Completeness Theorems for Syllogistic Fragments

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Comments/Corrections Welcome!

Abstract

Traditional syllogisms involve sentences of the following simple forms: *All X are Y*, *Some X are Y*, *No X are Y*; similar sentences with proper names as subjects, and identities between names. These sentences come with the natural semantics using subsets of a given universe, and so it is natural to ask about complete proof systems. Logical systems are important in this area due to the prominence of syllogistic arguments in human reasoning, and also to the role they have played in logic from Aristotle onwards. We present complete systems for the entire syllogistic fragment and many sub-fragments. These begin with the fragment of *All* sentences, for which we obtain one of the easiest completeness theorems in logic. The last system extends syllogistic reasoning with the classical boolean operations and cardinality comparisons.

1 Introduction: the program of natural logic

This particular project begins with the time-honored syllogisms. The completeness of various formulations of syllogistic logic has already been shown, for example by in Łukasiewicz [4] (in work with Słupecki), and in different formulations, by Corcoran [3] and Martin [5]. The technical part of this paper contains a series of completeness theorems for various systems as we mentioned in the abstract. In some form, two of them were known already: see van Benthem [2] and Westerståhl [14]. We are not aware of systematic studies of syllogistic fragments, and so this is a goal of the paper.

Perhaps the results and methods will be of interest primarily to specialists in logic, but we hope that the statements will be of wider interest. Even more, we hope that the project of *natural logic* will appeal to people in linguistic semantics, artificial intelligence, computational semantics, and cognitive science. This paper is not the place to give a full exposition of natural logic, and so we only present a few remarks here on it.

Textbooks on model theoretic semantics often say that the goal of the enterprise is to study entailment relations (or other related relations). So the question arises as to what *complete* logical systems for those fragments would look like. Perhaps formal reasoning in some system or other will be of independent interest in semantics. And if one has a complete logical system for some phenomenon, then one might well take the logical system to *be* the semantics in some sense. Even if one does not ultimately want to take a logical presentation as primary but treats

them as secondary, it still should be of interest to have completeness and decidability for as large a fragment of natural language as possible. As we found out by working on this topic, the technical work does not seem to be simply an adaptation of older techniques. So someone interested in pursuing that topic might find something of interest here.

Most publications on syllogistic-like fragments comes from either the philosophical or AI literatures. The philosophical work is generally concerned with the problem of modern reconstruction of Aristotle beginning with Łukasiewicz [4] and including papers which go in other directions, such as Corcoran [3] and Martin [5]. Our work is not reconstructive, however, and the systems from the past are not of primary interest here. The AI literature is closer to what we are doing in this paper; see for example Purdy [13]. (However, we are interested in completeness theorems, and the AI work usually concentrates on getting systems that work, and the meta-theoretic work considers decidability and complexity.) The reason is that it has proposals which go beyond the traditional syllogistic systems. This would be a primary goal of what we are calling natural logic. We take a step in this direction in this paper by adding expressions like *There are more As than Bs* to the standard syllogistic systems. This shows that it is possible to have complete syllogistic systems which are not sub-logics of first-order logic.

The next steps in this area can be divided into two groups, and we might call those the "conservative" and "radical" sub-programs. The conservative program is what we just mentioned: to expand the syllogistic systems but to continue to deal with extensional fragments of language. A next step in this direction would treat sentences with verbs other than the copula. There is some prior work on this: e.g., Nishihara, Morita, and Iwata [8] and McAllester and Givan [6]. In addition, Pratt-Hartmann [9, 10] and Pratt-Hartmann and Third [12] give several complexity-theoretic results in this direction. As soon as one has quantifiers and verbs, the phenomenon of quantifier-scope ambiguity suggests that some interaction with syntax will be needed.

Although the program of natural logic as I have presented it seems ineluctably model-theoretic, my own view is that this is a shortcoming that will have to be rectified. This leads to the more radical program. We also want to explore the possibility of having proof theory as the mathematical underpinning for semantics in the first place. This view is suggested in the literature on philosophy of language, but it is not well-explored in linguistic semantics because formal semantics is nowadays essentially the same as model-theoretic semantics. We think that this is only because nobody has yet made suggestions in the proof-theoretic direction. This is not quite correct, and one paper worth mentioning is Ben Avi and Francez [1]. In fact, Francez and his colleagues have begun to look at proof theoretic treatments of syllogistic fragments with a view towards what we are here calling the radical program. One can imagine several ways to "kick away the ladder" after looking at complete semantics for various fragments, incorporating work from several areas. But this paper is not concerned with any of these directions.

The results This paper proves completeness of the following fragments, written in notation which should be self-explanatory: (i) the fragment with *All X are Y*; (ii) the fragment with *Some X are Y*; (iii) = (i)+(ii); (iv) = (iii) + sentences involving proper names; (v) = (i) + *No X are Y*; (vi) *All + Some + No*; (vii)= (vi) + *Names*; (viii) boolean combinations of (vii); (ix)= (i) + *There are at least as many X as Y*; (x)= boolean combinations of (ix) + *Some + No*; In addition, we have a completeness for a system off the main track: (xi)

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All X which are Y are Z; (xii) *Most*; and (xiii) *Most + Some*.

For the most part, we work on systems that do not include sentential boolean operations. This is partly due to the intrinsic interest of the more spare systems. Also, we would like systems whose decision problem is polynomial-time computable. The existing (small) literature on logics for natural language generally works on top of propositional logic, and so their satisfiability problems are NP-hard. At the same time, adding propositional reasoning to the logics tends to make the completeness proofs *easier*, as we shall see: the closer a system is to standard first-order logic, the more applicable are well-known techniques. So from a logical point of view, we are interested in exploring systems which are quite weak.

A final point is that the work here should be of pedagogic interest: the simple completeness theorems in the first few sections of this paper are good vehicles for teaching students about logical systems, soundness, and completeness. This is because the presentation completely avoids all of the details of syntax such as substitution lemmas and rules with side conditions on free variables, and the mathematical arguments of this paper are absolutely elementary. At the same time, the techniques foreshadow what we find in the Henkin-style completeness proofs for first-order logic. So students would see the technique of syntactically defined models quite early on. (However, since we only have three sides of the classical square of opposition, one occasionally feels as if sitting on a wobbly chair.) This paper does not present natural deduction-style logics, but they do exist, and this would add to a presentation for novices. Overall, this material could be an attractive prelude to standard courses.

1.1   Getting started

We are concerned with logical system based on syllogistic reasoning. We interpret a syllogism such as the famous

<div align="center">

All men are mortal.

Socrates is a man.

Socrates is mortal.

</div>

(The first recorded version of this particular syllogism is due to Sextus Empiricus, in a slightly different form.) The interpretations use sets in the obvious way. The idea again is that the sentences above the line should *semantically entail* the one below the line. Specifically, in every context (or model) in which *All men are mortal* and *Socrates is a man* are true, it must be the case that *Socrates is mortal* is also true.

Here is another example, a bit closer to what we have in mind for the study:

<div align="center">

All xenophobics are yodelers.

John is a xenophobic.

Mary is a zookeeper.   (1)

John is Mary.

Some yodeler is a zookeeper.

</div>

To begin our study, we have the following definitions:

"Syntax"   We start with a set of *variables* X, Y, ..., representing plural common nouns. We also also *names* J, M, ... Then we consider sentences S of the following very restricted forms:

<div align="center">

All X are Y, *Some X are Y*, *No X are Y*, *J is an X*, *J is M*.

</div>

The reason we use scare quotes is that we only have five types of sentences, hence no recursion whatsoever. Obviously it would be important to propose complete systems for infinite fragments. The main example which I know of is that of McAllester and Givan [6]. Their paper showed a decidability result but was not concerned with logical completeness; for this, see [7].

Fragments As small as our language is, we shall be interested in a number of fragments of it. These include $\mathcal{L}(all)$, the fragment with *All* (and nothing else); and with obvious notation $\mathcal{L}(all, some)$, $\mathcal{L}(all, some, \text{names})$, and $\mathcal{L}(all, no)$. We also will be interested in extensions of the language and variations on the semantics.

