Syllogistic Logic With Comparative Adjectives

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Abstract

This paper adds comparative adjectives to the known systems of syllogistic logic. The comparatives are interpreted by transitive and irreflexive relations. The main point is to obtain sound and complete axiomatizations of the logics.

Contents

1 Introduction

This paper is concerned with comparative adjective phrases in syllogistic logics. That is, we study logical systems in which one can express arguments such as the following:

$$\begin{array}{l}\textsf{Every giraffe is taller than every gnu}\\ \textsf{Some gnu is taller than every lion}\\ \underline{\textsf{Some lion is taller than some zebra}}\\ \textsf{Every giraffe is taller than some zebra}\end{array} \tag{1}$$

Notice that the argument in (1) uses transitivity of is taller than. (It also uses the subject-wide scope reading in the second sentence, and indeed all sentences in this paper are read with the subject having wide scope.) Although transitivity is salient, it is also silent: it seems unnecessary to add the transitivity as an additional premise. Further, every comparative adjective phrase is most naturally interpreted by a transitive relation, and so this point will be of central importance in this paper. Further, the interpretations are *irreflexive*, corresponding to the fact that sentences like Every woman is not taller than some woman (with the subject wide scope reading) are *logical truths*. (This is because for every woman w, w is definitely not taller than herself.)

We are especially interested in *complete* and *decidable* logical system in which one can represent the main features of arguments such as (1). Here is a related point. One might take a formal representation of (1) to be a translation into first-order logic, together with a proof of the conclusion from the hypotheses in some formal system or other. For the purposes of this paper, this will not do: we are interested only in *decidable* logical systems. A more sophisticated point might be to translate (1) into the *two-variable fragment* FO^2 of first-order logic. This fragment is indeed decidable. But the requirement of transitivity cannot be expressed in FO^2, as shown in Theorem 20 of Purdy [6].

Furthermore, the same premises in (1) also entail Everything which is taller than some giraffe is taller than some zebra. This last sentence is important due to the relative clause in the subject. For another example, consider:

$$\frac{\begin{array}{l}\textsf{Every hyena is taller than some jackal}\\ \textsf{Everything taller than some jackal is not heavier than any warthog}\end{array}}{\textsf{Everything which is taller than some hyena is not heavier than any warthog}} \quad (2)$$

The reader wishing to get an idea of the subject is encouraged to devise a logical system in which one can carry out the reasoning in (1) and (2) and which has *no individual variables*.

The last topic of this paper is inference with *polar opposites*. For example, the following inference is intuitively valid:

$$\frac{\textsf{Some boulder is heavier than some car}}{\textsf{Some car is lighter than some boulder}} \quad (3)$$

We investigate logics in which (3) corresponds to an atomic step of reasoning.

Our plan in this paper is not to propose any new logical languages, but rather to use two systems studied in Pratt-Hartmann and Moss [5]. These are the languages $\mathcal{R}$ and $\mathcal{R}^*$ which we shall review in Section 1.1. These languages have "binary atoms" r which in $\mathcal{R}$ and $\mathcal{R}^*$ were taken to correspond to English transitive verbs (verbs taking direct objects). The difference between these is that $\mathcal{R}^*$ allows relative clauses in subject NP's, a point hinted at above. That is, the sentences in (1) are formalizable in $\mathcal{R}$, while those in (2) require the larger machinery of $\mathcal{R}^*$. In [5], binary atoms were interpreted as *arbitrary* relations on some domain. And [5] obtained logical completeness results for semantic interpretations where the verbs were interpreted as arbitrary relations. To pursue the topic of this paper, we therefore restrict attention to interpretations where binary atoms are required to be interpreted by *transitive and irreflexive* relations. This restriction means that more logical laws will be valid. The point of the paper is to say exactly which logical laws are valid in the new settings.

We find it convenient to separate transitivity from irreflexivity in our work. Accordingly, we study (i) $\mathcal{R}$ interpreted on transitive models (Sections 2.2 and 2.1), (ii) $\mathcal{R}^*$ on transitive models (Section 3.1), (iii) $\mathcal{R}^*$ on transitive and irreflexive models (Section 3.2), and polar opposites on top of everything else (Section 3.4).

Expression	Var's	Syntax	Example Glosses
unary atom	o, p, q		giraffe, lion, gnu
	x, y, z		
binary atom	r, s		taller than, shorter than
unary literal	l, m, n	$p \mid \bar{p}$	giraffe, non-giraffe
binary literal	t	$r \mid \bar{r}$	taller than, not taller than
positive c-term	b^+, c^+	$p \mid \exists(p, r) \mid \forall(p, r)$	shorter than all (some) giraffes
c-term	c, d	$l \mid \exists(p, t) \mid \forall(p, t)$	not smarter than all gnus
$\mathcal{R}$-formula	φ	$\exists(p, c) \mid \exists(c, p)$	Some/all/no lions are
		$\forall(p, c) \mid \forall(c, \bar{p})$	larger than all gnus
$\mathcal{R}^*$-formula	φ	$\exists(c^+, d) \mid \exists(d, c^+)$	Everything taller than some lion
		$\forall(c^+, d) \mid \forall(d, \overline{c^+})$	is (not) heavier than all hyenas

Figure 1: Syntax of the syllogistic fragments $\mathcal{R}$ and $\mathcal{R}^*$.

Incidentally, interpreting $\mathcal{R}$ on transitive and irrreflexive relations immediately gives an *axiom of infinity*: if every child is taller than some child, then there are infinitely many children. The model construction in Section 3.2 reflects this. To the best of my knowledge, this is the first time infinite models have turned up in the study of syllogistic systems.

We have not obtained all of the possible results on logical systems for comparative adjectives. It would of course be possible to study $\mathcal{R}$ on transitive and irreflexive models, and for that matter it would be possible to study the effect of irreflexivity alone. But we did not pursue any of these matters. We indicate a few open problems at the end, and also mention a consequence of this work for the way in which inference is thought about in natural language semantics.

Relation to other work As already mentioned, the particular logical systems studied in this paper derive from our earlier work with Ian Pratt-Hartmann [5]. We have tried to make this paper as self-contained as possible. At the same time, several results from [5] are quoted here. Further, several lemmas in this paper are minor variations on parallel results in [5], and we have not reproduced the proofs. So the interested reader would have to consult [5] for complete arguments at several points. The model construction in Section 2.2 is due to Ziegler [7]. Specifically, the relations in Figure 4 are essentially from [7].

1.1 Background on syllogistic languages and logics

The two logical languages studied in this paper are the languages $\mathcal{R}$ and $\mathcal{R}^*$ introduced in [5]. Their syntax is given in Figure 1. Although we shall use the same formal languages, we gloss them in English in a different way: whereas [5] took the binary atoms to represent English transitive verbs, in this paper we take them to be comparative adjective phrases.

Semantics A *model* $\mathfrak{M}$ is a pair $\langle M, [\![\ ]\!]\rangle$, where M is a non-empty set, $[\![p]\!] \subseteq M$ for all $p \in \mathbf{P}$, and $[\![r]\!] \subseteq M^2$ for all $r \in \mathbf{R}$. We call $[\![p]\!]$ the *interpretation of* p *in* $\mathfrak{M}$, and similarly for $[\![r]\!]$. We shall be especially interested in two classes of models: those in which all r are interpreted by transitive relations, and those in which all r are interpreted by transitive, irreflexive relations.

Given a model $\mathfrak{M}$, we extend the interpretation function $[\![\ ]\!]$ to the rest of the language by setting

$$
\begin{array}{lcl}
[\![\overline{p}]\!] & = & M \setminus [\![p]\!] \\
[\![\overline{r}]\!] & = & M^2 \setminus [\![r]\!] \\
[\![\exists(l,t)]\!] & = & \{x \in M : \langle x,y\rangle \in [\![t]\!] \text{ for some } y \in [\![l]\!]\} \\
[\![\forall(l,t)]\!] & = & \{x \in M : \langle x,y\rangle \in [\![t]\!] \text{ for all } y \in [\![l]\!]\}
\end{array}
$$

We define the truth relation $\models$ between models and formulas by: $\mathfrak{M} \models \forall(c,d)$ if and only if $[\![c]\!] \subseteq [\![d]\!]$, and $\mathfrak{M} \models \exists(c,d)$ if and only if $[\![c]\!] \cap [\![d]\!] \neq \emptyset$. If Γ is a set of formulas, we write $\mathfrak{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathfrak{M} \models \varphi$.

