor of a satisfying S_s. Then $t_b \vDash \chi_n(s)$ for all n; i.e., $s \sim_\omega t_b$ by Corollary 3.6.

(b) Let t be saturated. Let a be a node of t and S be a theory satisfying (3.1). For every natural number n define S_n to be a set of representatives of all $\psi \in S$ of depth at most n modulo logical equivalence in K. By Corollary 3.4 the sentences of depth n form a finite set (up to logical equivalence).So we have that S_n is finite. By (3.1) we see that for every n, there exists a child b_n of a such that

$$b_n \vDash \psi \qquad \text{for all } \psi \in S_n.$$

It is our task to prove that a has a child b satisfying S.

Let v be the graph whose nodes are all canonical sentences χ of depth any $n = 0, 1, 2, \dots$ such that $a \vDash \Diamond\chi$ and $\vdash \chi \to \psi$ for all $\psi \in S_n$. We make v a graph using the *converse* of logical implication in K. So the neighbors of the node χ are all the nodes χ' of depth $n+1$ with $\vdash \chi' \to \chi$. The root is $\top$, and every node χ' of v has indeed a unique parent (so v is a tree): since $a \vDash \Diamond\chi'$, we have a child c of a with $c \vDash \chi'$ which by Theorem 3.3 implies $\chi' = \chi_{n+1}(t_c)$. Put $\chi = \chi_n(t_c)$, then $\vdash \chi' \to \chi$. (This is because $\vdash \chi' \to \neg\chi$ cannot happen due to $c \vDash \chi'$ and $c \vDash \chi$. Now use Theorem 3.3). Consequently, χ is a parent of χ'. And the uniqueness of the parent is obvious: suppose $\vdash \chi \to \chi'$ where $\chi' \in v$ has depth n, then $t_c \vDash \chi'$, therefore $\chi' = \chi_n(t_c)$.

The tree v is obviously finitely branching. And since each $\chi_n(t_{b_n})$ lies in v and each of these formulas has a different depth, they form an infinite set of nodes of v. By König's Lemma, v has an infinite branch

$$\top = \chi_0 \leftarrow \chi_1 \leftarrow \chi_2 \dots$$

Each $S \cup \{\chi_n\}$ is consistent. Indeed, by compactness it is sufficient to verify that $S_k \cup \{\chi_n\}$ is consistent for every $k \geq n$: due to $a \vDash \Diamond\chi_k$ we have a child c of a satisfying χ_k, then t_c is a model of S_k (due to $\vdash \chi_k \to \psi$ for all $\psi \in S_k$) and of χ_n (due to $\vdash \chi_k \to \chi_n$). Consequently, $S \cup \{\chi_0, \chi_1, \chi_2, \dots\}$ is consistent: use compactness again. Let s be a tree which is model of the last theory. Then $s \vDash \chi_n$ which by Theorem 3.3 implies $\chi_n = \chi_n(s)$ for every n. On the other hand, since $a \vDash \Diamond\chi_n$, we have a child c_n of a with $c_n \vDash \chi_n$, thus, $\chi_n = \chi_n(t_{c_n})$. By Corollary 3.6 this proves $s \sim_n t_{c_n}$. Since t is saturated, there exists a child b of a with $s \sim_\omega t_b$. Then $S \subseteq S_s = S_{t_b}$ which concludes the proof: b satisfies S. ◀

4 Finite multisets with multiplicities in a commutative monoid

Here we follow the approach of P. Gumm and T. Schröder [12] who investigated finitely branching Kripke structures with transitions having weights from a given commutative monoid $(M, +, 0)$. These are just coalgebras for the functor $\mathscr{M}_f\colon \mathbf{Set} \to \mathbf{Set}$ (denoted by $\mathscr{M}_\omega$ in [12]) assigning to every set X the set $\mathscr{M}_f X$ of all finite multisets in X, i.e. all functions $A : X \to M$ with $A^{-1}[M \setminus \{0\}]$ finite. Given a function $h\colon X \to Y$, the functor $\mathscr{M}_f$ assigns to every finite multiset $A\colon X \to M$ the finite multiset $\mathscr{M}_f h(A)$ sending $y \in Y$ to $\sum_{x \in X, h(x)=y} A(x)$.

▶ **Example 4.1.** The Boolean monoid $P = \{0, 1\}$ yields the finite power-set functor $\mathscr{P}_f$. The cyclic group $C = \{0, 1\}$ yields a functor $\mathcal{C}_f$ which coincides with $\mathscr{P}_f$ on objects but is very different on morphisms.

▶ **Definition 4.2.** By an *M-labeled graph G* is meant a graph whose edges are labeled in $M \setminus \{0\}$. We denote by $w_G\colon G \times G \to M$ the corresponding "weight" function with $w_G(x, y) \neq 0$ iff y is a neighbor of x.

▶ **Remark 4.3.** (a) The coalgebras for $\mathscr{M}_f$ are precisely the finitely branching M-labeled graphs. Indeed, given such a graph G, define the coalgebra structure $G \to \mathscr{M}_f G$ by assigning to every vertex x the finite multiset $w_G(x, -)\colon G \to M$. Conversely, every finitely branching M-labeled graph is obtained from precisely one coalgebra of $\mathscr{M}_f$.

(b) Coalgebra homomorphisms between two finitely branching M-labeled graphs G and H are precisely the functions $f\colon G \to H$ between the vertex sets such that

$$w_H\big(f(x), y\big) = \sum_{x' \in X, f(x')=y} w_G(x, x') \qquad \text{for all } x \in G,\, y \in H. \tag{4.1}$$

(c) We identify, once again, two M-labeled trees whenever they are isomorphic (as coalgebras for $\mathscr{M}_f$).

▶ **Definition 4.4.** An M-labeled tree is *extensional* if distinct children of any node define non-isomorphic M-labeled subtrees.

We use $\sim_n$ and $\sim_\omega$ in an obvious analogy to Notation 2.4 and Definition 2.6.

▶ **Remark 4.5.** The concepts of tree bisimulation and strong extensionality (see Definitions 2.9 and 2.11) also immediately generalize to M-labeled trees. We now generalize Theorem 2.8, using B again for the coalgebra of all finitely branching extensional M-labeled trees.