Semantics One starts with a set M, a subset $[\![X]\!] \subseteq M$ for each variable X, and an element $[\![J]\!] \in \mathcal{M}$ for each name J. This gives a *model* $\mathcal{M} = (M, [\![\ ]\!])$. We then define

$$\begin{array}{lll}
\mathcal{M} \models All \ X \ are \ Y & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\
\mathcal{M} \models Some \ X \ are \ Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\
\mathcal{M} \models No \ X \ are \ Y & \text{iff} & [\![X]\!] \cap [\![Y]\!] = \emptyset \\
\mathcal{M} \models J \ is \ an \ X & \text{iff} & [\![J]\!] \in [\![X]\!] \\
\mathcal{M} \models J \ is \ M & \text{iff} & [\![J]\!] = [\![M]\!]
\end{array}$$

We allow $[\![X]\!]$ to be empty, and in this case, recall that $\mathcal{M} \models All \ X \ are \ Y$ vacuously. And if Γ is a finite or infinite set of sentences, then we write $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$.

Main semantic definition $\Gamma \models S$ means that every model which makes all sentences in the set Γ true also makes S true. This is the relevant form of semantic entailment for this paper.

Notation If Γ is a set of sentences, we write Γ_{all} for the subset of Γ containing only sentences of the form *All X are Y*. We do this for other constructs, writing Γ_{some}, Γ_{no} and Γ_{names}.

Inference rules of the logical system The complete set of rules for the syllogistic fragment may be found in Figure 6 below. But we are concerned with other fragments, especially in Sections 8 and onward. Rules for other fragments will be presented as needed.

Proof trees A *proof tree over* Γ is a finite tree $\mathcal{T}$ whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules. $\Gamma \vdash S$ means that there is a proof tree $\mathcal{T}$ for over Γ whose root is labeled S.

Example 1.1 Here is a proof tree formalizing the reasoning in (1):

$$\cfrac{\cfrac{All \ X \ are \ Y \quad J \ is \ an \ X}{J \ is \ a \ Y} \quad \cfrac{M \ is \ a \ Z \quad J \ is \ M}{J \ is \ a \ Z}}{Some \ Y \ are \ Z}$$

Example 1.2 We take

$$\Gamma = \{All \ A \ are \ B, All \ Q \ are \ A, All \ B \ are \ D, All \ C \ are \ D, All \ A \ are \ Q\}.$$

Let S be *All Q are D*. Here is a proof tree showing that $\Gamma \vdash S$:

$$
\cfrac{
  \text{\textit{All Q are A}} \qquad
  \cfrac{
    \cfrac{\text{\textit{All A are B} \quad \textit{All B are B}}}{\text{\textit{All A are B}}} \qquad \text{\textit{All B are D}}
  }{\text{\textit{All A are D}}}
}{\text{\textit{All Q are D}}}
$$

Note that all of the leaves belong to Γ except for one that is *All B are B*. Note also that some elements of Γ are not used as leaves. This is permitted according to our definition. The proof tree above shows that $\Gamma \vdash S$. Also, there is a smaller proof tree that does this, since the use of *All B are B* is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All A are A*.)

Lemma 1.3 (Soundness) *If $\Gamma \vdash S$, then $\Gamma \models S$.*

Proof   By induction on proof trees. $\dashv$

Example 1.4 One easy semantic fact is

$$\{\textit{Some X are Y}, \textit{Some Y are Z}\} \not\models \textit{Some X are Z}.$$

The smallest countermodel is $\{1, 2\}$ with $[\![X]\!] = \{1\}$, $[\![Y]\!] = \{1, 2\}$, and $[\![Z]\!] = \{2\}$. Even if we ignore the soundness of the logical system, an examination its proofs shows that

$$\{\textit{Some X are Y}, \textit{Some Y are Z}\} \not\vdash \textit{Some X are Z}$$

Indeed, the only sentences which follow from the hypotheses are those sentences themselves, the sentences *Some X are X*, *Some Y are Y*, *Some Z are Z*, *Some Y are X*, and *Some Z are Y*, and the axioms of the system: sentences of the form *All U are U* and *J is J*.

There are obvious notions of *submodel* and *homomorphism* of models.

Proposition 1.5 *Sentences in $\mathcal{L}(\text{all}, \text{no}, names)$ are preserved under submodels. Sentences in $\mathcal{L}(\text{some}, names)$ are preserved under homomorphisms. Sentences in $\mathcal{L}(\text{all})$ are preserved under surjective homomorphic images.*

2   *All*

This paper is organized in sections corresponding to different fragments. To begin, we present a system for $\mathcal{L}(all)$. All of our logical systems are sound by Lemma 1.3.

Theorem 2.1 *The logic of Figure 1 is complete for $\mathcal{L}(\text{all})$.*

Proof   Suppose that $\Gamma \models S$. Let S be *All X are Y*. Let $\{*\}$ be any singleton, and define a

$$\frac{}{All\ X\ are\ X} \qquad \frac{All\ X\ are\ Z \quad All\ Z\ are\ Y}{All\ X\ are\ Y}$$

Figure 1: The logic of *All X are Y*.

model $\mathcal{M}$ by $M = \{*\}$, and

$$\llbracket Z \rrbracket \quad = \quad \begin{cases} M & \text{if } \Gamma \vdash All\ X\ are\ Z \\ \emptyset & \text{otherwise} \end{cases} \tag{2}$$

It is important that in (2), X is the same variable as in the sentence S with which we began. We claim that if Γ contains *All V are W*, then $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$. For this, we may assume that $\llbracket V \rrbracket \neq \emptyset$ (otherwise the result is trivial). So $\llbracket V \rrbracket = M$. Thus $\Gamma \vdash All\ X\ are\ V$. So we have a proof tree as on the left below:

$$\frac{\overset{\vdots}{All\ X\ are\ V} \quad All\ V\ are\ W}{All\ X\ are\ W}$$

(The vertical dots $\vdots$ mean that there is some tree over Γ establishing the sentence at the bottom of the dots.) The tree overall has as leaves *All V are W* plus the leaves of the tree above *All X are V*. Overall, we see that all leaves are labeled by sentences in Γ. This tree shows that $\Gamma \vdash All\ X\ are\ W$. From this we conclude that $\llbracket W \rrbracket = M$. In particular, $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$.

Now our claim implies that the model $\mathcal{M}$ we have defined makes all sentences in Γ true. So it must make the conclusion true. Therefore $\llbracket X \rrbracket \subseteq \llbracket Y \rrbracket$. And $\llbracket X \rrbracket = M$, since we have a one-point tree for *All X are X*. Hence $\llbracket Y \rrbracket = M$ as well. But this means that $\Gamma \vdash All\ X\ are\ Y$, just as desired. $\dashv$

Remark The completeness of $\mathcal{L}(all)$ appears to be the simplest possible completeness result of any logical system! (One can also make this claim about the pure identity fragment, the one whose statements are of the form *J is M* and whose logical presentation amounts to the reflexive, symmetric, and transitive laws.) At the same time, we are not aware of any prior statement of its completeness.

2.1 The canonical model property

We introduce a property which some of the logical systems in this paper enjoy. First we need some preliminary points. For any set Γ of sentences, define $\leq_\Gamma$ on the set of variables by

$$U \leq_\Gamma V \quad \text{iff} \quad \Gamma \vdash All\ U\ are\ V \tag{3}$$

Lemma 2.2 *The relation $\leq_\Gamma$ is a preorder: a reflexive and transitive relation.*

We shall often use preorders $\leq_\Gamma$ defined by (3).

Also define a preorder $\preceq_\Gamma$ on the variables by: $U \preceq_\Gamma V$ if Γ contains *All U are V*. Let $\preceq_\Gamma^*$ be the reflexive-transitive closure of $\preceq_\Gamma$.

Usually we suppress mention of Γ and simply write $\leq$, $\preceq$, and $\preceq^*$.