A formula φ is *satisfiable* if there exists $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$; satisfiability of a set of formulas Γ is defined similarly. We write $\Gamma \models \varphi$ to mean that every model $\mathfrak{M}$ satisfying all sentences in Γ also satisfies φ. Similarly, we say that φ is *satisfiable on a transitive model* (or *on an irreflexive model*) if there is some $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$ and for all $r \in \mathbf{R}$, $[\![r]\!]$ is transitive (or irreflexive).

The main goal of this paper is to provide a proof-theoretic characterization of the consequence relation for the semantics on transitive models, and for the consequence on transitive and irreflexive models. That is, write $\mathcal{M} \models \Gamma$ when $\mathcal{M} \models S$ for all $S \in \Gamma$. Also write $\Gamma \models S$ to mean that for all $\mathcal{M} \models \Gamma$, we also have $\mathcal{M} \models S$.

Identifications We identify the formulas $\exists(c,d)$ and $\exists(d,c)$. We also identify $\forall(c,d)$ and $\forall(\bar{d},\bar{c})$. These identifications are not really necessary, but they simplify matters at a number of points.

Absurdity, negation and identifications We extend the "bar" notation to from the atoms to the rest of the syntax in the obvious way. So $\overline{\overline{p}} = p$, $\overline{\forall(l,r)} = \exists(l,\overline{r})$, $\overline{\exists(l,r)} = \forall(l,\overline{r})$, $\overline{\forall(c,d)} = \exists(c,\overline{d})$, and $\overline{\exists(c,d)} = \forall(c,\overline{d})$. For any model $\mathfrak{M}$ and any c-term c, $[\![\overline{c}]\!] = M \setminus [\![c]\!]$; hence, $\mathfrak{M} \not\models \exists(c,\bar{c})$. We refer to a formula of this form as an *absurdity*. Moreover, for all sentences φ, $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \not\models \bar{\varphi}$.

Positive and negative sentences A sentence φ is *positive* if the literal which occurs in φ is a unary or binary atom; otherwise φ is *negative*. Notice that for all φ, either φ is positive and $\overline{\varphi}$ is negative, or vice-versa.

Syllogistic rules and *reductio ad absurdum* We shall not enter into a general discussion of syllogistic rules and how sets of these rules generate a *deduction relation* $\Gamma \vdash \varphi$ between sets of sentences and individual sentences. For this, see [5]. Instead, we present two examples which illustrate all of the ideas needed in this paper.

Figure 2 shows a finite set R of rules in $\mathcal{R}$. In these, p and q range over unary atoms, c over c-terms, and t over binary literals. The idea is that we may form instances of them by substituting terms of the same syntactic type, and then chain those instances together in derivations. More formally, we say that $\Gamma \vdash \varphi$ in our $\mathcal{R}$ (or any other system) if there is a tree with root labeled φ and whose leaves are labeled by sentences in Γ such that all interior nodes are instances of some rule in the system.

Here is an example. The derivation below shows that $\{\forall(x,\forall(q,r)), \exists(q,p), \forall(p,y)\} \vdash \forall(x,\exists(y,r))$.

$$\frac{\exists(p,q) \qquad \forall(q,c)}{\exists(p,c)}\ \text{(D1)} \qquad \frac{\forall(p,q) \qquad \forall(q,c)}{\forall(p,c)}\ \text{(B)}$$

$$\frac{\forall(p,q) \qquad \exists(p,c)}{\exists(q,c)}\ \text{(D2)} \qquad \frac{}{\forall(p,p)}\ \text{(T)} \quad \frac{\exists(p,c)}{\exists(p,p)}\ \text{(I)}$$

$$\frac{\forall(q,\bar{c}) \qquad \exists(p,c)}{\exists(p,\bar{q})}\ \text{(D3)} \qquad \frac{\forall(p,\bar{p})}{\forall(p,c)}\ \text{(A)} \quad \frac{\exists(p,\exists(q,t))}{\exists(q,q)}\ \text{(II)}$$

$$\frac{\forall(p,\forall(q',t)) \qquad \exists(q,q')}{\forall(p,\exists(q,t))}\ (\forall\forall) \qquad \frac{\exists(p,\exists(q,t)) \qquad \forall(q,q')}{\exists(p,\exists(q',t))}\ (\exists\exists)$$

$$\frac{\forall(p,\exists(q,t)) \qquad \forall(q,q')}{\forall(p,\exists(q',t))}\ (\forall\exists)$$

Figure 2: The system $\mathcal{R}$. In addition to the rules above, the system also uses *reductio ad absurdum*.

$$\frac{\dfrac{\forall(x,\forall(q,r)) \qquad \exists(q,p)}{\forall(x,\exists(p,r))}\ (\forall\forall) \qquad \forall(p,y)}{\forall(x,\exists(y,r))}\ (\forall\exists) \tag{4}$$

Second, here is an example of a derivation using *reductio ad absurdum*:

$$\frac{\dfrac{\forall(y,\overline{y}) \qquad \dfrac{[\exists(x,\exists(y,\overline{r}))]}{\exists(y,y)}\ \text{(II)}}{\bot}\ \text{(D1)}}{\forall(x,\forall(y,r))}\ \text{(RAA)} \tag{5}$$

This corresponds to an assertion that if there are no lions, then every mouse is taller than every lion. What is going on in the formal derivation above is that in the presence of the premise $\forall(y,\overline{y})$, the assumption $\exists(x,\exists(y,\overline{r}))$ is shown to lead to a contradiction $\bot = \exists(y,\overline{y})$. And so this additional assumption $\exists(x,\exists(y,\overline{r}))$ is *withdrawn* (as indicated by the outer brackets in $[\exists(x,\exists(y,\overline{r}))]$ in the derivation), and its negation $\forall(x,\forall(y,r))$ is inferred.

We should mention that there is a difference between a set of rules and the derivation relation it generates. But this difference is not significant in this paper, and in general we use the same name for the set of rules and its associated derivation relation.

2 Requiring Transitivity in $\mathcal{R}$

The first system in this paper is an extension of $\mathcal{R}$ which enforces the transitivity of the relation. (We say "the" relation, because in fragments like $\mathcal{R}$, every sentence has at most one binary

$$\frac{\forall(x,\forall(y,r)) \quad \exists(y,\forall(z,r))}{\forall(x,\forall(z,r))}\ (\rho 1) \qquad \frac{\forall(x,\forall(y,r)) \quad \exists(y,\exists(z,r))}{\forall(x,\exists(z,r))}\ (\rho 2)$$

$$\frac{\forall(x,\exists(y,r)) \quad \forall(y,\exists(z,r))}{\forall(x,\exists(z,r))}\ (\rho 3) \qquad \frac{\forall(x,\exists(y,r)) \quad \forall(y,\forall(z,r))}{\forall(x,\forall(z,r))}\ (\rho 4)$$

$$\frac{\exists(x,\forall(y,r)) \quad \exists(y,\forall(z,r))}{\exists(x,\forall(z,r))}\ (\rho 5) \qquad \frac{\exists(x,\forall(y,r)) \quad \exists(y,\exists(z,r))}{\exists(x,\exists(z,r))}\ (\rho 6)$$

$$\frac{\exists(x,\exists(y,r)) \quad \forall(y,\forall(z,r))}{\exists(x,\forall(z,r))}\ (\rho 7) \qquad \frac{\exists(x,\exists(y,r)) \quad \forall(y,\exists(z,r))}{\exists(x,\exists(z,r))}\ (\rho 8)$$

Figure 3: Additions to R to get the system $\mathsf{R}(\mathsf{tr})$.

atom. Thus by Craig Interpolation Theorem (for example), if $\Gamma \models \varphi$, then φ indeed follows semantically from the sentences in Γ wit the same binary atom (or no atom). So we might as well formulate $\mathcal{R}$ in terms of a *single* binary atom. None of these points hold for $\mathcal{R}^*$ studied below; in that system one may reason about several relations.)

2.1 $\mathsf{R}(\mathsf{tr})$

In Figure 3 we introduce a set of eight rules $\mathsf{R}(\mathsf{tr})$ which are easily seen to be sound for transitive interpretations. These all make intuitively clear points that reflect transitivity. For example, ($\rho 6$) may be interpreted to say that if some girl is taller than all boys, and some boy is taller than some teacher, then some girl is taller than some teacher.