▶ **Theorem 4.6.** *Let M be a commutative monoid. The coalgebra $B/{\sim_\omega}$ of all finitely branching, extensional, M-labeled trees modulo Barr equivalence is final for $\mathscr{M}_f$.*

Sketch of proof. [2](1) B is weakly final. Indeed, for every finitely branching M-labeled graph (A, α) we define a coalgebra homomorphism $h\colon A \to B$ by assigning to every vertex $a \in A$ the extensional modification of the tree expansion of a. Recall that the nodes of the tree expansion of a are the paths $a_0, a_1, \dots, a_k$ of A starting in a, including the empty path, a, which is the root. A child of $a_1, \dots, a_k$ is any extension $a_1, \dots, a_k, a_{k+1}$ and its weight in the tree expansion of a is $w_G(a_k, a_{k+1})$, see Definition 4.2.

(2) The final coalgebra is obtained from B by the quotient modulo the largest congruence. This follows from the fact that the category of coalgebras for $\mathscr{M}_f$ is complete. Thus, by Freyd's Adjoint Functor Theorem a weakly final object always has the final object as the quotient modulo the greatest congruence.

(3) The Barr equivalence is a congruence on B. That is, the quotient $B/{\sim_\omega}$ carries a coalgebra structure for $\mathscr{M}_f$ such that the quotient map $q\colon B \to B/{\sim_\omega}$ is a coalgebra homomorphism.

(4) Every congruence $\approx$ on B is contained in $\sim_\omega$. That is, our task is to prove the implication

$$t \approx t' \qquad \text{implies} \qquad \partial_n t = \partial_n t'.$$

This follows by induction on n since $\approx$ being a congruence means that for the quotient map $q\colon B \to B/{\approx}$ we have a coalgebra structure $B/{\approx}$ making q a homomorphism. ◀

▶ **Definition 4.7** (See [12]). A commutative monoid M is called

(a) *positive* if $a + b = 0$ implies $a = 0 = b$ and

(b) *refinable* if $a_1 + a_2 = b_1 + b_2$ implies that there exists a 2×2 matrix with row sums a_1 and a_2, respectively, and column sum b_1 and b_2, respectively.

---

[2] Full proofs are in the appendix.

▶ **Theorem 4.8.** *The following conditions on a commutative monoid M are equivalent*

(a) *The functor $\mathscr{M}_f$ weakly preserves pullbacks*

(b) *M is positive and refinable, and*

(c) *whenever $a_1 + \cdots + a_n = b_1 + \cdots + b_k$, there exists an $n \times k$-matrix whose vector of row sums is $a_1, \ldots, a_n$ and the vector of column sums is $b_1, \ldots, b_k$.*

In [12] this theorem is proved except that in lieu of (a) weak preservation of *non-empty* pullbacks is requested. However, the functor $\mathscr{M}_f$ has a unique distinguished point in the sense of V. Trnková [20], namely, the empty set $\emptyset \in \mathscr{M}_f X$. Since $\mathscr{M}_f \emptyset = \{\emptyset\}$, it follows from the result in [20] that $\mathscr{M}_f$ preserves weak pullbacks iff it preserves the nonempty ones. Now for (a) $\iff$ (b), see [12], Theorem 5.13, and concerning (b) $\iff$ (c), Proposition 5.10 of *loc. cit.* states that refinability is equivalent to condition (c) with $n, k > 1$ and positivity of M is equivalent to condition (c) with $n > 1$ and $k = 0$. For $n = 1$, condition (c) is trivial.

▶ **Example 4.9** (See [12]). The Boolean monoid $P = \{0, 1\}$ and the monoids $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$ are positive and refinable. The cyclic group $\mathscr{C} = \{0, 1\}$ is refinable but not positive. For every lattice L the monoid $\mathscr{L} = (L, \vee, 0)$ is positive, and it is refinable iff L is a distributive lattice.

▶ **Theorem 4.10.** *Let M be a positive and refinable monoid. The coalgebra B_s of all strongly extensional, finitely branching M-labeled trees is final for $\mathscr{M}_f$.*

▶ **Remark 4.11.** For the coalgebra B of extensional M-labeled trees all strongly extensional trees clearly form a subcoalgebra $m \colon B_s \hookrightarrow B$. We prove that the composite of m with the quotient homomorphism $q \colon B \to B/{\sim_\omega}$ is an isomorphism $q{\cdot}m \colon B_s \to B/{\sim_\omega}$. This proves that B_s is final.

Sketch of proof. Since $q{\cdot}m$ is a homomorphism of coalgebras, it is sufficient to prove that it is a bijection, then it is an isomorphism. In other words: we are to prove that B_s is a choice class of $\sim_\omega$ on the set B.

(1) Every tree t in B is Barr equivalent to a strongly extensional tree $t/\bar{R}$. Indeed, recall from Theorem 4.8 that $\mathscr{M}_f$ weakly preserves pullbacks. Thus the greatest bisimulation $R \subseteq t \times t$ is an equivalence relation which is also the greatest congruence. And for the greatest tree bisimulation $\bar{R}$ contained in R, the strongly extensional tree $t/\bar{R}$ is bisimilar to t. Since $B/{\sim_\omega}$ is the final coalgebra, this proves that $t \sim_\omega t/\bar{R}$.

(2) If two strongly extensional trees are Barr equivalent, then they are equal. Instead, we prove in items (3) and (4) below that given trees $t, s \in B$ then if $t \sim_\omega s$ then t is bisimilar to s. Thus, we obtain a tree bisimulation $R \subseteq t \times s$ and, by symmetry, a tree bisimulation $S \subseteq s \times t$. Since $\mathscr{M}_f$ weakly preserves pullbacks, $S \circ R \subseteq t \times t$ is a tree bisimulation. This proves in the case t and s are strongly extensional, that $S \circ R = \Delta$. By symmetry, $R \circ S = \Delta$. Then R is a graph of an isomorphism from t to s, i.e., $t = s$.

(3) We consider the given trees $t \sim_\omega s$ as elements of the coalgebra B of Theorem 4.6. We know that $\sim_\omega$ is the greatest congruence, hence, the greatest bisimulation on B. As proved in [12], Lemma 5.5, this means that there exists a matrix $m \colon B \times B \to M$ such that

(a) $w_B(t, t') = \sum\limits_{s' \in B} m(t', s')$ for all $t' \in B$

(b) $w_B(s, s') = \sum\limits_{t' \in B} m(t', s')$ for all $s' \in B$, and

(c) $m(t', s') \neq 0$ implies $t' \sim_\omega s'$.