Proposition 2.3 *Let Γ be any set of sentences in this fragment, let $\preceq^*$ be defined from Γ as above. Let X and Y be any variables. Then the following are equivalent:*

1. *$\Gamma \vdash$ All X are Y.*

2. *$\Gamma \models$ All X are Y.*

3. *$X \preceq^* Y$.*

Proof (1)$\Longrightarrow$(2) is by soundness, and (3)$\Longrightarrow$(1) is by induction on $\preceq^*$. The most significant part is (2)$\Longrightarrow$(3). We build a model $\mathcal{M}$. As in the proof of Theorem 2.1, we take $M = \{*\}$. But we modify (2) by taking $[\![Z]\!] = M$ iff $X \preceq^* Z$. We claim that $\mathcal{M} \models \Gamma$. Consider *All V are W* in Γ. We may assume that $[\![V]\!] = M$, or else our claim is trivial. Then $X \preceq^* V$. But $V \preceq W$, so we have $X \preceq^* W$, as desired. This verifies that $\mathcal{M} \models \Gamma$. But $[\![X]\!] = M$, and therefore $[\![Y]\!] = M$ as well. Hence $X \preceq^* Y$, as desired. $\dashv$

Definition Let $\mathcal{F}$ be a fragment, let Γ be a set of sentences in $\mathcal{F}$, and consider a fixed logical system for $\mathcal{F}$. A model $\mathcal{M}$ is *canonical for* Γ if for all $S \in \mathcal{F}$, $\mathcal{M} \models S$ iff $\Gamma \vdash S$. A fragment $\mathcal{F}$ has the *canonical model property* (for the given logical system) if every set $\Gamma \subseteq \mathcal{F}$ has a canonical model.

(For example, in $\mathcal{L}(\textit{all})$, $\mathcal{M}$ is canonical for Γ provided: $X \leq Y$ iff $[\![X]\!] \subseteq [\![Y]\!]$.)

Notice, for example, that classical propositional and first-order logic do not have the canonical model property. A model of $\Gamma = \{p\}$ will have to commit to a value on a different propositional symbol q, and yet neither q nor $\neg q$ follow from Γ. These systems do have the property that every *maximal consistent* set has a canonical model. Since they also have negation, this last fact leads to completeness. As it turns out, fragments in this paper exhibit differing behavior with respect to the canonical model property. Some have it, some do not, and some have it for certain classes of sentences.

Proposition 2.4 *$\mathcal{L}(\textrm{all})$ has the canonical model property with respect to our logical system for it.*

Proof Given Γ, let $\mathcal{M}$ be the model whose universe is the set of variables, and with $[\![U]\!] = \{Z : Z \leq U\}$. Consider a sentence $S \equiv \textit{All X are Y}$. Then $[\![X]\!] \subseteq [\![Y]\!]$ in $\mathcal{M}$ iff $X \leq Y$. (Both rules of the logic are used here.) $\dashv$

The canonical model property is stronger than completeness. To see this, let $\mathcal{M}$ be canonical for a fixed set Γ. In particular $\mathcal{M} \models \Gamma$. Hence if $\Gamma \models S$, then $\mathcal{M} \models S$; so $\Gamma \vdash S$.

2.2   A digression: *All X which are Y are Z*

At this point, we digress from our main goal of the examination of the syllogistic system of Section 1.1. Instead, we consider the logic of *All X which are Y are Z*. To save space, we abbreviate this by (X, Y, Z). We take this sentence to be true in a given model $\mathcal{M}$ if $[\![X]\!] \cap [\![Y]\!] \subseteq [\![Z]\!]$. Note that *All X are Y* is semantically equivalent to (X, X, Y).

First, we check that the logic is genuinely new. The result in Proposition 2.5 clearly also holds for the closure of $\mathcal{L}(\textit{all}, \textit{some}, \textit{no})$ under (infinitary) boolean operations.

$$\frac{}{(X,Y,X)} \qquad \frac{}{(X,Y,Y)} \qquad \frac{(X,Y,U) \quad (X,Y,V) \quad (U,V,Z)}{(X,Y,Z)}$$

Figure 2: The logic of *All X which are Y are Z*, written here (X,Y,Z).

Proposition 2.5 *Let R be* All X which are Y are Z. *Then R cannot be expressed by any set in the language $\mathcal{L}(\mathrm{all}, \mathrm{some}, \mathrm{no})$. That is, there is no set Γ of sentences in $\mathcal{L}(\mathrm{all}, \mathrm{some}, \mathrm{no})$ such that for all $\mathcal{M}$, $\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models R$.*

Proof Consider the model $\mathcal{M}$ with universe $\{x, y, a\}$ with $[\![X]\!] = \{x, a\}$, $[\![Y]\!] = \{y, a\}$, $[\![Z]\!] = \{a\}$, and also $[\![U]\!] = \emptyset$ for other variables U. Consider also a model $\mathcal{N}$ with universe $\{x, y, a, b\}$ with $[\![X]\!] = \{x, a, b\}$, $[\![Y]\!] = \{y, a, b\}$, $[\![Z]\!] = \{a\}$, and the rest of the structure the same as in $\mathcal{M}$. An easy examination shows that for all sentences $S \in \mathcal{L}(\mathit{all}, \mathit{some}, \mathit{no})$, $\mathcal{M} \models S$ iff $\mathcal{N} \models S$.

Now suppose towards a contradiction that we could express R, say by the set Γ. Then since $\mathcal{M}$ and $\mathcal{N}$ agree on $\mathcal{L}(\mathit{all}, \mathit{some}, \mathit{no})$, they agree on Γ. But $\mathcal{M} \models R$ and $\mathcal{N} \not\models R$, a contradiction.
$\dashv$

Theorem 2.6 *The logic of* All X which are Y are Z *in Figure 2 is complete.*

Proof Suppose $\Gamma \models (X, Y, Z)$. Consider the interpretation $\mathcal{M}$ given by $M = \{*\}$, and for each variable W, $[\![W]\!] = \{*\}$ iff $\Gamma \vdash (X, Y, W)$. We claim that for $(U, V, W) \in \Gamma$, $[\![U]\!] \cap [\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that $M = [\![U]\!] \cap [\![V]\!]$. So we use the proof tree

$$\frac{(X,Y,U) \quad (X,Y,V) \quad (U,V,W)}{(X,Y,W)}$$

This shows that $[\![W]\!] = M$, as desired.

Returning to our sentence (X, Y, Z), our overall assumption that $\Gamma \models (X, Y, Z)$ tells us that $\mathcal{M} \models (X, Y, Z)$. The first two axioms show that $* \in [\![X]\!] \cap [\![Y]\!]$. Hence $* \in [\![Z]\!]$. That is, $\Gamma \vdash (X, Y, Z)$.
$\dashv$

Remark Instead of the axiom (X, Y, Y), we could have taken the symmetry rule

$$\frac{(Y,X,Z)}{(X,Y,Z)}$$

The two systems are equivalent.

Remark The fragment with (X, X, Y) is a conservative extension of the fragment with *All*, via the translation of *All X are Y* as (X, X, Y).

3   *All* and *Some*

We enrich our language with sentences *Some X are Y* and our rules with those of Figure 3. The symmetry rule for *Some* may be dropped if one 'twists' the transitivity rule to read

$$\frac{All\ Y\ are\ Z \quad Some\ X\ are\ Y}{Some\ Z\ are\ X}$$

Then symmetry is derivable. We will use the twisted form in later work, but for now we want the three rules of Figure 3 because the first two alone are used in Theorem 3.2 below.

Example 3.1 Perhaps the first non-trivial derivation in the logic is the following one:

$$\frac{All\ Z\ are\ Y \quad \dfrac{\dfrac{All\ Z\ are\ X \quad Some\ Z\ are\ Z}{Some\ Z\ are\ X}}{Some\ X\ are\ Z}}{Some\ X\ are\ Y}$$

That is, if there is a Z, and if all Zs are Xs and also Ys, then some X is a Y.

In working with *Some* sentences, we adopt some notation parallel to (3): for *All*

$$U \uparrow_\Gamma V \quad \text{iff} \quad \Gamma \vdash Some\ U\ are\ V \tag{4}$$

Usually we drop the subscript Γ. Using the symmetry rule, $\uparrow$ is symmetric.

The next result is essentially due to van Benthem [2], Theorem 3.3.5.