We should mention that some variations on the rules in Figure 3 which appear to be sound really are unsound. For example, consider

$$\frac{\forall(x,\forall(y,r)) \quad \forall(y,\forall(z,r))}{\forall(x,\forall(z,r))}$$

If there is one x, one y, and no z, then the premises hold vacuously. But the conclusion fails when the x does not have the relation r to the z. Indeed worries about the emptiness of interpretations is one of the main complications in proofs in this subject.

We use $\mathsf{R}(\mathsf{tr})$ for the set of rules containing those in Figures 2 and 3, and we also use the same notation for the derivation relation determined by this set. (That is, we shall refer to the logical system by the same name as its set of rules.)

We shall prove the completeness result for $\mathsf{R}(\mathsf{tr})$ in Theorem 2.5 below. The reader may wish to look at is proof in the course of reading the work which leads up to it.

The first order of business is to check that the system can represent valid arguments such as that found in (1) and repeated below:

Every giraffe is taller than every gnu
Some gnu is taller than every lion
<u>Some lion is taller than some zebra</u>
Every giraffe is taller than some zebra

Here is the relevant derivation:

$$\dfrac{\dfrac{\forall(\mathsf{giraffe},\forall(\mathsf{gnu},\mathsf{taller})) \quad \exists(\mathsf{gnu},\forall(\mathsf{lion},\mathsf{taller}))}{\forall(\mathsf{giraffe},\forall(\mathsf{lion},\mathsf{taller}))}\,(\rho 1) \qquad \exists(\mathsf{lion},\exists(\mathsf{zebra}),\mathsf{taller})}{\forall(\mathsf{giraffe},\exists(\mathsf{zebra},\mathsf{taller}))}\,(\rho 2)$$

2.2 Model construction

We shall show that every set Γ of sentences which is consistent in $\mathsf{R}(\mathsf{tr})$ is satisfiable on a transitive model. By Lemma 2.2 of [5], we may assume that Γ is $\mathcal{R}$-complete; that is, for every sentence θ, either θ or $\overline{\theta}$ belongs to Γ.

For such a consistent set Γ, we shall define a model $\mathcal{M} = \mathcal{M}(\Gamma)$ as follows: we let

$$M \quad = \quad \{x_1, x_2 : \Gamma \vdash \exists(x,x)\} \ \cup \ \{\{p,q\} : p \neq q \text{ and } \Gamma \vdash \exists(p,q)\}.$$

So we have two copies of every variable x such that Γ entails the existence of x, and also pairs of the form $\{p,q\}$ corresponding to sentences of the form $\exists(p,q)$ which are provable from Γ and such that $p \neq q$. Our semantics will insure that the element $\{p,q\}$ belongs to $[\![p]\!] \cap [\![q]\!]$, and so this element will witness the truth of $\exists(p,q)$ in the model which we are constructing.) We always assume that the union above is a disjoint union. Also, we call the elements $\{p,q\}$ *pair-elements*.

The unary variables x are interpreted in our models as follows:

$$\begin{array}{ll} w_i \in [\![x]\!] & \text{iff } \Gamma \vdash \forall(w,x) \\ \{p,q\} \in [\![x]\!] & \text{iff } \Gamma \vdash \forall(p,x), \text{ or } \Gamma \vdash \forall(q,x) \end{array}$$

For the binary variable, we first need a definition. For two variables u and w (possibly identical) we shall need a certain set

$$R^{\Gamma}_{x,y} \subseteq \{x_1,x_2\} \times \{y_1,y_2\}.$$

We present the definition in Figure 4.

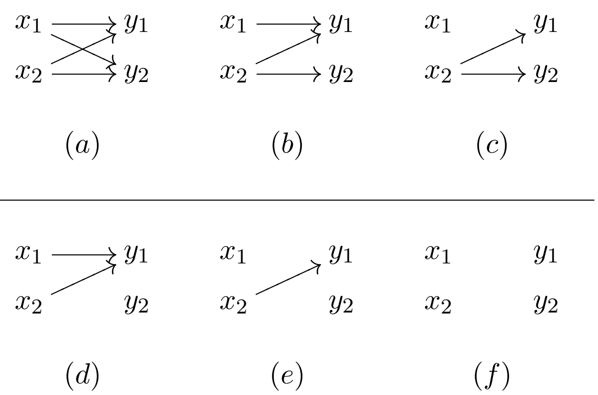
We return to the definition of the models $\mathcal{M}(\Gamma)$. The semantics $[\![r]\!]$ of a binary atom r is given as follows:

$$\begin{array}{lll} x_i [\![r]\!] y_j & \text{iff} & x_i \to y_j \text{ according to } R^{\Gamma}_{x,y} \text{ in Figure 4} \\ \{x,y\}[\![r]\!] w_2 & \text{iff} & \Gamma \vdash \forall(x,\forall(w,r)) \text{ or } \Gamma \vdash \forall(x,\forall(y,r)) \\ \{x,y\}[\![r]\!] w_1 & \text{iff} & \{x,y\}[\![r]\!] w_2, \text{ or } \Gamma \vdash \forall(x,\exists(w,r)), \text{ or } \Gamma \vdash \forall(y,\exists(w,r)) \\ u_1 [\![r]\!] \{x,y\} & \text{iff} & \Gamma \vdash \forall(u,\forall(x,r)) \text{ or } \Gamma \vdash \forall(u,\forall(y,r)) \\ u_2 [\![r]\!] \{x,y\} & \text{iff} & u_1[\![r]\!]\{x,y\}, \text{ or } \Gamma \vdash \exists(u,\forall(x,r)), \text{ or } \Gamma \vdash \exists(u,\forall(y,r)) \\ \{x,y\}[\![r]\!]\{p,q\} & \text{iff} & \Gamma \vdash \forall(x,\forall(p,r)), \text{ or } \Gamma \vdash \forall(x,\forall(q,r)), \text{ or} \\ & & \Gamma \vdash \forall(y,\forall(p,r)), \text{ or } \Gamma \vdash \forall(y,\forall(q,r)) \end{array}$$

Proposition 2.1 *Concerning the relations* $R^{\Gamma}_{u,w}$*:*

1. $x_1[\![r]\!]y_2$ *iff* $\Gamma \vdash \forall(x,\forall(y,r))$.

2. $x_1[\![r]\!]y_1$ *iff* $\Gamma \vdash \forall(x,\forall(y,r))$ *or* $\Gamma \vdash \forall(x,\exists(y,r))$.



Use (a) when $\Gamma \vdash \forall(x, \forall(y, r))$;
(b) when $\Gamma \nvdash \forall(x, \forall(y, r))$, but $\Gamma \vdash \forall(x, \exists(y, r))$, and $\Gamma \vdash \exists(x, \forall(y, r))$;
(c) when $\Gamma \nvdash \forall(x, \forall(y, r))$, but $\Gamma \vdash \exists(x, \forall(y, r))$ and $\Gamma \nvdash \forall(x, \exists(y, r))$;
(d) when $\Gamma \nvdash \forall(x, \forall(y, r))$, but $\Gamma \vdash \forall(x, \exists(y, r))$ and $\Gamma \nvdash \exists(x, \forall(y, r))$;
(e) when $\Gamma \nvdash \forall(x, \exists(y, r))$ and $\Gamma \nvdash \exists(x, \forall(y, r))$, but $\Gamma \vdash \exists(x, \exists(y, r))$;
and (f) when none of (a)–(e) hold.

Figure 4: The relation $R^{\Gamma}_{x,y}$.

3. $x_2 [\![r]\!] y_2$ *iff* $\Gamma \vdash \forall(x, \forall(y, r))$ *or* $\Gamma \vdash \exists(x, \forall(y, r))$.

4. $x_2 [\![r]\!] y_1$ *iff for some* i *and* j, $x_i [\![r]\!] y_j$.

Lemma 2.2 (Transitivity) *Let* Γ *be complete. Then* $[\![r]\!]$ *is transitive in* $\mathcal{M}(\Gamma)$.

Proof We are first going to check the transitivity of $[\![r]\!]$ when restricted to the elements x_i; after this we shall consider also the pair-elements $\{p, q\}$.