Since M is positive, whenever $m(t', s') \neq 0$, we have $w_B(t, t') \neq 0$, that is, there exists a child x of the root x_0 of t with $t' = t_x$ and $w_B(t, t') = w_t(x_0, x)$. Analogously, $m(t', s') \neq 0$

implies $s' = s_y$ for some child y of the root y_0 of s with $w_B(s, s') = w_s(y_0, y)$. Since t and s are extensional, the trees $t' \in B$ with $w_B(t, t') \neq 0$ are in bijective correspondence with the children x of x_0 in t via $x \mapsto t_x$. Analogously for s. Thus we can translate (a)–(c) as follows:

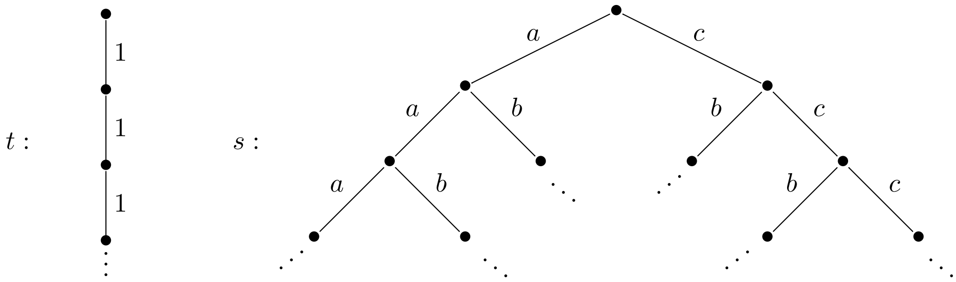
(a*) $w_t(x_0, x) = \sum_{y \in s} m(t_x, t_y)$ for all $x \in t$

(b*) $w_s(y_0, y) = \sum_{x \in t} m(t_x, t_y)$ for all $y \in s$, and

(c*) $m(t', s') \neq 0$ implies that there exists a unique child x of x_0 in t and a unique child y of y_0 in s with $t_x \sim_\omega t_y$, $t' = t_x$ and $s' = s_y$.

(4) We prove that given trees $\bar{t}, \bar{s} \in B$ with $\bar{t} \sim_\omega \bar{s}$, it follows that the relation $R \subseteq \bar{t} \times \bar{s}$ defined recursively by $x \, R \, y$ iff $\bar{t}_x \sim_\omega \bar{s}_y$ and x and y are roots or have R-related parents is a tree bisimulation. ◀

▶ **Example 4.12.** The above theorem does not generalize to all positive monoids. Indeed, consider the monoid $\mathscr{L} = (L, \vee, 0)$ for the lattice $\{0, a, b, c, 1\}$ where a, b, c are pairwise incomparable. Then strongly extensional finitely branching $\mathscr{L}$-labeled trees do not form a final coalgebra, since they are not a choice class of the Barr equivalence. The following trees are easily seen to be Barr equivalent:



Here s has as nodes the binary words, and the weights are, for all $x \in \{0, 1\}^*$, defined by $w_s(x0, x00) = a$, $w_s(x1, x11) = c$ and $w_s(x0, x01) = b = w_s(x1, x10)$. It is obvious that t is strongly extensional. To prove that so is s, let $R \subseteq s \times s$ be a tree bisimulation. Using the conditions (a)–(c) in the preceding proof it is easy to verify that $R \subseteq \Delta_s$.

5 The final chain of $\mathscr{P}$

Although the power-set functor $\mathscr{P}$ has no final coalgebra, one can describe its final chain concretely: we now prove that the i-th step $\mathscr{P}^i 1$ consists of all i-saturated, strongly extensional trees. This generalization from saturated to i-saturated turns out the be quite nontrivial. In addition, this allows us to describe the final chain of the restricted power set functors $\mathscr{P}_\lambda$ (see Corollary 5.13).

▶ **Notation 5.1.** Recall that the subtree of t rooted at the node x is denoted by t_x. We define equivalences $\sim_i$ on the class of all trees for every ordinal i (cf. Notation 2.4 and Definition 2.6) by transfinite induction:

$s \sim_0 t$	holds for all pairs s, t;
$s \sim_{i+1} t$	holds iff for every child x of the root of s there is a child y of the root of t with $s_x \sim_i t_y$, and vice versa

and for limit ordinals i, $s \sim_i t$ means $s \sim_j t$ for all $j < i$.

▶ **Example 5.2.** $\sim_\omega$ is Barr equivalence, see Definition 2.6. The first and second trees in Example 2.10 are trees s and t with $s \nsim_{\omega+1} t$ which are Barr equivalent.

There exist, for every ordinal i, trees s and t with $s \sim_i t$ but $s \nsim_{i+1} t$, see [5].

▶ **Definition 5.3.** We define the concept of *i-saturated tree* for every ordinal i by transfinite induction: A tree t is i-saturated iff

(a) $i = 0$: t consists of the root only

(b) $i = j + 1$: t_x is j-saturated for every child x of the root

(c) i a limit ordinal: given a tree s and a node x of t having children x_j with $s \sim_j t_{x_j}$ $(j < i)$, then x has a child y with $s \sim_i t_y$.

▶ **Examples 5.4.** (a) For i finite, a tree is i-saturated iff it has height at most i. And $s \sim_i t$ holds iff $\partial_i s = \partial_i t$. Therefore, a tree is ω-saturated iff it is saturated in the sense of Definition 2.13.

(b) An example of an $(\omega + 1)$-saturated tree which is not ω-saturated is the left-hand tree in Example 2.10.

(c) For every infinite cardinal λ, all λ-branching trees t are λ-saturated.

▶ **Remark 5.5.** For every tree s there exists a Barr-equivalent tree which is ω-saturated and strongly extensional. We denote it by $\partial_\omega s$ and call it the ω-saturation of s. In fact, for the sequence $r^n = \partial_n s$ of trees in $\mathscr{P}_f^n 1$, which is clearly compatible, apply the construction in the proof of Theorem 2.16: the resulting tree t in $\mathscr{P}_f^\omega 1$ is Barr equivalent to s because $\partial_n t = r^n = \partial_n s$ for every $n < \omega$. Put $\partial_\omega s = t$. We generalize this to all ordinals:

▶ **Definition 5.6.** By the *i-saturation* of a given tree s is meant an i-saturated, strongly extensional tree $\partial_i s$ with $s \sim_i \partial_i s$.