Theorem 3.2 *The first two rules in Figure 3 give a logical system with the canonical model property for $\mathcal{L}(\text{some})$. Hence the system is complete.*

Proof   Let $\Gamma \subseteq \mathcal{L}(some)$. Let $\mathcal{M} = \mathcal{M}(\Gamma)$ be the set of sets of unordered pairs (i.e., sets with one or two elements) of variables. Let

$$[\![U]\!] \;=\; \{\{U, V\} : U \uparrow V\}.$$

Observe that the elements of $[\![U]\!]$ are unordered pairs with one element being U. If $U \uparrow V$, then $\{U, V\} \in [\![U]\!] \cap [\![V]\!]$. Assume first $X \neq Y$ and that Γ contains $S = Some\ X\ are\ Y$. Then $\{X, Y\} \in [\![X]\!] \cap [\![Y]\!]$, so $\mathcal{M} \models S$. Conversely, if $\{U, V\} \in [\![X]\!] \cap [\![Y]\!]$, then by what we have said above $\{U, V\} = \{X, Y\}$. In particular, $\{X, Y\} \in M$. So $X \uparrow Y$. Second, we consider the situation when $X = Y$. If Γ contains $S = Some\ X\ are\ X$, then $\{X\} \in [\![X]\!]$. So $\mathcal{M} \models S$. Conversely, if $\{U, V\} \in [\![X]\!]$, then (without loss of generality) $U = X$, and $X \uparrow V$. Using our second rule of *Some*, we see that $X \uparrow X$. $\dashv$

The rest of this section is devoted to the combination of *All* and *Some*.

Lemma 3.3 *Let $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some})$. Then there is a model $\mathcal{M}$ with the following properties:*

1. If $X \leq Y$, then $[\![X]\!] \subseteq [\![Y]\!]$.

2. $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ iff $X \uparrow Y$.

In particular, $\mathcal{M} \models \Gamma$.

Proof   Let $N = |\Gamma_{some}|$. We think of N as the ordinal number $\{0, 1, \ldots, N-1\}$. For $i \in N$,

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$$\frac{Some\ X\ are\ Y}{Some\ Y\ are\ X} \qquad \frac{Some\ X\ are\ Y}{Some\ X\ are\ X} \qquad \frac{All\ Y\ are\ Z \quad Some\ X\ are\ Y}{Some\ X\ are\ Z}$$

Figure 3: The logic of *Some* and *All*, in addition to the logic of *All*.

let U_i and V_i be such that

$$\Gamma_{some} \quad = \quad \{Some\ V_i\ are\ W_i : i \in I\} \tag{5}$$

Note that for $i \neq j$, we might well have $V_i = V_j$ or $W_i = W_j$. For the universe of $\mathcal{M}$ we take the set N. For each variable Z, we define

$$[\![Z]\!] \quad = \quad \{i \in N : \text{either } V_i \leq Z \text{ or } W_i \leq Z\}. \tag{6}$$

(As in (3), the relation $\leq$ is: $X \leq Y$ iff $\Gamma \vdash All\ X\ are\ Y$.) This defines the model $\mathcal{M}$.

For the first point, suppose that $X \leq Y$. It follows from (6) and Lemma 2.2 that $[\![X]\!] \subseteq [\![Y]\!]$.

Second, take a sentence *Some V_i are W_i* on our list in (5) above. Then i itself belongs to $[\![V_i]\!] \cap [\![W_i]\!]$, so this intersection is not empty. At this point we know that $\mathcal{M} \models \Gamma$, and so by soundness, we then get half of the second point in this lemma.

For the left-to-right direction of the second point, assume that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. Let $i \in [\![X]\!] \cap [\![Y]\!]$. We have four cases, depending on whether $V_i \leq X$ or $V_i \leq Y$, and whether $W_i \leq X$ or $W_i \leq Y$. In each case, we use the logic to see that $X \uparrow Y$. The formal proofs are all similar to what we saw in Example 3.1 above. ⊣

Theorem 3.4 *The logic of Figures 1 and 3 is complete for $\mathcal{L}(\text{all}, \text{some})$.*

Proof Suppose that $\Gamma \models S$. There are two cases, depending on whether S is of the form *All X are Y* or of the form *Some X are Y*. In the first case, we claim that $\Gamma_{all} \models S$. To see this, let $\mathcal{M} \models \Gamma_{all}$. We get a new model $\mathcal{M}' = \mathcal{M} \cup \{*\}$ via $[\![X]\!]' = [\![X]\!] \cup \{*\}$. The model $\mathcal{M}'$ so obtained satisfies Γ_{all} and all *Some* sentences whatsoever in the fragment. Hence $\mathcal{M}' \models \Gamma$. So $\mathcal{M}' \models S$. And since S is a universal sentence, $\mathcal{M} \models S$ as well. This proves our claim that $\Gamma_{all} \models S$. By Theorem 2.1, $\Gamma_{all} \vdash S$. Hence $\Gamma \vdash S$.

The second case, where S is of the form *Some X are Y*, is an immediate application of Lemma 3.3. ⊣

Remark Let $\Gamma \subseteq \mathcal{L}(\text{all}, \text{some})$, and let $S \in \mathcal{L}(\text{some})$. As we know from Lemma 3.3, if $\Gamma \nvdash S$, there is a $\mathcal{M} \models \Gamma$ which makes S false. The proof gets a model $\mathcal{M}$ whose size is $|\Gamma_{some}|$. We can get a countermodel of size at most 2. To see this, let $\mathcal{M}$ be as in Lemma 3.3, and let S be *Some X are Y*. If either $[\![X]\!]$ or $[\![Y]\!]$ is empty, we can coalesce all the points in $\mathcal{M}$ to a single point $*$, and then take $[\![U]\!]' = \{*\}$ iff $[\![U]\!] \neq \emptyset$. So we assume that $[\![X]\!]$ and $[\![Y]\!]$ are non-empty. Let $\mathcal{N}$ be the two-point model $\{1, 2\}$. Define $f : \mathcal{M} \to \mathcal{M}$ by $f(x) = 1$ iff $x \in [\![X]\!]$. The structure of $\mathcal{N}$ is that $[\![U]\!]_{\mathcal{N}} = f[[\![U]\!]_{\mathcal{N}}]$. This makes f a surjective homomorphism. By Proposition 1.5, $\mathcal{N} \models \Gamma$. And the construction insures that in $\mathcal{N}$, $[\![X]\!] \cap [\![Y]\!] = \emptyset$.

Note that 2 is the smallest we can get, since on models of size 1,

$$\{Some\ X\ are\ Y, Some\ Y\ are\ Z\} \models Some\ X\ are\ Z.$$

$$\frac{}{J \ is \ J} \qquad\qquad \frac{J \ is \ M \quad M \ is \ F}{F \ is \ J} \qquad \frac{J \ is \ an \ X \quad J \ is \ a \ Y}{Some \ X \ are \ Y}$$

$$\frac{All \ X \ are \ Y \quad J \ is \ an \ X}{J \ is \ a \ Y} \qquad \frac{M \ is \ an \ X \quad J \ is \ M}{J \ is \ an \ X}$$

Figure 4: The logic of names, on top of the logic of *All* and *Some*.

Remark $\mathcal{L}(all, some)$ does not have the canonical model property with respect to any logical system. To see this, let Γ be the set $\{All \ X \ are \ Y\}$. Let $\mathcal{M} \models \Gamma$. Then either $\mathcal{M} \models All \ Y \ are \ X$, or $\mathcal{M} \models Some \ Y \ are \ Y$. But neither of these sentences follows from Γ. We cannot hope to avoid the split in the proof of Theorem 3.4 due to the syntax of S.

Remark Suppose that one wants to say that *All X are Y* is true when $[\![X]\!] \subseteq [\![Y]\!]$ and also $[\![X]\!] \neq \emptyset$. Then the following rule becomes sound:

$$\frac{All \ X \ are \ Y}{Some \ X \ are \ Y} \tag{7}$$

On the other hand, is is no longer sound to take *All X are X* to be an axiom. So we drop that rule in favor of (7). In this way, we get a complete system for the modified semantics. Here is how one sees this. Given Γ, let $\overline{\Gamma}$ be Γ with all sentences *Some X are Y* such that *All X are Y* belongs to Γ. An easy induction on proofs shows that $\Gamma \vdash S$ in the modified system iff $\overline{\Gamma} \vdash S$ in the old system.

4   Adding Proper Names

In this section we obtain completeness for sentences in $\mathcal{L}(all, some, \text{names})$. The proof system adds rules in Figure 4 to what we already have seen in Figures 1 and 3.

Fix a set $\Gamma \subseteq \mathcal{L}(all, some, \text{names})$. Let $\equiv$ and $\in$ be the relations defined from Γ by

$$\begin{aligned} J \equiv M \quad & \text{iff} \quad \Gamma \vdash J \ is \ M \\ J \in X \quad & \text{iff} \quad \Gamma \vdash J \ is \ an \ X \end{aligned}$$

Lemma 4.1 $\equiv$ *is an equivalence relation. And if* $J \equiv M \in X \leq Y$, *then* $J \in Y$.