The idea is to look at all of the possible ways that instances of our six types of diagrams in Figure 4 can appear next to each other. We need not worry about (f), but we still must examine $5 \times 5 = 25$ cases. These are shown in the table below:

	(a)	(b)	(c)	(d)	(e)
(a)	$\rho_4, \exists$	$\rho_3, \rho_5, \exists$	ρ_1	$\rho_3, \exists$	ρ_2
(b)	ρ_4	ρ_3, ρ_5	ρ_5	ρ_3	ρ_6
(c)	$\rho_5, \exists$	ρ_5	ρ_5	$\rho_6, \exists$	ρ_6
(d)	ρ_4	ρ_3	$-$	ρ_3	$-$
(e)	ρ_8, ρ_7	ρ_8	$-$	ρ_8	$-$

The entry in the ith row and jth column indicates which rule(s) are needed in the verification of transitivity when an instance of the ith letter of the alphabet is placed next to an instance of the $j$the letter. A dash $-$ in the table means that no instances of transitivity arise. For the other entries, it is best to give some examples; these also explain the notation. Consider first an instance of (c) to the left of an instance of (b), shown on the left below:

$$\begin{tikzcd} x_1 & y_1 \arrow[r] & z_1 \\ x_2 \arrow[ur] \arrow[r] & y_2 \arrow[ur] \arrow[r] & z_2 \end{tikzcd} \qquad \begin{tikzcd} x_1 & y_1 \arrow[r] & z_1 \\ x_2 \arrow[ur] \arrow[r] & y_2 \arrow[ur] & z_2 \end{tikzcd} \tag{6}$$

We need to check that $x_2 \to z_1$ and $x_2 \to z_2$. But we have $\Gamma \vdash \exists(x, \forall(y, r))$; and also $\Gamma \vdash \exists(y, \forall(z, r))$. We use ($\rho$5) to see that $\Gamma \vdash \exists(x, \forall(z, r))$. This implies that $x_2 \to z_1$ and $x_2 \to z_2$, as desired.

Next, consider an instance of (c) to the left of an instance of (d); this is shown in the right of (6) above. This time, $\Gamma \vdash \exists(x, \forall(y, r))$ and also $\Gamma \vdash \forall(y, \exists(z, r))$. It does *not* follow from these alone that $\Gamma \vdash \exists(x, \exists(z, r))$. However, since the y points belong to the model, $\Gamma \vdash \exists(y, y)$. So $\Gamma \vdash \exists(y, \exists(z, r))$. Now we may use ($\rho$6) to see that $\Gamma \vdash \exists(x, \exists(z, r))$. (In the chart, the steps of the argument where we need to know that Γ derives the existence of some object are indicated with a $\exists$ symbol.)

We have only given two of the 25 cases, but the rest are very similar. It remains to verify transitivity when one or two pair-elements $\{p, q\}$ are involved. Here is one of the many similar verifications. Suppose that $u_2 \to \{p, q\} \to w_1$. There are several ways that this could happen, and to be concrete, let us assume that from Γ we have $\exists(u, \forall(z, r))$,$\exists(p, q)$, and $\forall(q, \exists(w, r))$. From the first two of these, we have $\exists(u, \exists(q, r))$. Then with the last we indeed have $\exists(u, \forall(w, r))$. This means that $u_2 \to w_1$, just as desired.

All of the remaining points are similar. $\dashv$

Lemma 2.3 *Let Γ be an arbitrary set of $\mathcal{R}$-sentences. Let φ be a positive sentence in Γ. Then $\mathcal{M}(\Gamma) \models \varphi$.*

Proof We argue by cases on φ.

Case 1: φ is $\forall(x, y)$. If $z_i \in [\![x]\!]$, then $z \leq x$. By monotonicity, $z \leq y$. So $z_i \in [\![y]\!]$. If $\{p, q\} \in [\![x]\!]$, then without loss of generality $\forall(p, x)$. Again, we see that $\{p, q\} \in [\![y]\!]$.

Case 2: φ is $\exists(x, y)$. This time $\{x, y\}$ is an element of our model. Our logic contains the identity axioms *All x are x*. By our semantics, $\{x, y\} \in [\![x]\!] \cap [\![y]\!]$. Thus the model overall satisfies *Some x are y*.

Case 3: φ is $\forall(x, \forall(y, r))$. Let $z_i \in [\![x]\!]$ and $w_j \in [\![y]\!]$; so we have $z \leq x$ and $w \leq y$. By monotonicity, $\Gamma \vdash \forall(z, \forall(w, r))$. So $z_i [\![r]\!] w_j$ for $1 \leq i, j \leq 2$. we also must consider pair-elements $\{p, q\} \in [\![x]\!]$. Without loss of generality, assume $\forall(p, x)$. By monotonicity, $\Gamma \vdash \forall(p, \forall(w, r))$. Again, our semantics insures that $\{p, q\} [\![r]\!] w_j$ for all j. There are two other cases are argued similarly; these show that if $z_i \in [\![x]\!]$ and $\{u, w\} \in [\![y]\!]$, then $z_i [\![r]\!] \{u, w\}$; also, if $\{p, q\} \in [\![x]\!]$ and $\{u, w\} \in [\![y]\!]$, then $\{p, q\} [\![r]\!] \{u, w\}$.

Case 4: φ is $\forall(x, \exists(y, r))$. In this case, we can assume that $[\![x]\!] \neq \emptyset$. That is, $\Gamma \vdash \exists(x, x)$. Then $\Gamma \vdash \exists(y, y)$ as well. we shall show that every element of $[\![x]\!]$ is related to y_1. Let $z_i \in [\![x]\!]$, so that $z \leq x$. By monotonicity, $\Gamma \vdash \forall(z, \exists(y, r))$. Then by Proposition 2.1, parts (2) and (4), we indeed have $z_i [\![r]\!] y_1$ and $z_i [\![r]\!] y_2$. Further, let $\{p, q\} \in [\![x]\!]$. Without loss of generality, $p \leq x$. By monotonicity, $\Gamma \vdash \forall(p, \exists(y, r))$. Then by the definition of $[\![r]\!]$, $\{p, q\} [\![r]\!] y_1$.

Case 5: φ is $\exists(x, \forall(y, r))$. By rule (I) of our logic, $\Gamma \vdash \exists(x, x)$. Let $w_j \in [\![y]\!]$. Then $\Gamma \vdash \forall(w, y)$. Hence $\Gamma \vdash \exists(x, \forall(w, r))$. By Proposition 2.1, parts (3) and (4), $x_2 [\![r]\!] w_1$, and also $x_2[\![r]\!]w_2$. We must also consider pair-elements of $[\![y]\!]$. Let $\{p,q\} \in [\![y]\!]$ so that $\Gamma \vdash \exists(p,q)$; and assume $\Gamma \vdash \forall(p,y)$. By monotonicity, $\Gamma \vdash \exists(x, \forall(p,r))$. By construction, $x_2[\![r]\!]\{p,q\}$. We conclude that x_2 is the required witness to $\exists(x, \forall(y,r))$.

Case 6: φ is $\exists(x, \exists(y,r))$. Here both $\Gamma \vdash \exists(x,x)$ and also $\exists(y,y)$. By Proposition 2.1, part (4), $x_2[\![r]\!]y_1$. So $\mathcal{M} \models \exists(x, \exists(y,r))$. ⊣

Lemma 2.4 *Let Γ be complete and consistent. Let φ be a positive sentence. If $\mathcal{M}(\Gamma) \models \varphi$, then $\varphi \in \Gamma$.*

Proof We argue by cases on φ. In each case, we assume that $\mathcal{M}(\Gamma) \models \varphi$, and we then show $\Gamma \vdash \varphi$. Since Γ is complete, we indeed have $\varphi \in \Gamma$.

One fact which we shall use frequently is that if $[\![x]\!] \neq \emptyset$ in $\mathcal{M}(\Gamma)$, then $\Gamma \vdash \exists(x,x)$. For if $y_j \in [\![x]\!]$, they by the structure of the model, $\Gamma \vdash \exists(y,y)$ and also $\forall(y,x)$. Similar considerations apply to a pair-element $\{u,w\} \in [\![x]\!]$.

Case 1: φ is $\forall(x,y)$. We may assume that $\Gamma \vdash \exists(x,x)$; if not, then $\Gamma \vdash \varphi$ using (A). And then the structure of the model easily tells us that $\Gamma \vdash \varphi$.

Case 2: φ is $\exists(x,y)$. The argument is very close to what we do concerning $\exists(x, \exists(y,r))$ in Case 6 below.