▶ **Example 5.7.** (1) For i finite, Notation 2.4 yields the desired tree.

(2) An example of ω-saturation can be seen in Example 2.10: for the left-hand tree s the second tree is $\partial_\omega s$.

▶ **Remark 5.8.** If t and u are bisimilar trees, then they are equivalent under all of the above equivalences $\sim_i$. This is easy to see by transfinite induction.

Also, if t is i-saturated, then every tree bisimilar to t is i-saturated (in fact, every tree equivalent under $\sim_i$ is). In particular, the *strongly extensional quotient* of a tree t, which is the quotient modulo the largest tree bisimulation $R \subseteq t \times t$, is i-saturated whenever t is.

So even though a tree t is i-saturated it might not be its own i-saturation. Indeed, the third tree in Example 2.10 is ω-saturated but its ω-saturation is the fourth tree.

▶ **Proposition 5.9.** *Every tree has for every ordinal i a unique i-saturation.*

▶ **Remark 5.10.** For infinite ordinals we will see that there is a canonical tree morphism d_i from a tree s to its i-saturation $\partial_i s$. Moreover, if $i = \alpha + n$ where α is a limit ordinal and $n < \omega$, then d_i is surjective when restricted to nodes of depths at most n.

Sketch of proof. (1) Uniqueness. Given i-saturated, strongly extensional trees t and u, we need to prove that $t \sim_i u$ implies $t = u$. The step from i to $i + 1$ is easy. For $i = \omega$ this was established in Theorem 2.16. For all other limit ordinals i, this is proved analogously.

(2) Existence. For $i < \omega$ we have $\partial_i s$ as in Section 2. The isolated steps are trivial: given ∂_i we define ∂_{i+1} by taking a tree s and letting s' be the tree-tupling of all $\partial_i s_x$, where x ranges over the children of the root of s. Then the strong extensional quotient of s' is $\partial_{i+1} s$, see Remark 5.8.

The canonical morphism is the composite $d_{i+1} = e \cdot d'_i$ where $d'_i \colon s \to s'$ is the tree morphism acting on every maximal subtree s_x as the corresponding canonical morphism $d_i^{(s_x)} \colon s_x \to \partial_i s_x$, and e is the strong extensional quotient. This composite is, in the case $i = \alpha + n$ for a limit ordinal α, surjective on nodes of depths at most $n+1$ since each $d_i^{(s_x)}$ is surjective on nodes of depths at most n.

For limit ordinals i we construct, for every tree t, the i-saturation in two steps: first t' will be a possibly large tree (with a class of nodes) which is i-saturated and fulfils $t \sim_i t'$. Then $\partial_i t$ is the strong extensional quotient. This is an i-saturation of t because Remark 5.8 holds clearly for large trees, too. And the resulting tree $\partial_i t$ is small, in fact, it is λ_i-branching for a specific cardinal λ_i analogously to $2^{\aleph_0}$ for $i = \omega$ (see Lemma 2.15). ◀

▶ **Theorem 5.11.** *The final chain of $\mathscr{P}$ can be described for all ordinals i by*

$$\mathscr{P}^i 1 = \textit{all } i\textit{-saturated, strongly extensional trees}$$

with the connecting maps into $\mathscr{P}^j 1$ given by ∂_j for all $j < i$.

Sketch of proof. For i finite this is obvious since $\mathscr{P}^i 1 = \mathscr{P}_f^i 1$, for $i = \omega$ use Theorem 2.16.

We proceed by transfinite induction for all infinite ordinals. If the statement holds for i then it holds for $i+1$ provided that every set $M \subseteq \mathscr{P}(\mathscr{P}^i 1) = \mathscr{P}^{i+1} 1$ of trees is identified with the tree-tupling t_M of all members of M. Obviously, t_M is $(i+1)$-saturated and strongly extensional. Conversely, every $(i+1)$-saturated strongly extensional tree is obtained by tree tupling via a unique set M. Thus, $\mathscr{P}^{i+1} 1 = \mathscr{P}(\mathscr{P}^i 1)$ is the set of all $(i+1)$-saturated, strongly extensional trees. If $\partial_j \colon \mathscr{P}^i 1 \to \mathscr{P}^j 1$ is the given connecting map, then the connecting map $\mathscr{P}\partial_j$ corresponds to ∂_{j+1} when the above identification of M and t_M is performed.

Thus, it remains to prove, for every limit ordinal $\alpha > \omega$, that the cone $\partial_i \colon \mathscr{P}^\alpha 1 \to \mathscr{P}^i 1$ $(i < \alpha)$ is a limit cone. This is technically more involved than the proof of Theorem 2.16 but the ideas are similar. ◀

▶ **Remark 5.12.** J. Worrell [21] proved that since $\mathscr{P}_\lambda$ preserves intersections of chains of subobjects and is λ-accessible, the final chain converges in $\lambda + \omega$ steps, where all steps after λ are given by monomorphisms. We can describe the individual steps $\mathscr{P}_\lambda^i 1$ for all $i < \lambda$ in terms of trees. If i is an infinite ordinal then it has the form $i = \alpha + n$, $n < \omega$ and α a limit ordinal.

▶ **Corollary 5.13.** *The final chain of $\mathscr{P}_\lambda$ has the steps $\mathscr{P}_\lambda^i 1$ for all ordinals $i < \lambda$ given by*

$$\begin{aligned} \mathscr{P}_\lambda^i 1 = {} & \textit{all } i\textit{-saturated, strongly extensional trees whose } i\textit{-saturation} \\ & \textit{is } \lambda\textit{-branching at the first } n \textit{ levels for all } i = \alpha + n. \end{aligned}$$

The connecting maps are $\partial_j \colon \mathscr{P}_\lambda^i 1 \to \mathscr{P}_\lambda^j 1$ for all $j < i$.