Lemma 4.2 *Let* $\Gamma \subseteq \mathcal{L}(all, some, names)$. *Then there is a model* $\mathcal{N}$ *with the following properties:*

1. *If* $X \leq Y$, *then* $[\![X]\!] \subseteq [\![Y]\!]$.

2. $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ *iff* $X \uparrow Y$.

3. $[\![J]\!] = [\![M]\!]$ *iff* $J \equiv M$.

4. $[\![J]\!] \in [\![X]\!]$ *iff* $J \in X$.

Proof Let $\mathcal{M}$ be any model satisfying the conclusion of Lemma 3.3 for $\Gamma_{all} \cup \Gamma_{some}$. Let $\mathcal{N}$

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$$\frac{All\ X\ are\ Z \quad No\ Z\ are\ Y}{No\ Y\ are\ X} \qquad \frac{No\ X\ are\ X}{No\ X\ are\ Y} \qquad \frac{No\ X\ are\ X}{All\ X\ are\ Y}$$

Figure 5: The logic of *No X are Y* on top of *All X are Y*.

be defined by

$$
\begin{aligned}
N &= M + \{[J] : J \text{ a name}\} \\
[\![X]\!] &= [\![X]\!]_M + \{[J] : \Gamma \vdash J \text{ is an } X\}
\end{aligned}
\tag{8}
$$

The $+$ here denotes a disjoint union. It is easy to check that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences in *All*, that the *Some* sentences true in $\mathcal{M}$ are still true in $\mathcal{N}$, and that points (3) and (4) in our lemma hold. So what remains is to check that if $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ in $\mathcal{N}$, then $X \uparrow Y$. The only interesting case is when $J \in [\![X]\!] \cap [\![Y]\!]$ for some name J. So $J \in X$ and $J \in Y$. Using the one rule of the logic which has both names and *Some*, we see that $X \uparrow Y$. $\qquad\dashv$

Theorem 4.3 *The logic of Figures 1, 3, and 4 is complete for* $\mathcal{L}(\text{all}, \text{some}, names)$.

Proof The proof is nearly the same as that of Theorem 3.4. In the part of the proof dealing with *All* sentences, we had a construction taking a model $\mathcal{M}$ to a one-point extension $\mathcal{M}'$. To interpret names in $\mathcal{M}'$, we let $[\![J]\!] = *$ for all names J. Then all sentences involving names are automatically true in $\mathcal{M}'$. $\qquad\dashv$

5 *All* and *No*

In this section, we consider $\mathcal{L}(all, no)$. Note that *No X are X* just says that there are no Xs. In addition to the rules of Figure 1, we take the rules in Figure 5. As in (3) and (4), we write

$$U \perp_\Gamma V \quad \text{iff} \quad \Gamma \vdash No\ U\ are\ V \tag{9}$$

This relation is symmetric.

Lemma 5.1 $\mathcal{L}(\text{all}, \text{no})$ *has the canonical model property with respect to our logic.*

Proof Let Γ be any set of sentences in *All* and *No*. Let

$$
\begin{aligned}
M &= \{\{U, V\} : U \not\perp V\} \\
[\![W]\!] &= \{\{U, V\} \in M : U \leq W \text{ or } V \leq W\}
\end{aligned}
\tag{10}
$$

The semantics is monotone, and so if $X \leq Y$, then $[\![X]\!] \subseteq [\![Y]\!]$. Conversely, suppose that $[\![X]\!] \subseteq [\![Y]\!]$. If $[\![X]\!] = \emptyset$, then $X \perp X$, for otherwise $\{X\} \in [\![X]\!]$. From the last rule in Figure 5, we see that $X \leq Y$, as desired. In the other case, $[\![X]\!] \neq \emptyset$, Fix $\{V, W\} \in [\![X]\!]$ so that $V \not\perp W$, and either $V \leq X$ or $W \leq X$. Without loss of generality, $V \leq X$. We cannot have $X \perp X$, or else $V \perp V$ and then $V \perp W$. So $\{X\} \in [\![X]\!] \subseteq [\![Y]\!]$. Thus $X \leq Y$.

We have shown $X \leq Y$ iff $[\![X]\!] \subseteq [\![Y]\!]$. This is half of the canonical model property, the other half being $X \perp Y$ iff $[\![X]\!] \cap [\![Y]\!] = \emptyset$. Suppose first that $[\![X]\!] \cap [\![Y]\!] = \emptyset$. Then $\{X, Y\} \notin M$, lest it belong to both $[\![X]\!]$ and $[\![Y]\!]$. So $X \perp Y$. Conversely, suppose that $X \perp Y$. Suppose

towards a contradiction that $\{V, W\} \in [\![X]\!] \cap [\![Y]\!]$. There are four cases, and two representative ones are (i) $V \leq X$ and $W \leq Y$, and (ii) $V \leq X$ and $V \leq Y$. In (i), we have the following tree over Γ:

$$
\cfrac{\cfrac{\vdots}{All\ W\ are\ Y} \quad \cfrac{\cfrac{\vdots}{All\ V\ are\ X} \quad \cfrac{\vdots}{No\ X\ are\ Y}}{No\ Y\ are\ V}}{No\ V\ are\ W}
$$

This contradicts $\{V, W\} \in M$. In (ii), we replace W by V in the tree above, so that the root is *No V are V*. Then we use one of the rules to conclude that *No V are W*, again contradicting $\{V, W\} \in M$. $\dashv$

Since the canonical model property is stronger than completeness, we have shown the following result:

Theorem 5.2 *The logic of Figures 1 and 5 is complete for* All *and* No.

6 $\mathcal{L}(\boldsymbol{all}, \boldsymbol{some}, \boldsymbol{no}, \mathrm{names})$

At this point, we put together our work on the previous systems by proving a completeness result for $\mathcal{L}(\mathit{all}, \mathit{some}, \mathit{no}, \mathrm{names})$. For the logic, we take all the rules in Figure 6. This includes the all rules from Figures 1, 3, 4, and 5. But we also must add a principle relating *Some* and *No*. For the first time, we face the problem of potential inconsistency: there are no models of *Some X are Y* and *No X are Y*. Hence any sentence S whatsoever follows from these two. This explains the last rule, a new one, in Figure 6.

Definition A set Γ is *inconsistent* if $\Gamma \vdash S$ for all S. Otherwise, Γ is *consistent*.

Before we turn to the completeness result in Theorem 6.2 below, we need a result specifically for $\mathcal{L}(\mathit{all}, \mathit{no}, \mathrm{names})$.

Lemma 6.1 *Let $\Gamma \subseteq \mathcal{L}(\mathrm{all}, \mathrm{no}, \mathit{names})$ be a consistent set. Then there is a model $\mathcal{N}$ such that*

1. $[\![X]\!] \subseteq [\![Y]\!]$ iff $X \leq Y$.

2. $[\![X]\!] \cap [\![Y]\!] = \emptyset$ iff $X \perp Y$.

3. $[\![J]\!] = [\![M]\!]$ iff $J \equiv M$.

4. $[\![J]\!] \in [\![X]\!]$ iff $J \in X$.