Case 3: φ is $\forall(x, \forall(y,r))$. By completeness, either $\Gamma \vdash \forall(x, \overline{x})$; or $\Gamma \vdash \forall(y, \overline{y})$; or else both $\Gamma \vdash \exists(x,x)$ and $\Gamma \vdash \exists(y,y)$. In the first case, $\Gamma \vdash \varphi$ using the rule (A). In the second case, $\Gamma \vdash \varphi$ as we have seen in (5). In the last, consider $\mathcal{M} = \mathcal{M}(\Gamma)$. By Lemma 2.3, $\mathcal{M}(\Gamma) \models S$. In $\mathcal{M}$, $x_1 \in [\![x]\!]$ and $y_2 \in [\![y]\!]$. Since $\mathcal{M} \models \varphi$, $x_1[\![r]\!]y_2$. By Proposition 2.1, part (1), $\Gamma \vdash \forall(x, \forall(y,r))$.

Case 4: φ is $\forall(x, \exists(y,r))$. As in the previous case, we may assume that $\Gamma \vdash \exists(x,x)$. Consider $\mathcal{M} = \mathcal{M}(\Gamma)$. In the model, $[\![x]\!] \neq \emptyset$ by definition of the model, and because $\mathcal{M} \models \varphi$, the same is true of y. In particular, x_1 is related to some element of $[\![y]\!]$. Say $x_1[\![r]\!]z_j$ where $z \leq y$. If $j = 1$, then by Proposition 2.1, part (2), we $\Gamma \vdash \forall(x, \exists(z,r))$ or $\Gamma \vdash \forall(x, \forall(z,r))$. In the first case, we are done by monotonicity. So we shall assume that $\Gamma \vdash \forall(x, \forall(z,r))$. Since z_j belongs to the model $\Gamma \vdash \exists(z,z)$. Therefore $\Gamma \vdash \forall(x, \exists(z,r))$, and as above we are done.

Now if $j = 2$, then by Proposition 2.1, part (1), we have $\Gamma \vdash \forall(x, \forall(z,r))$. Exactly as above, we reason that $\Gamma \vdash \forall(x, \exists(y,r))$.

The last possibility is that $x_1[\![r]\!]\{p,q\}$, where $\Gamma \vdash \exists(p,q)$. Without loss of generality, suppose that $\Gamma \vdash \forall(x, \forall(q,r))$ and also that $\Gamma \vdash \forall(p,y)$. The derivation in (4) in Section 1.1 shows that $\Gamma \vdash \forall(x, \exists(y,r))$.

Case 5: φ is $\exists(x, \forall(y,r))$. In this case, $\mathcal{M} \models \exists(x,x)$, and so $\Gamma \vdash \exists(x,x)$. We may assume that $\Gamma \vdash \exists(y,y)$, since otherwise we get $\Gamma \vdash \varphi$ as in Case 3.

Some element of $[\![x]\!]$ is related to y_2. In particular, $\Gamma \vdash \exists(x,x)$. Suppose first that $x_1[\![r]\!]y_2$. By Proposition 2.1, part (1), $\Gamma \vdash \forall(x, \forall(y,r))$. And as $\Gamma \vdash \exists(x,x)$, we have $\Gamma \vdash \exists(x, \forall(y,r))$.

Suppose next that it is $x_2 \in [\![x]\!]$ which is related by $[\![r]\!]$ to y_2. By Proposition 2.1, part (3), we have two alternatives: $\Gamma \vdash \exists(x, \forall(y,r))$ (and we are done); or else $\Gamma \vdash \forall(x, \forall(y,r))$. In this last case, we have already seen $\Gamma \vdash \exists(x,x)$, and we now have $\Gamma \vdash \exists(x, \forall(y,r))$.

Finally, suppose that $\{p,q\} \in [\![x]\!]$ and also that $\{p,q\}[\![r]\!]y_2$. Either $\Gamma \vdash \forall(p, \forall(y,r))$ or $\Gamma \vdash \forall(q, \forall(y,r))$. Since this pair-element $\{p,q\}$ belongs to our model, $\Gamma \vdash \exists(p,q)$. So either

$\Gamma \vdash \exists(p, \forall(y,r))$ or $\Gamma \vdash \exists(q, \forall(y,r))$. But also, either $p \leq x$ or $q \leq x$. Without loss of generality, $p \leq x$. By monotonicity, $\Gamma \vdash \exists(x, \forall(y,r))$.

Case 6: φ is $\exists(x, \exists(y,r))$. In our final case, we must have $\Gamma \vdash \exists(x,x)$; also $\Gamma \vdash \exists(y,y)$.

Suppose that $z_1 \in [\![x]\!]$ and $w_1 \in [\![y]\!]$ are related by $[\![r]\!]$. Thus $z \leq x$ and $w \leq y$. By Proposition 2.1, part (2), $\Gamma \vdash \forall(z, \exists(w,r))$. Since z_1 belongs to our model, $\Gamma \vdash \exists(z,z)$. Thus $\Gamma \vdash \exists(z, \exists(w,r))$. And by monotonicity again, $\Gamma \vdash \exists(x, \exists(y,r))$.

Next, suppose that $z_1 \in [\![x]\!]$ and $w_2 \in [\![y]\!]$ are related by $[\![r]\!]$. The work here is quite similar, and we omit all the details. The same goes for the case of $z_2 \in [\![x]\!]$ and $w_1 \in [\![y]\!]$, and also for the case of $z_2 \in [\![x]\!]$ and $w_2 \in [\![y]\!]$. There are several more cases, owing to the possibility that the witnesses to $\exists(x, \exists(y,r))$ might include pair-elements. These are all routine, and we omit these details.

This concludes the proof. $\dashv$

Theorem 2.5 *For $\Gamma \cup \{\varphi\} \subseteq \mathcal{R}$, $\Gamma \models \varphi$ on transitive models iff $\Gamma \vdash \varphi$ in* $\mathsf{R}(\mathsf{tr})$.

Proof The soundness is an easy induction on derivations. For the completeness, we need only show that a consistent set Γ is satisfiable. By Lemma 2.2 of [5], we may assume that Γ is $\mathcal{R}$-complete. Consider $\mathcal{M} = \mathcal{M}(\Gamma)$ as defined in Section 2.2. By Lemma 2.3, $\mathcal{M}$ satisfies the positive sentences in Γ. We claim that $\mathcal{M}$ satisfies the negative sentences in Γ as well. For suppose that ψ is positive and $\overline{\psi}$ belongs to Γ. If $\mathcal{M} \not\models \overline{\psi}$, we would have $\mathcal{M} \models \psi$. By Lemma 2.3, $\Gamma \vdash \psi$. But then Γ is inconsistent, a contradiction. The claim shown, we now see that $\mathcal{M} \models \Gamma$. $\dashv$

3 Requiring Transitivity and Irreflexivity in $\mathcal{R}^*$

Section 2 studied what happens when one requires the interpretations of binary atoms in $\mathcal{R}$ to be interpreted by transitive relations. In this section, we carry out the same study for the system $\mathcal{R}^*$ shown in Figure 6. This system was introduced in Pratt-Hartmann and Moss [5]; it was the strongest logical system in the paper.

3.1 $\mathsf{R}^*(\mathsf{tr})$

The requirement of transitivity adds the additional set of rules shown in Figure 5. We use $\mathsf{R}^*(\mathsf{tr})$ for the set of rules in Figures 5 and 6, and we use the same symbol for the logical system associated to it.

The rules are similar to the ones we have seen in $\mathsf{R}(\mathsf{tr})$, except that the subject noun phrases are allowed to have relative clauses in the conclusion. More formally, the fragment $\mathcal{R}^*$ is larger and so the rules are simpler. As we'll see shortly, $\mathsf{R}^*(\mathsf{tr})$ derives the rules of $\mathsf{R}(\mathsf{tr})$.