▶ **Corollary 5.14.** *Let λ be an infinite cardinal. The final coalgebra for $\mathscr{P}_\lambda$ can be described as the coalgebra of all strongly extensional, λ-branching trees.*

Indeed, each such tree is λ-saturated, see Example 5.4. Thus, it lies in $\mathscr{P}_\lambda^{\lambda+\omega} 1$. Conversely, every tree in $\mathscr{P}_\lambda^{\lambda+\omega} 1$ is λ-branching because the connecting map $\mathscr{P}_\lambda^{\lambda+n+1} 1 \to \mathscr{P}_\lambda^{\lambda+n} 1$ is a monomorphism for every $n < \omega$: this follows from $\mathscr{P}_\lambda$ being a λ-accessible functor, see [21]. Consequently, the limit $\mathscr{P}_\lambda^{\lambda+\omega} 1 = \lim_{n<\omega} \mathscr{P}_\lambda^{\lambda+n} 1$ is the intersection of the subsets $\mathscr{P}_\lambda^{\lambda+n} 1$ of $\mathscr{P}_\lambda^{\lambda} 1$. Since all tree in $\mathscr{P}_\lambda^{\lambda+n} 1$ are λ-branching at the first n levels, it follows that every tree in $\mathscr{P}_\lambda^{\lambda+\omega} 1$ is λ-branching.

A completely different proof has been presented by D. Schwencke [19].

6 Conclusions and future work

We proved several results which generalize Worrell's description [21] of the final coalgebra of $\mathscr{P}_f$ as the coalgebra of all strongly extensional, finitely branching trees. We described the final coalgebra for the functor $\mathscr{M}_f$ of finite multisets with weights from a given commutative monoid M. This final coalgebra consists of all finitely branching strongly extensional M-labeled trees. This holds for all positive and refinable monoids. Our proof is substantially different from Worrell's, since it is based on congruences on the coalgebra of all extensional trees. We would like to generalize our work on saturated trees to the case of functors $\mathscr{M}_f$. And we plan to apply our methods to probabilistic transition systems.

For $\mathscr{P}_f$ we also described the limit $\mathscr{P}_f^\omega 1$ of the final ω^{op}-chain as the set of all saturated strongly extensional trees, or the set of all maximal consistent theories in the modal logic K. We proved that saturated trees are precisely those trees which are modally saturated in the sense of K. Fine [10]. This also is related to (but quite different from) Worrell's description of $\mathscr{P}_f^\omega 1$ by means of compactly branching trees. We then generalized saturatedness to i-saturatedness and proved that the final chain $\mathscr{P}^i 1$ of the full power-set functor consists of all strongly extensional i-saturated trees. From this we derived e.g. that the countable power-set functor $\mathscr{P}_c$ has the final coalgebra consisting of all strongly extensional countably branching trees. We leave as open problem the decision whether $\omega_1 + \omega$ is the smallest ordinal for the convergence of the final chain of $\mathscr{P}_c$.

We have characterizations of the initial algebras of the functors $\mathscr{M}_f$. There are open questions concerning the final chains for these functors: for the Boolean monoid, yielding $\mathscr{P}_f$, the chain needs $\omega + \omega$ steps, as proved by Worrell. However, for the monoid $(\mathbb{N}, +, 0)$, the corresponding functor $\mathscr{N}_f$ is analytic, hence, the convergence needs only ω steps. We do not know what happens in the case of a general monoid.

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A Proof of Theorem 4.6

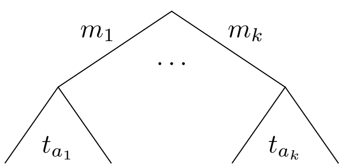
(1) B is weakly final. Indeed, for every finitely branching M-labeled graph (A, α) we define a coalgebra homomorphism $h\colon A \to B$ by assigning to every vertex $a \in A$ the extensional modification t_a of the tree expansion of a. Recall that the nodes of the tree expansion of a are the paths $a_0, a_1, \dots, a_k$ of A starting in a, including the empty path, a, which is the root. A child of $a_1, \dots, a_k$ is any extension $a_1, \dots, a_k, a_{k+1}$ and its weight in the tree expansion of a is $w_G(a_k, a_{k+1})$, see Definition 4.2. We need to prove that the square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathscr{M}_f A \arrow[d, "\mathscr{M}_f h"] \\ B \arrow[r, "\beta"'] & \mathscr{M}_f B \end{tikzcd}$$

commutes. Let $a \in A$ be a vertex with

$$\alpha(a) = \big\{(a_1, m_1), \dots, (a_k, m_k)\big\}.$$

Then $\beta \cdot h(a)$ is the extensional modification of the following tree r:



Since t_{a_i} are all extensional, the extensional modification $\bar{r}$ of r is the tree whose neighbors in B are precisely the trees $s \in B$ for which the sum

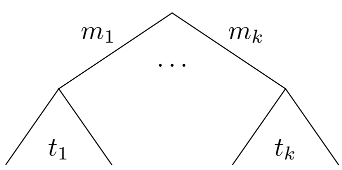
$$\sum_{\substack{i=1 \\ s=t_{a_i}}}^{k} m_i = \sum_{\substack{i=1 \\ h(a_i)=s}}^{k} m_i$$

is nonzero; the sum is then the weight of the edge from $\bar{r}$ to s. But this precisely describes the finite multiset $\mathscr{M}_f h(\alpha(a))$.

(2) The final coalgebra is obtained from B by the quotient modulo the largest congruence. This follows from the fact that the category of coalgebras for $\mathscr{M}_f$ is complete. Thus, by Freyd's Adjoint Functor Theorem a weakly final object has always the final object as the quotient modulo the greatest congruence.

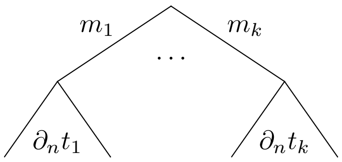
(3) The Barr equivalence is a congruence on B. That is, the quotient $B/{\sim_\omega}$ carries a coalgebra structure for $\mathscr{M}_f$ such that the quotient map $q\colon B \to B/{\sim_\omega}$ is a coalgebra homomorphism. To prove this, all we need to verify is that given a tree

$t:$



then the class $[t]$ modulo $\sim_\omega$ is independent of the choice of representatives of the classes $[t_1], \dots, [t_k]$. That is, given trees $s_1 \sim_\omega t_1, \dots, s_k \sim_\omega t_k$ and forming their tree-tupling s with the given weights $m_1, \dots, m_k$, then $s \sim_\omega t$.