Proof Let $\mathcal{M}$ be from Lemma 5.1 for $\Gamma_{all} \cup \Gamma_{no}$. Let $\mathcal{N}$ come from $\mathcal{M}$ by the definitions in (8) in Lemma 4.2. (That is, we add the equivalence classes of the names in the natural way.) It is easy to check all of the parts above except perhaps for the second. If $[\![X]\!] \cap [\![Y]\!] = \emptyset$ in $\mathcal{N}$, then the same holds in its submodel $\mathcal{M}$. And so $X \perp Y$. In the other direction, assume that $X \perp Y$ but towards a contradiction that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. There are no points in the intersection in $M \subseteq N$. So let J be such that $[J] \in [\![X]\!] \cap [\![Y]\!]$. Then by our last point, $J \in X$ and $J \in Y$. Using the one rule of the logic which has both names and *Some*, we see that $\Gamma \vdash$ *Some X are Y*. Since $X \perp Y$, we see that Γ is inconsistent. $\dashv$

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$$\frac{}{All\ X\ are\ X} \qquad \frac{All\ X\ are\ Z \quad All\ Z\ are\ Y}{All\ X\ are\ Y}$$

$$\frac{Some\ X\ are\ Y}{Some\ X\ are\ X} \qquad \frac{All\ Y\ are\ Z \quad Some\ X\ are\ Y}{Some\ Z\ are\ X}$$

$$\frac{}{J\ is\ J} \qquad \frac{J\ is\ M \quad M\ is\ F}{F\ is\ J}$$

$$\frac{J\ is\ an\ X \quad J\ is\ a\ Y}{Some\ X\ are\ Y} \qquad \frac{All\ X\ are\ Y \quad J\ is\ an\ X}{J\ is\ a\ Y}$$

$$\frac{M\ is\ an\ X \quad J\ is\ M}{J\ is\ an\ X} \qquad \frac{All\ X\ are\ Z \quad No\ Z\ are\ Y}{No\ Y\ are\ X}$$

$$\frac{No\ X\ are\ X}{No\ X\ are\ Y} \qquad \frac{No\ X\ are\ X}{All\ X\ are\ Y}$$

$$\frac{Some\ X\ are\ Y \quad No\ X\ are\ Y}{S}$$

Figure 6: A complete set of rules for $\mathcal{L}(all, some, no, \text{names})$.

Theorem 6.2 *The logic in Figure 6 is complete for* $\mathcal{L}(\text{all}, \text{some}, \text{no}, names)$.

Proof   Suppose that $\Gamma \models S$. We show that $\Gamma \vdash S$. We may assume that Γ is consistent, or else our result is trivial. There are a number of cases, depending on S.

First, suppose that $S \in \mathcal{L}(some, \text{names})$. Let $\mathcal{N}$ be from Lemma 4.2 for $\Gamma_{all} \cup \Gamma_{some} \cup \Gamma_{names}$. There are two cases. If $\mathcal{N} \models \Gamma_{no}$, then by hypothesis, $\mathcal{N} \models S$. Lemma 4.2 then shows that $\Gamma \vdash S$, as desired. Alternatively, there may be some *No A are B* in Γ_{no} such that $[\![A]\!] \cap [\![B]\!] \neq \emptyset$. And again, Lemma 4.2 shows that $\Gamma_{all} \cup \Gamma_{some} \cup \Gamma_{names} \vdash$ *Some A are B*. So Γ is inconsistent.

Second, suppose that $S \in \mathcal{L}(all, no)$. Let $\mathcal{N}$ come from Lemma 6.1 for $\mathcal{N} \models \Gamma_{all} \cup \Gamma_{names}$. If $\mathcal{N} \models \Gamma_{some}$, then by hypothesis $\mathcal{N} \models S$. By Lemma 6.1, $\Gamma \vdash S$. Otherwise, there is some sentence *Some A are B* in Γ_{some} such that $[\![A]\!] \cap [\![B]\!] = \emptyset$. And then $\mathcal{N} \models$ *No A are B*. By Lemma 6.1, $\Gamma \vdash$ *No A are B*. Again, Γ is inconsistent. ⊣

7   Adding Boolean Operations

The classical syllogisms include sentences *Some X is not a Y*. In our setting, it makes sense also to add other sentences with negative verb phrases: *J is not an X*, and *J is not M*. It is possible to consider the logical system that is obtained by adding just these sentences. But it is also possible to simply add the boolean operations on top of the language which we have already considered. So we have atomic sentences of the kinds we have already seen (the sentences in $\mathcal{L}(all, some, no, \text{names})$), and then we have arbitrary conjunctions, disjunctions, and negations of sentences. We present a Hilbert-style axiomatization of this logic in Figure 7. The completeness of it appears in Łukasiewicz [4] (in work with Słupecki; they also showed

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1. All substitution instances of propositional tautologies.

2. *All X are X*

3. $(All\ X\ are\ Z) \wedge (All\ Z\ are\ Y) \rightarrow All\ X\ are\ Y$

4. $(All\ Y\ are\ Z) \wedge (Some\ X\ are\ Y) \rightarrow Some\ Z\ are\ X$

5. $Some\ X\ are\ Y \rightarrow Some\ X\ are\ X$

6. $No\ X\ are\ X \rightarrow All\ X\ are\ Y$

7. $No\ X\ are\ Y \leftrightarrow \neg(Some\ X\ are\ Y)$

8. *J is J*

9. $(J\ is\ M) \wedge (M\ is\ F) \rightarrow F\ is\ J$

10. $(J\ is\ an\ X) \wedge (J\ is\ a\ Y) \rightarrow Some\ X\ are\ Y$

11. $(All\ X\ are\ Y) \wedge (J\ is\ an\ X) \rightarrow J\ is\ a\ Y$

12. $(M\ is\ an\ X) \wedge (J\ is\ M) \rightarrow J\ is\ an\ X$

Figure 7: Axioms for boolean combinations of sentences in $\mathcal{L}(all, some, no, \text{names})$.

decidability), and also by Westerståhl [14], and axioms 1–6 are essentially the system SYLL. We include Theorem 7.2 in this paper because it is a natural next step, because the techniques build on what we have already seen, and because we shall generalize the result in Section 8.3.

It should be noted that the axioms in Figure 7 are not simply transcriptions of the rules from our earlier system in Figure 6. The biconditional (7) relating *Some* and *No* is new, and using it, one can dispense with two of the transcribed versions of the *No* rules from earlier. Similarly, we should emphasize that the pure syllogistic logic is computationally much more tractable than the boolean system, being in polynomial time.

As with any Hilbert-style system, the only rule of the system in this section is modus ponens. (We think of the other systems in this paper as having many rules.) We define $\vdash \varphi$ in the usual way, and then we say that. $\Gamma \vdash \varphi$ if there are $\psi_1, \ldots, \psi_n$ from Γ such that $\vdash (\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \varphi$.

The soundness of this system is routine.

Proposition 7.1 *If $\Gamma_0 \cup \{\chi\} \subseteq \mathcal{L}(all, some, no, names)$, and if $\Gamma_0 \vdash \chi$ using the system of Figure 6, then $\Gamma_0 \vdash \chi$ in the system of Figure 7.*

The proof is by induction on proof trees in the previous system. We shall use this result above frequently in what follows, without special mention.

Theorem 7.2 *The logic of Figure 7 is complete for assertions $\Delta \models \varphi$ in the language of boolean combinations from $\mathcal{L}(all, some, no, names)$.*

The rest of this section is devoted to proof of Theorem 7.2. As usual, the presence of

negation in the language allows us to prove completeness by showing that every consistent Δ in the language of this section has a model. We may as well assume that Δ is *maximal* consistent.

Definition The *basic* sentences are those of the form *All X are Y*, *Some X and Y*, *J is M*, and *J is an X* or their negations. Let

$$\Gamma \quad = \quad \{S : \Delta \models S \text{ and } S \text{ is basic}\}.$$

Note that Γ might contain sentences $\neg(All\ X\ are\ Y)$ which do not belong to the syllogistic language $\mathcal{L}(all, some, no, \text{names})$.

Claim 7.3 $\Gamma \models \Delta$. *That is, every model of Γ is a model of Δ.*

To see this, let $\mathcal{M} \models \Gamma$ and let $\varphi \in \Delta$. We may assume that φ is in disjunctive normal form. It is sufficient to show that some disjunct of φ holds in $\mathcal{M}$. By maximal consistency, let ψ be a disjunct of φ which also belongs to Δ. Each conjunct of ψ belongs to Γ and so holds in $\mathcal{M}$.

The construction of a model of Γ is similar to what we saw in Theorem 4.3. Define $\leq$ to be the relation on variables given by $X \leq Y$ if the sentence *All X are Y* belongs to Γ. We claim that $\leq$ is reflexive and transitive. We'll just check the transitivity. Suppose that *All X are Y* and *All Y are Z* belong to Γ. Then they belong to Δ. Using Proposition 7.1, we see that $\Delta \vdash All\ X\ are\ Z$. Since Δ is maximal consistent, it must contain *All X are Z*; thus so must Γ.

Define the relation $\equiv$ on names by $J \equiv M$ iff the sentence *J is M* belongs to Γ. Then $\equiv$ is an equivalence relation, just as we saw above for $\leq$. Let the set of equivalence classes of $\equiv$ be $\{[J_1], \ldots, [J_m]\}$. (Incidentally, this result does not need Γ to be finite, and we are only pretending that it is finite to simplify the notation a bit.)