Here is an example of a derivation, corresponding to the informal example presented in (2) in the Introduction:

$$\dfrac{\dfrac{\forall(\mathsf{hyena}, \exists(\mathsf{jackal}, \mathsf{taller}))}{\forall(\exists(\mathsf{hyena}, \mathsf{taller}), \exists(\mathsf{jackal}, \mathsf{taller}))}\,(\mathrm{tr1}) \quad \forall(\exists(\mathsf{jackal}, \mathsf{taller}), \forall(\mathsf{warthog}, \overline{\mathsf{heavier}}))}{\forall(\exists(\mathsf{hyena}, \mathsf{taller}), \forall(\mathsf{warthog}, \overline{\mathsf{heavier}}))}\,(\mathrm{B})$$

$$\frac{}{\forall(c^+,c^+)}\ (\mathrm{T}) \qquad \frac{\exists(c^+,d)}{\exists(c^+,c^+)}\ (\mathrm{I}) \qquad \frac{\forall(b^+,c^+) \qquad \forall(c^+,d)}{\forall(b^+,d)}\ (\mathrm{B})$$

$$\frac{\exists(b^+,c^+) \qquad \forall(c^+,d)}{\exists(b^+,d)}\ (\mathrm{D1}) \qquad \frac{\forall(b^+,c^+) \qquad \exists(b^+,d)}{\exists(c^+,d)}\ (\mathrm{D2})$$

$$\frac{\forall(p,q)}{\forall(\forall(q,r),\forall(p,r))}\ (\mathrm{J}) \quad \frac{\forall(p,q)}{\forall(\exists(p,r),\exists(q,r))}\ (\mathrm{K}) \quad \frac{\exists(p,q)}{\forall(\forall(p,r),\exists(q,r))}\ (\mathrm{L})$$

$$\frac{\exists(q,\exists(p,r))}{\exists(p,p)}\ (\mathrm{II}) \qquad \frac{\forall(p,\bar{p})}{\forall(c^+,\forall(p,r))}\ (\mathrm{Z}) \qquad \frac{\forall(p,\bar{p})}{\exists((\forall(p,r),\forall(p,r))}\ (\mathrm{W})$$

Figure 5: The logical system R^*

$$\frac{\forall(p,\exists(q,r))}{\forall(\exists(p,r),\exists(q,r))}\ (\mathrm{tr1}) \qquad \frac{\forall(p,\forall(q,r))}{\forall(\exists(p,r),\forall(q,r))}\ (\mathrm{tr2})$$

$$\frac{\exists(p,\forall(q,r))}{\forall(\forall(p,r),\forall(q,r))}\ (\mathrm{tr3}) \qquad \frac{\exists(p,\exists(q,r))}{\forall(\forall(p,r),\exists(q,r))}\ (\mathrm{tr4})$$

Figure 6: The four transitivity rules. $\mathsf{R}^*(\mathsf{tr})$ is R^* together with these rules.

The application of (tr1) corresponds to using the premise every hyena is taller than some jackal to derive everything which is taller than some hyena is taller than some jackal. The second step corresponds to the transitivity of predication (is).

Our completeness result is that the logical system $\mathsf{R}^*(\mathsf{tr})$ is complete for the class of transitive models. The proof is a modification of the corresponding result for $\mathcal{R}^*$ in [5]. That result showed that $\mathcal{R}^*$ is complete for the class of *all* models. Rather than reproduce all of the details, we review the main idea and quote without proofs two lemmas that are In the sequel, the variables b^+, c^+ range over positive c-terms, and d over the larger class of all c-terms. As earlier, p, q, x, y, and z range over unary atoms and r over binary atoms.

The relation between $\mathsf{R}^*(\mathsf{tr})$ and $\mathsf{R}(\mathsf{tr})$ It was noted in [5] that R^* derives all of the rules in R. This observation extends to the systems in this paper. Specifically, the eight rules in Figure 3 are all derivable in $\mathsf{R}^*(\mathsf{tr})$. Here is one example, a derivation of (ρ3):

$$\frac{\forall(x,\exists(z,r)) \qquad \dfrac{\forall(y,\exists(z,r))}{\forall(\exists(y,r),\exists(z,r))}\ (\mathrm{tr1})}{\forall(x,\exists(z,r))}\ (\mathrm{B}) \tag{7}$$

The derivations of all the other rules in Figure 3 are similar.

We should also add that $\mathsf{R}^*(\mathsf{tr})$ derives versions of (ρ1)–(ρ8) in which the subject noun phrase might contain a relative clause; that is, where x in (7) is replaced by any c-term.

Theorem 3.1 *For $\Gamma \cup \{\varphi\} \subseteq \mathcal{R}^*$, $\Gamma \models \varphi$ on transitive models iff $\Gamma \vdash \varphi$ in* $\mathsf{R}(\mathsf{tr})$.

The rest of this section is devoted to the proof of Theorem 3.1. It is based on the proof from [5] of the same result for $\mathcal{R}^*$. Fortunately, the same proof idea works. (This contrasts with our work earlier in the paper; the constructions in Section 2 were new, and they made for a much longer argument than what we shall see below.) We need only show that every consistent set Γ in $\mathsf{R}^*(\mathsf{tr})$ is satisfiable on a transitive model. Also, by Lemma 2.2 of [5], we may assume that Γ is $\mathsf{R}^*(\mathsf{tr})$-complete; that is, every sentence or its negation belongs to Γ.

We construct a model $\mathfrak{M}$ and prove that it satisfies Γ. Since we need the details on this, we must review the construction. First, let $\mathbf{C}^+$ be the set of positive c-terms. Then we define $\mathfrak{M}$ by setting:

$$\begin{array}{rcl} M & = & \{\langle c_1, c_2, Q\rangle \in \mathbf{C}^+ \times \mathbf{C}^+ \times \{\forall, \exists\} : \Gamma \vdash \exists(c_1, c_2)\} \\ [\![p]\!] & = & \{\langle c_1, c_2, Q\rangle \in M : \Gamma \vdash \forall(c_1, p) \text{ or } \Gamma \vdash \forall(c_2, p)\} \end{array}$$

and $\langle c_1, c_2, Q_1\rangle [\![r]\!] \langle d_1, d_2, Q_2\rangle$ if and only if either:

(a) for some i, j, and $q \in \mathbf{P}$, $\Gamma \vdash \forall(c_i, \forall(q, r))$ and $\Gamma \vdash \forall(d_j, q)$; or

(b) $Q_2 = \exists$, and for some i and $q \in \mathbf{P}$, $d_1 = d_2 = q$, and $\Gamma \vdash \forall(c_i, \exists(q, r))$.

One then checks easily that A is non-empty (using (W)). The main work is given by two lemmas. The first shows that for all $c \in \mathbf{C}^+$,

$$[\![c]\!] = \{\langle d_1, d_2, Q\rangle \in M : \text{either } \Gamma \vdash \forall(d_1, c), \text{ or } \Gamma \vdash \forall(d_2, c)\}.$$

The second uses the first to show that $\mathfrak{M} \models \Gamma$. These two results show that every consistent set in R^* is satisfiable; this is the completeness of $\mathcal{R}^*$.

Extra rules insure transitivity At this point, we return to the topic of this section, the logic of the same fragment $\mathcal{R}^*$ when interpreted on transitive relations. This additional requirement leads to some new rules, as shown in Figure 6. We write $\mathsf{R}^*(\mathsf{tr})$ for the logical system consisting of R from Figure 5, together with the extra rules in Figure 6, one for each binary atom. To show the completeness of $\mathsf{R}^*(\mathsf{tr})$ for transitive models, we follow the same strategy as for R. Further, we use *the very same model construction*. We only need to show that the interpretation of each r is transitive. This is the content of Lemma 3.4.

Lemma 3.2 *For all $c \in \mathbf{C}^+$,*

$$[\![c]\!] = \{\langle d_1, d_2, Q\rangle \in M : \textit{either } \Gamma \vdash \forall(d_1, c), \textit{ or } \Gamma \vdash \forall(d_2, c)\}.$$

Lemma 3.3 $\mathfrak{M} \models \Gamma$.

These results are taken directly from [5], and so we shall not reproduce the proofs. We might mention that all of the $\mathcal{R}^*$ rules from Figure 5 are used in them.

Lemma 3.4 *The interpretation of each binary atom r is transitive.*

Proof Assume that

$$\langle b_1, b_2, Q_1\rangle \ [\![r]\!] \ \langle c_1, c_2, Q_2\rangle \ [\![r]\!] \ \langle d_1, d_2, Q_3\rangle.$$

We shall use several times the fact that since $\langle c_1, c_2, Q_2\rangle$ belongs to the model, $\Gamma \vdash \exists(c_1, c_2)$. We have four cases.