Indeed, for every n the extensional modification $\partial_{n+1}t$ of the cutting of t at depth $n+1$ is obtained by forming the extensional modifications $\partial_n t_1, \dots, \partial_n t_k$ of the cuttings of the children and then taking the extensional modification of the following tree



Analogously for s. Therefore $\partial_n t_i = \partial_n s_i$ for all $i = 1, \dots, k$ implies $\partial_{n+1} t = \partial_{n+1} s$, as required.

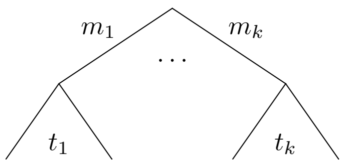
(4) Every congruence $\approx$ on B is contained in $\sim_\omega$. That is, our task is to prove the implication

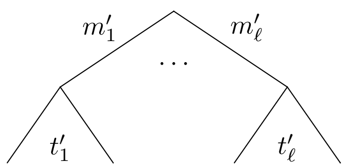
$$t \approx t' \qquad \text{implies} \qquad \partial_n t = \partial_n t' \text{ for all } n \in \mathbb{N}.$$

To be a congruence means that for the quotient map $q\colon B \to B/{\approx}$ there is a commutative square

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "q"'] & \mathscr{M}_f B \arrow[d, "\mathscr{M}_f q"] \\ B/{\approx} \arrow[r, dashed] & \mathscr{M}_f(B/{\approx}) \end{tikzcd}$$

In other words, $\mathscr{M}_f q \cdot \beta$ factorizes through q. More detailed: for every pair $t \approx t'$ of trees of the form

$t =$ 

$t' =$ 

the multiset given by $[t_i]$ and m_i is the same one as that given by $[t'_j]$ and m'_j. This means that for every tree $s \in B$ the two sums below are equal:

$$\sum_{\substack{i=1 \\ s \approx t_i}}^{k} m_i = \sum_{\substack{j=1 \\ s \approx t'_j}}^{\ell} m'_j. \tag{A.1}$$

From this we derive $\partial_n t = \partial_n t'$ as follows.

Case $n = 0$ is trivial: $\partial_n t$ is the root-only tree.

Case $n = 1$. We are to prove $m_1 + \dots + m_k = m'_1 + \dots + m'_l$. For every $s = t_{i_0}$, $i_0 = 1, \dots, k$, we have the equality in (A.1). In case the left-hand sum is nonzero, we thus have some j with $t_{i_0} \approx t'_j$. And we can express $m_1 + \dots + m_k$ as the sum of all non-zero sums $\sum_{s \approx t_i} m_i$ where i_0 ranges over a set of representatives (for $\approx$) of all indexes $1, \dots, k$ making the above left-hand sum nonzero. By symmetry, this yields $m_1 + \dots + m_k = m'_1 + \dots + m'_l$, as desired.

Analogously for $n = 2$: here we take any t_{i_0} with $\sum_{t_{i_0} \approx t_i} m_i \neq 0$ and find a corresponding $t'_j \approx t_{i_0}$ (and vice versa). Then we apply the case $n = 1$ to the pairs t_{i_0}, t'_j. Etc. ◀

B Proof of Theorem 4.10

Since $q{\cdot}m$ is a homomorphism of coalgebras, it is sufficient to prove that it is a bijection, then it is an isomorphism. In other words: we are to prove that B_s is a choice class of $\sim_\omega$ on the set B.

(1) Every tree t in B is Barr equivalent to a strongly extensional tree $t/\bar{R}$. Indeed, recall that since $\mathscr{M}_f$ weakly preserves pullbacks, the greatest bisimulation $R \subseteq t \times t$ is an equivalence relation which is also the greatest congruence. The corresponding tree bisimulation $\bar{R}$ (with $x\bar{R}y$ iff xRy and x and y are roots or have R-related parents) is also a congruence and the quotient coalgebra $t/\bar{R}$ is a tree. This yields a quotient homomorphism $q\colon t \to t/\bar{R}$ whose kernel is the greatest tree bisimulation on t. Consequently, $t/\bar{R}$ is strongly extensional: given the greatest tree bisimulation S on $t/\bar{R}$, the corresponding quotient $q'\colon t/\bar{R} \to (t/\bar{R})/S$ has the property that q and $q'{\cdot}q$ have the same kernel equivalence, hence, $S = \Delta$. And for every coalgebra homomorphism q the elements x and $q(x)$ are bisimilar. In particular, the roots of t and $t/\bar{R}$ are bisimilar. Since $B/{\sim_\omega}$ is the final coalgebra, this proves $t \sim_\omega t/\bar{R}$.

(2) If two strongly extensional trees are Barr equivalent, then they are equal. Instead, we prove in items (3) and (4) below that given trees $t, s \in B$ then

if $t \sim_\omega s$ then t is bisimilar to s.

Thus, we obtain a tree bisimulation $R \subseteq t \times s$ and, by symmetry, a tree bisimulation $S \subseteq s \times t$. Since $\mathscr{M}_f$ weakly preserves pullbacks, $S \circ R \subseteq t \times t$ is a tree bisimulation. This proves in case t and s are strongly extensional, that $S \circ R = \Delta$. By symmetry, $R \circ S = \Delta$. Then R is a graph of an isomorphism $t \xrightarrow{\sim} s$.

(3) We consider the given trees $t \sim_\omega s$ as elements of the coalgebra B. We know that $\sim_\omega$ is the greatest congruence, hence, the greatest bisimulation on B. By Lemma 5.5 in [12] there exists a matrix

$$m\colon B \times B \to M$$

such that

(a) $w_B(t, t') = \sum_{s' \in B} m(t', s')$ for all $t' \in B$

(b) $w_B(s, s') = \sum_{t' \in B} m(t', s')$ for all $s' \in B$, and

(c) $m(t', s') \neq 0$ implies $t' \sim_\omega s'$.