Let the set of *Some X are Y* sentences in Γ be $S_1, \ldots, S_n$, and for $1 \leq i \leq n$, let U_i and V_i be such that S_i is *Some U_i are V_i*. So

$$\Gamma_{some} \quad = \quad \{Some\ U_i\ are\ V_i : i = 1, \ldots, n\} \tag{11}$$

Let the set of $\neg(All\ X\ are\ Y)$ sentences in Γ be $T_1, \ldots, T_p$. For $1 \leq i \leq p$, let W_i and X_i be such that T_i is $\neg(All\ W_i\ are\ X_i)$. So this time we are concerned with

$$\{\neg(All\ W_i\ are\ X_i) : i = 1, \ldots, p\} \tag{12}$$

Note that for $i \neq j$, we might well have $U_i = U_j$ or $U_i = W_j$, or some other such equation. (This is the part of the structure that goes beyond what we saw in Theorem 4.3.)

We take $\mathcal{M}$ to be a model with M the following set

$$\{(a, 1), \ldots, (a, m)\} \cup \{(b, 1), \ldots, (b, n)\} \cup \{(c, 1), \ldots, (c, p)\}.$$

Here m, n, and p are the numbers we saw in the past few paragraphs. The purpose of a, b, and c is to make a disjoint union. Let $[\![J]\!] = (a, i)$, where i is the unique number between 1 and m such that $J \equiv J_i$. And for a variable Z we set

$$
\begin{aligned}
[\![Z]\!] \quad = \quad & \{(a, i) : 1 \leq i \leq n \text{ and } J_i \text{ is a } Z \text{ belongs to } \Gamma\} \\
\cup \quad & \{(b, i) : 1 \leq i \leq m \text{ and either } U_i \leq Z \text{ or } V_i \leq Z\} \\
\cup \quad & \{(c, i) : 1 \leq i \leq p \text{ and } W_i \leq Z\}
\end{aligned}
\tag{13}
$$

This completes the specification of $\mathcal{M}$. The rest of our work is devoted to showing that all sentences in Γ are true in $\mathcal{M}$. We must argue case-by-case, and so we only give the parts of the arguments that differ from what we have seen in Theorem 4.3.

Consider the sentence T_i, that is $\neg(All\ W_i\ are\ X_i)$. We want to make sure that $[\![W_i]\!] \setminus [\![X_i]\!] \neq \emptyset$. For this, consider (c, i). This belongs to $[\![W_i]\!]$ by the last clause in (13). We want to be sure that $(c, i) \notin [\![X_i]\!]$. For if $(c, i) \in [\![X_i]\!]$, then Γ would contain $All\ W_i\ are\ X_i$. And then Γ would be inconsistent in our previous system, so our original Δ would be inconsistent in our Hilbert-style system.

Continuing, consider a sentence $\neg(Some\ P\ are\ Q)$ in Γ. We have to make sure that $[\![P]\!] \cap [\![Q]\!] = \emptyset$. We argue by contradiction. There are three cases, depending on the first coordinate of a putative element of the intersection. Perhaps the most interesting case is when $(c, i) \in [\![P]\!] \cap [\![Q]\!]$ for $1 \leq i \leq p$. Then Γ contains both $All\ W_i\ are\ P$ and $All\ W_i\ are\ Q$. Now the fact that Γ contains $\neg(All\ W_i\ are\ X_i)$ implies that it must contain $Some\ W_i\ are\ W_i$. For if not, then it would contain $No\ W_i\ are\ W_i$ and hence $All\ W_i\ are\ X_i$; as always, this would contradict the consistency of Δ. Thus Γ contains $All\ W_i\ are\ P$, $All\ W_i\ are\ Q$ and $Some\ W_i\ are\ W_i$. Using our previous system, we see that Γ contains $Some\ P\ are\ Q$ (see Example 3.1). This contradiction shows that $[\![P]\!] \cap [\![Q]\!]$ cannot contain any element of the form (c, i). The other two cases are similar, and we conclude that the intersection is indeed empty.

This concludes our outline of the proof of Theorem 7.2.

8   *There are at least as many X as Y*

In our final section, we show that it is possible to have complete syllogistic systems for logics which go are not first-order. We regard this as a proof-of-concept; it would be of interest to get complete systems for richer fragments, such the ones in Pratt-Hartmann [11].

We write $\exists^{\geq}(X, Y)$ for *There are at least as many X as Y*, and we are interested in adding these sentences to our fragments. We are usually interested in sentences in this fragment on *finite* models. We write $|S|$ for the cardinality of the set S. The semantics is that $\mathcal{M} \models \exists^{\geq}(X, Y)$ iff $|[\![X]\!]| \geq |[\![Y]\!]|$ in $\mathcal{M}$.

$\mathcal{L}(all, \exists^{\geq})$ does not have the canonical model property of Section 2.1. We show this via establishing that the semantics is not compact. Consider

$$\Gamma \quad = \quad \{\exists^{\geq}(X_1, X_2), \exists^{\geq}(X_2, X_3), \ldots, \exists^{\geq}(X_n, X_{n+1}), \ldots\}$$

Suppose towards a contradiction that $\mathcal{M}$ were a canonical model for Γ. In particular, $\mathcal{M} \models \Gamma$. Then $|[\![X_1]\!]| \geq |[\![X_2]\!]| \geq \ldots$. For some n, we have $|[\![X_n]\!]| = |[\![X_{n+1}]\!]|$. Thus $\mathcal{M} \models \exists^{\geq}(X_{n+1}, X_n)$. However, this sentence does not follow from Γ.

Remark In the remainder of this section, Γ denotes a *finite* set of sentences.

In this section, we consider $\mathcal{L}(all, \exists^{\geq})$. For proof rules, we take the rules in Figure 8 together with the rules for *All* in Figure 1. The system is sound. The last rule is perhaps the most interesting, and it uses the assumption that our models are finite. That is, if all Y are X, and there are at least as many elements in the bigger set Y as in X, then the sets have to be the same.

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$$\frac{\textit{All Y are X}}{\exists^{\geq}(X,Y)} \qquad \frac{\exists^{\geq}(X,Y) \quad \exists^{\geq}(Y,Z)}{\exists^{\geq}(X,Z)} \qquad \frac{\textit{All Y are X} \quad \exists^{\geq}(Y,X)}{\textit{All X are Y}}$$

Figure 8: Rules for $\exists^{\geq}(X,Y)$ and *All*.

We need a little notation at this point. Let Γ be a (finite) set of sentences. We write $X \leq_c Y$ for $\Gamma \vdash \exists^{\geq}(Y,X)$. We also write $X \equiv_c Y$ for $X \leq_c Y \leq_c X$, and $X <_c Y$ for $X \leq_c Y$ but $X \not\equiv_c Y$. We continue to write $X \leq Y$ for $\Gamma \vdash \textit{All X are Y}$. And we write $X \equiv Y$ for $X \leq Y \leq X$.

Proposition 8.1 *Let $\Gamma \subseteq \mathcal{L}(\text{all}, \exists^{\geq})$ be a (finite) set. Let $\mathcal{V}$ be the set of variables in Γ.*

1. *If $X \leq Y$, then $X \leq_c Y$.*

2. *$(\mathcal{V}, \leq_c)$ is a preorder: a reflexive and transitive relation.*

3. *If $X \leq_c Y \leq X$, then $X \leq Y$.*

4. *If $X \leq_c Y$, $X \equiv X'$, and $Y \equiv Y'$, then $X' \leq_c Y'$.*

5. *$(\mathcal{V}, \leq_c)$ is pre-wellfounded: a preorder with no descending sequences in its strict part.*

Proof Part (1) uses the first rule in Figure 8. In part (2), the reflexivity of $\leq_c$ comes from that of $\leq$ and part (1); the transitivity is by the second rule of $\exists^{\geq}$. Part (3) is by the last rule of $\exists^{\geq}$. Part 4 uses part (1) and transitivity. Part 5 is just a summary of the previous parts. ⊣

Theorem 8.2 *The logic of Figures 1 and 8 is complete for $\mathcal{L}(\text{all}, \exists^{\geq})$.*

Proof Suppose that $\Gamma \models \exists^{\geq}(Y,X)$. Let $\{*\}$ be any singleton, and define a model $\mathcal{M}$ by taking M to be a singleton $\{*\}$, and

$$[\![Z]\!] \quad = \quad \begin{cases} M & \text{if } \Gamma \vdash \exists^{\geq}(Z,X) \\ \emptyset & \text{otherwise} \end{cases} \tag{14}$$

We claim that if Γ contains $\exists^{\geq}(W,V)$ or *All V are W*, then $[\![V]\!] \subseteq [\![W]\!]$. We only verify the second assertion. For this, we may assume that $[\![V]\!] \neq \emptyset$ (otherwise the result is trivial). So $[\![V]\!] = M$. Thus $\Gamma \vdash \exists^{\geq}(V,X)$. So we see that $\Gamma \vdash \exists^{\geq}(W,X)$. From this we conclude that $[\![W]\!] = M$. In particular, $[\![V]\!] \subseteq [\![W]\!]$.