Case 1: for some i, j, k, l, and $q_1, q_2 \in \mathbf{P}$, $\Gamma \vdash \forall(b_i, \forall(q_1, r))$, $\Gamma \vdash \forall(c_j, q_1)$, $\Gamma \vdash \forall(c_k, \forall(q_2, r))$, and $\Gamma \vdash \forall(d_l, \forall(q_2, r))$. We have $\Gamma \vdash \exists(c_j, c_k)$ also. Now we have derivation from Γ:

$$\dfrac{\begin{array}{c}\vdots\\ \forall(b_i,\forall(q_1,r))\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall(c_k,\forall(q_2,r))\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \forall(c_j,q_1)\end{array} \quad \begin{array}{c}\vdots\\ \exists(c_j,c_k)\end{array}}{\exists(q_1,c_k)}}{\exists(q_1,\forall(q_2,r))}}{\forall(\forall(q_1,r),\forall(q_2,r))}\ (\text{tr3})}{\forall(b_i,\forall(q_2,r))}\ (\text{B})$$

And now we see that $\langle b_1, b_2, Q_1\rangle [\![r]\!] \langle d_1, d_2, Q_3\rangle$, using alternative (a) in the definition of $[\![r]\!]$.

Case 2: for some i, j, and $q_1 \in \mathbf{P}$, $\Gamma \vdash \forall(b_i, \forall(q_1, r))$ and $\Gamma \vdash \forall(c_j, q_1)$; $Q_3 = \exists$, and for some k and $q_2 \in \mathbf{P}$, $d_1 = d_2 = q_2$, and $\Gamma \vdash \forall(c_k, \exists(q_2, r))$. This time, we have $\Gamma \vdash \exists(q_1, c_k)$, and we also have the following from Γ:

$$\dfrac{\begin{array}{c}\vdots\\ \forall(b_i,\forall(q_1,r))\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall(c_k,\exists(q_2,r))\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \forall(c_j,q_1)\end{array} \quad \begin{array}{c}\vdots\\ \exists(c_j,c_k)\end{array}}{\exists(q_1,c_k)}}{\exists(q_1,\exists(q_2,r))}}{\forall(\forall(q_1,r),\exists(q_2,r))}\ (\text{tr4})}{\forall(b_i,\exists(q_2,r))}\ (\text{B})$$

In this case, alternative (b) shows that $\langle b_1, b_2, Q_1\rangle [\![r]\!] \langle d_1, d_2, Q_3\rangle$.

Case 3: $Q_2 = \exists$, and for some i and $q_1 \in \mathbf{P}$, $c_1 = c_2 = q_1$, and $\Gamma \vdash \forall(b_i, \exists(q_1, r))$; and also for some j, k, and $q_2 \in \mathbf{P}$, $\Gamma \vdash \forall(c_j, \forall(q_2, r))$ and $\Gamma \vdash \forall(d_k, q_2)$. Thus c_j at the end is the same as q_1.

$$\dfrac{\begin{array}{c}\vdots\\ \forall(b_i,\exists(q_1,r))\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \forall(q_1,\forall(q_2,r))\end{array}}{\forall(\exists(q_1,r),\forall(q_2,r))}\ (\text{tr2})}{\forall(b_i,\forall(q_2,r))}\ (\text{B})$$

We again finish the case with an application of (a).

Case 4: $Q_2 = \exists$, and for some i and $q_1 \in \mathbf{P}$, $c_1 = c_2 = q_1$, and $\Gamma \vdash \forall(b_i, \exists(q_1, r))$; and also $Q_3 = \exists$, and for some j and $q_2 \in \mathbf{P}$, $d_1 = d_2 = q_2$, and $\Gamma \vdash \forall(c_l, \exists(q_2, r))$.

$$\dfrac{\begin{array}{c}\vdots\\ \forall(b_i,\exists(q_1,r))\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \forall(q_1,\exists(q_2,r))\end{array}}{\forall(\exists(q_1,r),\exists(q_2,r))}\ (\text{tr1})}{\forall(b_i,\exists(q_2,r))}\ (\text{B})$$

In this case, we use (b) to see that $\langle b_1, b_2, Q_1\rangle [\![r]\!] \langle d_1, d_2, Q_3\rangle$.

This completes the proof. $\dashv$

And with the work of Lemma 3.4 done, we have also completed the proof of Theorem 3.1.

3.2 Irreflexivity: the system $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$

We now continue the study of the syllogistic logic of comparative adjectives by adding a requirement to the semantics that the interpretation of each binary atom be irreflexive. Notice that as soon as we add this requirement, the logic no longer has the finite model property. Specifically, $\forall(p, \exists(p, r))$ is satisfiable, but not by any finite (transitive and irreflexive) model. (In Section 3.3 below, we discuss what happens if we want to restrict to transitive, irreflexive, and *finite* models.)

The additional requirement also results in the soundness of the following *irreflexivity axiom*

$$\overline{\forall(p, \exists(p, \overline{r}))} \text{ (Irr)}$$

Let $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$ be the logical system whose rules are those of Figures 5 and 6, and also (Irr). The main work of this section is the completeness of $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$ for transitive and irreflexive models.

Theorem 3.5 *For $\Gamma \cup \{\varphi\} \subseteq \mathcal{R}^*$, $\Gamma \models \varphi$ on transitive, irreflexive models iff $\Gamma \vdash \varphi$ in $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$.*

Towards this end, let Γ be complete and consistent in $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$. Define a model $\mathcal{M} = \mathcal{M}(\Gamma)$ in several steps, as follows: First, let

$$\begin{array}{rcl} M & = & \{\langle c_1, c_2, n\rangle \in \mathbf{C}^+ \times \mathbf{C}^+ \times N : \Gamma \vdash \exists(c_1, c_2), \\ & & \text{and if } n \geq 0, \text{ then } c_1 = c_2, \text{ and } c_1 \text{ is a unary atom } \} \\ [\![p]\!] & = & \{\langle c_1, c_2, n\rangle \in M : \Gamma \vdash \forall(c_1, p) \text{ or } \Gamma \vdash \forall(c_2, p)\} \\ & & \cup \ \{\langle q, n\rangle \in M : \Gamma \vdash \forall(q, p)\} \end{array} \quad (8)$$

In what follows, we use x, y, and z to range over A, and we speak of elements of type I and II.

Notation Let c and d be such that $\Gamma \vdash \exists(c, c)$ and also $\Gamma \vdash \exists(d, d)$. We write $c \sqsupseteq d$ if d is a unary atom, and $\Gamma \vdash \forall(c, \exists(d, r))$. The relation $\sqsupseteq$ is not reflexive, but it is transitive. We write $c \equiv d$ for $c \sqsubseteq d \sqsubseteq c$. We also write $c \sqsupset d$ if $c \sqsupseteq d$ and not $c \equiv d$. We write $c \Rightarrow d$ if d is a unary atom and $\Gamma \vdash \forall(c, \forall(d, r))$. Notice that if $c \Rightarrow d$ and d is an atom, then $c \sqsupset d$. (This uses ($\rho 7$) and the assumption that $\Gamma \vdash \exists(c, c)$ and also $\Gamma \vdash \exists(d, d)$.) And recall the convention to write $c \leq d$ for $\Gamma \vdash \forall(c, d)$.

The interpretation of binary atoms We say $\langle c_1, c_2, n\rangle [\![r]\!] \langle d_1, d_2, m\rangle$ if and only if one of the conditions below holds:

(a) for some i and j and some unary atom p, $c_i \Rightarrow p$ and $d_j \leq p$.

(b) $d_1 = d_2$ is a unary atom p, and for some i, $c_i \sqsupset p$ and $m > 0$.

(c) $d_1 = d_2$ is a unary atom p, and for some i, $c_i \equiv p$ and $m > n$.

Note that the conditions above are not exclusive.

Lemma 3.6 *The interpretation of each binary atom r is transitive.*

Proof Assume that

$$\langle b_1, b_2, n_1\rangle \ \llbracket r\rrbracket \ \langle c_1, c_2, n_2\rangle \ \llbracket r\rrbracket \ \langle d_1, d_2, n_3\rangle,$$

and write these three triples as $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{c}$, respectively.

Case 1: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (a), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (a).* Let i, j, p, and q be such that $b_i \Rightarrow p$, $c_j \leq p$, $c_k \Rightarrow q$, and $d_l \leq q$. See Case 1 of Lemma 3.4.