Since M is positive, whenever $m(t', s') \neq 0$ we have $w_B(t, t') \neq 0$, that is, there exists a child x of the root x_0 of t with

$$t' = t_x \qquad \text{and} \qquad w_B(t, t') = w_t(x_0, x).$$

Analogously, $m(t', s') \neq 0$ implies $s' = s_y$ for some child y of the root y_0 of s with

$$w_B(s, s') = w_s(y_0, y).$$

Since t and s are extensional, the trees $t' \in B$ with $w_B(t, t') \neq 0$ are in bijective correspondence with the children x of x_0 in t via $x \mapsto t_x$. Analogously for s. Therefore we can translate (a)–(c) as follows:

(a*) $w_t(x_0, x) = \sum_{y \in s} m(t_x, t_y)$ for all $x \in t$

(b*) $w_s(y_0, y) = \sum_{x \in t} m(t_x, t_y)$ for all $y \in s$, and (c*) $m(t', s') \neq 0$ implies that there exists a unique child x of x_0 in t and a unique child y of y_0 in s with $t_x \sim_\omega t_y$, $t' = t_x$ and $s' = s_y$.

(4) We prove that given trees $\bar{t}, \bar{s} \in B$ with $\bar{t} \sim_\omega \bar{s}$, it follows that the relation $R \subseteq \bar{t} \times \bar{s}$ defined recursively by

$$x \; R \; y \qquad \text{iff} \qquad \bar{t}_x \sim_\omega \bar{s}_y \text{ and } x \text{ and } y \text{ are roots or have } R\text{-related parents}$$

is a tree bisimulation. The roots are clearly related. If $x \; R \; y$ then put $t := \bar{t}_x$ and $s := \bar{s}_y$ and let $\bar{m} \colon \bar{t} \times \bar{s} \to M$ be the following matrix

$$\bar{m}(x', y') = \begin{cases} m(t_{x'}, t_{y'}) & \text{if } x' \text{ is a child of } x \text{ and } y' \text{ a child of } y \\ 0 & \text{else} \end{cases}$$

The property (c*) tells us that $\bar{m}$ is obtained from the matrix m by removing all zero columns and zero rows. Therefore, (a*) and (b*) imply that $\bar{m}$ has the desired row and column sums:

$$w_{\bar{t}}(x, x') = \sum_{y' \in \bar{s}} \bar{m}(x', y') \qquad \text{for all } x \in t$$

$$w_{\bar{t}}(y, y') = \sum_{x' \in \bar{t}} \bar{m}(\bar{x}', \bar{y}') \qquad \text{for all } y' \in s.$$

Moreover, since x and y have equal depth, say, k, we conclude that

$$\bar{m}(x', y') \neq 0 \qquad \text{implies} \qquad x', y' \text{ have depth } k + 1.$$

And (c*) yields $t_{x'} \sim_\omega s_{y'}$. Therefore $\bar{m}(x', y') \neq 0$ implies $x' \; R \; y'$. ◀

C Proof of Proposition 5.9

(1) Uniqueness. Given i-saturated, strongly extensional trees t and u, we need to prove that

$$t \sim_i u \qquad \text{implies} \qquad t = u.$$

This is obvious if i is finite. For $i = \omega$ this was established in Theorem 2.16. For all other limit ordinals i this is proved analogously: if $t \sim_i u$ then the relation $R \subseteq t \times u$ given recursively by

$$y \; R \; z \qquad \text{iff} \qquad y \text{ and } z \text{ are roots or have } R\text{-related parents and } t_y \sim_i t_z$$

is a tree bisimulation. Thus $t = u$ by strong extensionality. It remains to prove that if the statement holds for i, then it holds for $i + 1$. From $t \sim_{i+1} u$ we conclude that for every child x of the root of t there exists a child y of the root of u with $t_x = u_y$ (due to $t_x \sim_i u_y$) and vice versa. This implies $t = u$.

(2) Existence. For $i < \omega$ we have $\partial_i s$ as in Section 2. The isolated steps are trivial: given ∂_i we define ∂_{i+1} by taking a tree s and putting

$$s' = \text{tree-tupling of all } \partial_i s_x$$

where x ranges over the children of the root of s. Then the strong extensional quotient of s' is $\partial_{i+1} s$, see Remark 5.8.

The canonical morphism is the composite

$$d_{i+1} = (s \xrightarrow{d_i'} s' \xrightarrow{e} \partial_{i+1} s),$$

where d'_i is the tree morphism acting on every maximal subtree s_x as the corresponding canonical morphism $d_i^{(s_x)}: s_x \to \partial_i s_x$, and e is the strong extensional quotient. This composite is, in case $i = \alpha + n$ for a limit ordinal α, surjective on nodes of depths at most $n+1$ since each $d_i^{(s_x)}$ is surjective on nodes of depths at most n.

We now turn to limit ordinals i.

(2a) The ω-saturation $\partial_\omega s$. In the proof of Theorem 2.16 we provided for the sequence $r^n = \partial_n s$, which is clearly compatible, an ω-saturated, strongly extensional tree t mapped by the limit projections to r^n for every $n < \omega$. Let us denote this tree t by $\partial_\omega s$. Its nodes of depth n have the form $x = (\bar{x}^n, \bar{x}^{n+1}, \bar{x}^{n+2}, \dots)$ of nodes $\bar{x}^n \in \partial_n s$. The canonical tree morphism

$$d_\omega : s \to \partial_\omega(s)$$

assigns to every node x of s of depth n the node $(\bar{x}^n, \bar{x}^{n+1}, \bar{x}^{n+2}, \dots)$, where for $k \geq n$ the node $\bar{x}^k$ corresponds to x in the extensional quotient $\partial_n s$ of the cutting of s.

(2b) The i-saturation $\partial_i s$ where $i > \omega$ is a limit ordinal. Observe first that the number of all j-saturated strongly extensional trees is bound by the cardinal λ_j, where

$$\lambda_0 = 1, \qquad \lambda_{j+1} = 2^{\lambda_j} \qquad \text{and } \lambda_j = \prod_{k<j} \lambda_k \text{ if } j \text{ is a limit ordinal.}$$

(Indeed, for $j = \omega$ we have $\lambda_\omega = 2^{\aleph_0}$ and we use Lemma 2.15. The argument for general λ_j, $j \geq \omega$, is analogous.) We construct, for every tree t, the i-saturation in two steps: first t' will be a possibly large tree (with a class of nodes) which is i-saturated and fulfils $t \sim_i t'$. Then $\partial_i t$ is the strong extensional quotient of t'. This is an i-saturation of t because Remark 5.8 holds clearly for large trees, too. And the resulting tree $\partial_i t$ is small.

The tree t' is obtained from t by systematically adding new subtrees to given nodes x: whenever we find a tree s and children x_j of x with $t_{x_j} \sim_j s$ for all $j < i$ but with no child y of x with $t_y \sim_i s$, we add s as a new subtree of x. The resulting tree is equivalent to the original one under $\sim_i$. We perform this extension as long as a pair (x, s) with the above property can be found. The resulting (possibly large) tree t' is i-saturated and $t \sim_i t'$.

The morphism $d_i : t \to \partial_i t$ is the composite of the embedding $t \hookrightarrow t'$ and the quotient map $t' \to \partial_i t$. ◀

D Proof of Theorem 5.11

For i finite this is obvious since $\mathscr{P}^i 1 = \mathscr{P}_f^i 1$, for $i = \omega$ use Theorem 2.16.

We proceed by transfinite induction for all infinite ordinals. If the statement holds for i then it holds for $i+1$ provided that every set $M \subseteq \mathscr{P}(\mathscr{P}^i 1) = \mathscr{P}^{i+1} 1$ of trees is identified with the tree-tupling t_M of all members of M. Obviously, t_M is $(i+1)$-saturated and strongly extensional. Conversely, every $(i+1)$-saturated strongly extensional tree is obtained by tree tupling via a unique set M. Thus, $\mathscr{P}^{i+1} 1 = \mathscr{P}(\mathscr{P}^i 1)$ is the set of all $(i+1)$-saturated, strongly extensional trees. If $\partial_j : \mathscr{P}^i 1 \to \mathscr{P}^j 1$ is the given connecting map, then the connecting map $\mathscr{P} \partial_j$ corresponds to ∂_{j+1} when the above identification of M and t_M is performed.

Thus, it remains to prove, for every limit ordinal $\alpha > \omega$, that the cone

$$\partial_i : \mathscr{P}^\alpha 1 \to \mathscr{P}^i 1 \qquad (i < \alpha)$$

is a limit cone. This cone is clearly compatible and collectively monic: if t and u are α-saturated, strongly extensional trees with $\partial_i t = \partial_i u$ for all $i < \alpha$ (that is, with $t \sim_\alpha u$), then

$t = u$. Indeed, to prove this it suffices to present a tree bisimulation $R \subseteq t \times u$. This is completely analogous to the proof of Lemma 2.15: put $y \mathrel{R} z$ iff $t_y \sim_\alpha u_z$, first for the roots, then successively for all nodes $y \in t$ and $z \in u$ with R-related parents.

Finally, given a compatible family r^i of i-saturated, strongly extensional trees for $\omega \leq i < \alpha$, we will present an α-saturated, strongly extensional tree t with $r^i = \partial_i t$ for all $\omega \leq i < \alpha$. Compatibility means $\partial_j r^i = r^j$ $(i \geq j)$. Let $d_{ij} \colon r^i \to r^j$ be the corresponding canonical morphisms (see Remark 5.10). They form a chain of tree homomorphisms, and we denote its limit by

$$s = \lim_{\omega \leq i < \alpha} r^i.$$

(1) Let us prove $\partial_i s = r^i$ for all infinite ordinals $i < \alpha$. Since r^i is i-saturated and strongly extensional, all we need proving is

$$s \sim_i r^i \qquad (\omega \leq i < \alpha).$$

We proceed by transfinite induction.

(1a) To prove $s \sim_\omega r^\omega$, we establish

$$s \sim_n r^{\omega+n} \qquad \text{for every } n < \omega.$$

This proves the statement because from $r^\omega = \partial_\omega r^{\omega+n}$ we conclude $r^\omega \sim_\omega r^{\omega+n}$ and then $s \sim_n r^{\omega+n} \sim_\omega r^\omega$. Consider the set A_n of all infinite ordinals $i < \alpha$ with distance at least n from a limit ordinal, i.e., $A_n = \{i < \alpha; i = \beta + k$ where β is a limit ordinal and $n \leq k < \omega\}$. Then s is a limit of all r^i, $i \in A_n$, and by Remark 5.10 the connecting morphisms $d_{ij} : r^i \to r^j$ have, for all $i \geq j$ in A_n, the property that they are surjective when restricted to nodes of depth at most n. Therefore, the cuttings of r^i at level n have, for all $i \in A_n$, the same extensional quotient. Consequently, the cutting of s at level n also has this extensional quotient. This proves $s \sim_n r^{\omega+n}$.

(1b) If the statement holds for i, we prove it for $i+1$. For every child x of the root of s where $x = (x_k)$, the subtrees $r^{k+1}_{x_{k+1}}$ of r^{k+1} $(\omega \leq k < \alpha)$ are k-saturated and form a chain whose limit is s_x. By induction hypothesis, $r^{i+1}_{x_{i+1}} \sim_i s_x$. Thus, in order to establish $r^{i+1} \sim_{i+1} s$, we only need to prove that for every child y of the root of r^{i+1} there exists a child x of the root of s whose component $i+1$ is y: we then have $r^{i+1}_y \sim_i s_x$ as above. In fact, express s as a limit of all r^{k+1} for $k < \alpha$. By Remark 5.10 the connecting maps are surjective on nodes of depth 1. Therefore, we can find a compatible collection of nodes $x_{k+1} \in r^{k+1}$ of depth 1 with $x_{i+1} = y$. This gives us the desired node x of $s = \lim r^{k+1}$ with component y.

(1c) If the statement holds for all $i < \beta$, where $\beta < \alpha$ is a limit ordinal, we prove $r^\beta \sim_\beta s$. That is, $r^\beta \sim_i s$ for every $i < \beta$. This follows from $r^i \sim_i s$ (induction hypothesis) since $r^\beta \sim_i r^i$ (recall that $i < \beta$ implies that r^i is the i-saturation of r^β).

(2) Finally, let t be the α-saturation of s. Then for every $\omega \leq i < \alpha$ we have $t \sim_i r^i$ because $t \sim_\alpha s \sim_i r^i$. Thus, t is the desired α-saturated, strongly extensional tree (cf. Definition 5.6) with $\partial_i t = r^i$ for all infinite $i < \alpha$. ◀