Case 2: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (a), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (b).* This time, $d_1 = d_2$ is an atom, and we have i, j, k and p so that $b_i \Rightarrow p$, $c_j \leq p$, $c_k \sqsupset d$. Also, $m_3 > 0$. The derivation below shows that $b_i \sqsupseteq d$:

$$\dfrac{\begin{matrix}\vdots\\ \forall(b_i,\forall(p,r))\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \forall(c_i,\exists(d,r))\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \exists(c_j,c_k)\end{matrix}\quad\begin{matrix}\vdots\\ \forall(c_j,p)\end{matrix}}{\exists(p,c_k)}}{\exists(p,\exists(d,r))}}{\forall(b_i,\exists(d,r))}\ (\rho 2)$$

(The rule we are quoting as ($\rho 2$)) here is more general than the rule we used earlier in connection with $\mathbf{R}^*$ in that we are allowing the subject noun phrase to be complex. But the form above is again derivable in two steps using (tr 2) and (B). The same comments apply below in several other cases of this proof.) An argument with ($\rho 7$) now shows that $b_i \sqsubset d$. (NEEDED) Recalling that $m_3 > 0$, we see that $\mathbf{b}\llbracket r\rrbracket\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ via alternative (b) in the definition of $\llbracket r\rrbracket$.

Case 3: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (a), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (c).* The argument is as in Case 2.

Case 4: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (b), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (a).* Write c for $c_1 = c_2$; this is an atom. And the same holds for d. We have i and j and p such that $b_i \sqsupset c$, $c \Rightarrow p$, $p \leq d_j$. (Also, $n_2 > 0$, but this is irrelevant.) The following derivation shows that $b_i \sqsupseteq p$:

$$\dfrac{\forall(b_i,\exists(c,r)) \qquad \forall(c,\forall(d,r))}{\forall(b_i,\forall(d,r))}\ (\rho 4)$$

From this we see that $\mathbf{b}\llbracket r\rrbracket\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ via alternative (a) in the definition of $\llbracket r\rrbracket$.

Case 5: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (b), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (b).* Write c for $c_1 = c_2$; this is an atom. And the same holds for d. For some i, we have $b_i \sqsupset c \sqsupset d$. Using ($\rho 3$), we see that $b_i \sqsupset d$. Also, $m_3 > 0$. It now follows that $\mathbf{b}\llbracket r\rrbracket\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ via alternative (b) in the definition of $\llbracket r\rrbracket$.

Case 6: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (b), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (c).* The argument is as in Case 5. (The only difference is that this time we have $b_i \sqsupset c \sqsupset d$. But this is sufficient to imply $b_i \sqsupset d$.)

Case 7: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (c), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (a).* The argument is as in Case 4.

Case 8: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (c), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (b).* The argument is as in Case 5.

Case 9: $\mathbf{b}\llbracket r\rrbracket\mathbf{c}$ *via (c), and* $\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ *via (c).* As in Case 5, we write c and d for the evident atoms. For some i, we have $b_i \equiv c \equiv d$, and we also have $n_1 > n_2 > n_3$. Since the order on natural numbers is transitive, $n_1 > n_3$. Thus $\mathbf{b}\llbracket r\rrbracket\mathbf{c}\llbracket r\rrbracket\mathbf{d}$ via alternative (c) in the definition of $\llbracket r\rrbracket$.

This concludes the proof. $\dashv$

Lemma 3.7 *The interpretation of each binary atom r is irreflexive.*

Proof Suppose towards a contradiction that

$$\langle c_1, c_2, n\rangle [\![r]\!] \langle c_1, c_2, n\rangle. \tag{9}$$

We have $\Gamma \vdash \exists(c_i, c_j)$ for all i and j. The reason for (9) cannot be (a) in the definition of $[\![r]\!]$ because if it were, we would have $\Gamma \vdash \forall(c_i, (\forall(c_j, r))$ for some i and j. With the irreflexivity rule (Irr), this would contradict the consistency of Γ.

Suppose that the reason for (9) were (b). Then $c_1 = c_2$ is a unary atom, say c, and $c \sqsupset c$. But this is impossible: if $c \sqsupseteq c$, then $c \equiv c$.

Finally, it is impossible that (c) be the reason for (9), since $n \not> n$. ⊣

Lemmas 3.6 and 3.7 show that $\mathfrak{M}$ is a transitive and irreflexive model for the interpretation of our fragment. Recall that our completeness proof for the logic $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$ on the intended class reduces to showing that every complete, consistent set Γ has a transitive and irreflexive model. The next two results show that $\mathfrak{M}$ serves as such a model.

Lemma 3.8 *For all $c \in \mathbf{C}^+$,*

$$[\![c]\!] = \{\langle d_1, d_2, Q\rangle \in M : \textit{either } \Gamma \vdash \forall(d_1, c), \textit{ or } \Gamma \vdash \forall(d_2, c)\}.$$

Lemma 3.9 $\mathfrak{M} \models \Gamma$.

The proof of Lemmas 3.8 and 3.9 are nearly the same as that of Lemmas 5.3 and 5.4 in [5], so we shall not reproduce them. The main notational change is that where [5] mentions an element of the model such as $\langle c, c, \forall\rangle$, we would use the element $\langle c, c, 0\rangle$ in this paper.

3.3 Finite models

We have seen that the requirements of transitivity and irreflexivity lead us to infinite models, because with these requirements $\forall(p, \exists(p, r))$ only has infinite models. (Note that in the presence of transitivity alone, we have no axiom of infinity. This is because the proof of Theorem 3.1 shows that every finite consistent set has a finite model.) If we want to add a finiteness requirement, then we must also add a *finiteness axiom*

$$\frac{}{\exists(p, \forall(p, \overline{r}))}\ (\mathrm{Fin})$$

Let $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr}, \mathsf{fin})$ be the logic which adds this new axiom to $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr})$.

Theorem 3.10 *For $\Gamma \cup \{\varphi\} \subseteq \mathcal{R}^*$, $\Gamma \models \varphi$ on transitive, irreflexive and finite models iff $\Gamma \vdash \varphi$ in $\mathsf{R}^*(\mathsf{tr}, \mathsf{irr}, \mathsf{fin})$.*

Proof We use the same construction as in Theorem 3.5 but with one change. The definition of the model $\mathcal{M}$ began in (8), and of course the structure is infinite. To keep straight the old and new models, we'll now define a model to be called $\mathcal{N}$. The domain of the model is the set N given by

$$\begin{aligned} N \quad = \quad & \{\langle c_1, c_2, n\rangle \in \mathbf{C}^+ \times \mathbf{C}^+ \times \{0, 1\} : \Gamma \vdash \exists(c_1, c_2), \\ & \text{and if } n \geq 0, \text{ then } c_1 = c_2, \text{ and } c_1 \text{ is a unary atom } \} \end{aligned} \tag{10}$$

The rest of the structure is defined in the same way as with $\mathcal{M}$: the unary atoms are interpreted as in (8), and the binary ones from the three alternatives (a)–(c) stated before Lemma 3.6. This way, $\mathcal{N}$ is a submodel of $\mathcal{M}$, so it is transitive, irreflexive, and (clearly) finite. The only point to check is that the analog of Lemma 3.8 goes through. And for this, the only point to check is that for c a positive c-term of the form $\exists(p, r)$,

$$[\![c]\!] \quad = \quad \{\langle d_1, d_2, Q\rangle \in N : \text{either } \Gamma \vdash \forall(d_1, c), \text{ or } \Gamma \vdash \forall(d_2, c)\}.$$

And here, only half needs to be checked. Suppose that $\langle d_1, d_2, Q\rangle \in N$ and $\Gamma \vdash \forall(d_1, \exists(p, r))$. We need to see that $\langle d_1, d_2, Q\rangle \in [\![\exists(p, r)]\!]$. That is, we need some $\mathbf{b} \in [\![p]\!]$ such that $\langle d_1, d_2, Q\rangle [\![r]\!] \mathbf{b}$. We claim that $\mathbf{b} = \langle p, p, 1\rangle$ works using alternative (b) in the definition of $[\![r]\!]$. We have $d \sqsupseteq p$, and we only need to see that the converse assertion $p \sqsupseteq d$ does not hold. Suppose toward a contradiction that it did. Then d would be a unary atom, and also we would have $\forall(p, \exists(p, r))$ using (ρ3). But this directly contradicts the finiteness axiom.

The remaining verifications are as above. $\dashv$

3.4 Opposites

4 Conclusion

References

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