Now our claim implies that $\mathcal{M} \models \Gamma$. Therefore $|[\![X]\!]| \leq |[\![Y]\!]|$. And $[\![X]\!] = M$, since $\Gamma \vdash \exists^{\geq}(X,X)$. Hence $[\![Y]\!] = M$ as well. But this means that $\Gamma \vdash \exists^{\geq}(Y,X)$, just as desired.

We have shown one case of the general completeness theorem that we are after. In the other case, we have $\Gamma \models \textit{All X are Y}$. We construct a model $\mathcal{M} = \mathcal{M}_\Gamma$ such that for all A and B,

(α) $[\![A]\!] \subseteq [\![B]\!]$ iff $A \leq B$.

(β) If $A \leq_c B$, then $|[\![A]\!]| \leq |[\![B]\!]|$.

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Let $\mathcal{V}/\equiv_c$ be the (finite) set of equivalences classes of variables in Γ under $\equiv_c$. This set is then well-founded by the natural relation induced on it by $\leq_c$. It is then standard that we may list the elements of $\mathcal{V}/\equiv_c$ in some order

$$[U_0], [U_2], \ldots, [U_k]$$

with the property that if $U_i <_c U_j$, then $i < j$. (But if $i < j$, then it might be the case that $U_i \equiv_c U_j$.)

We define by recursion on $i \leq k$ the interpretation $[\![V]\!]$ of all $V \in [U_i]$. Suppose we have $[\![W]\!]$ for all $j < i$ and all $W \equiv_c U_j$. Let

$$\mathfrak{X}_i \quad = \bigcup_{j<i, W\equiv_c U_j} [\![W]\!],$$

and note that this is the set of all points used in the semantics of any variable so far. Let $n = 1 + |\mathfrak{X}_i|$. For all $V \equiv_c U_i$, we shall arrange that $[\![V]\!]$ be a set of size n.

Now $[U_i]$ is the equivalence class of U_i under $\equiv_c$. It splits into equivalence classes of the finer relation $\equiv$. For a moment, consider one of those finer classes, say $[A]_\equiv$. We must interpret each variable in this class by the same set. For this A, let

$$\mathfrak{Y}_A \quad = \bigcup\{[\![B]\!] : (\exists j < i)V_j \equiv_c B \leq A\}.$$

Note that $\mathfrak{Y}_A \subseteq \mathfrak{X}_i$ so that $|\mathfrak{Y}_A| < n$. We set $[\![A]\!]$ to be $\mathfrak{Y}_A$ together with $n - |\mathfrak{Y}_A|$ fresh elements. Moving on to the other $\equiv$-classes which partition the $\equiv_c$-class of U_i, we do the same thing. We must insure that for $A \not\equiv A'$, the fresh elements added into $[\![A']\!]$ are disjoint from the fresh elements added into $[\![A]\!]$.

This completes the definition of $\mathfrak{M}$. We check that so that conditions (α) and (β) are satisfied. It is easy to first check that for $i < j$, $|[\![U_i]\!]| < |[\![U_j]\!]|$. It might also be worth noting that $[\![U_0]\!] \neq \emptyset$, so no $[\![A]\!]$ is empty.

For (β), let $A \leq_c B$. Let i and j be such that $A \equiv_c U_i$ and $B \equiv_c U_j$. If $A \equiv_c B$, then $i = j$ and the construction arranged that $[\![A]\!]$ and $[\![B]\!]$ be sets of the same cardinality. If $A <_c B$, then $i < j$ by the way we enumerated the U's, and so $|[\![A]\!]| = |[\![U_i]\!]| < |[\![U_j]\!]| = |[\![B]\!]|$.

Turning to (α), we argue the two directions separately. Suppose first that $A \leq B$. Then $A \leq_c B$. If $A <_c B$, then $[\![A]\!] \subseteq \mathfrak{Y}_B \subseteq [\![B]\!]$. If $A \equiv_c B$, then we also have $A \equiv B$. The construction has then arranged that $[\![A]\!] = [\![B]\!]$. In the other direction, assume that $[\![A]\!] \subseteq [\![B]\!]$, and let i and j be such that $A \equiv_c U_i$ and $B \equiv_c U_j$. On cardinality grounds, $i \leq j$. If $i < j$, then the construction shows that $A \leq B$. (For if not, $[\![A]\!]$ would be a non-empty set disjoint from $[\![B]\!]$, and this contradicts $[\![A]\!] \subseteq [\![B]\!]$.) Finally (for perhaps the most interesting point), if $i = j$, then we must have $A \equiv B$: otherwise, the construction arranged that both A and B have at least one point that is not in the other, due to the "1+" in the definition of n.

Since (α) and (β), we know that $\mathfrak{M} \models \Gamma$. Recall that we are assuming that $X \leq Y$ holds semantically from Γ; we need to show that this assertion is derivable in the logic. But $[\![X]\!] \subseteq [\![Y]\!]$ in the model, and so by (α), we indeed have $\Gamma \vdash X \leq Y$. $\quad\dashv$

8.1 Larger syllogistic fragments

Having worked through $\mathcal{L}(all, \exists^\geq)$, it is natural to go on to further syllogistic fragments. We are not going to do this in detail. Instead, we simply state the results and move ahead in our final

section to the largest system in the paper, the one that adds $\exists^{\geq}$ to the system from Section 7. We would need two rules:

$$\frac{\textit{Some Y are Y} \quad \exists^{\geq}(X,Y)}{\textit{Some X are X}} \qquad \frac{\textit{No Y are Y}}{\exists^{\geq}(X,Y)}$$

The rule on the left should be added to our existing system for $\mathcal{L}(\textit{all}, \textit{some})$ (and adding the rules in Figure 8), and the resulting system would be complete for $\mathcal{L}(\textit{all}, \textit{some}, \exists^{\geq})$. Similarly, the rule on the right can be added to the system for $\mathcal{L}(\textit{all}, \textit{no})$ to get a completeness result. Finally, adding both rules to $\mathcal{L}(\textit{all}, \textit{some}, \textit{no})$ would again result in a complete system.

8.2 Digression: *Most*

The semantics of *Most* is that *Most X are Y* are that this is true iff $|[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![X]\!]|$. So if $[\![X]\!]$ is empty, then *Most X are Y* is false.

As an example of what is going on, consider the following. Assume that *All X are Z*, *All Y are Z*, *Most Z are Y*, and *Most Y are X*. Does it follow that *Most X are Y*? As it happens, the conclusion does not follow. One can take $X = \{a, b, c, d, e, f, g\}$, $Y = \{e, f, g, h, i\}$, and $Z = \{a, b, c, d, e, f, g, h, i\}$. Then $|X| = 7$, $|Y| = 5$, $|Z| = 9$, $|Y \cap Z| = 5 > 9/5$, $|X \cap Y| = 3 > 5/2$, but $|X \cap Y| = 3 < 7/2$. (Another countermodel: let $X = \{1, 2, 4, 5\}$, $Y = \{1, 2, 3\}$, and $Z = \{1, 2, 3, 4, 5\}$. Then $|Y \cap Z| = 3 > 5/2$, $|Y \cap X| = 2 > 3/2$, but $|X \cap Y| = 2 \not> 4/2$.)

On the other hand, the following is a sound rule:

$$\frac{\textit{All U are X} \quad \textit{Most X are V} \quad \textit{All V are Y} \quad \textit{Most Y are U}}{\textit{Some U are V}}$$

Here is the reason for this. Assume our hypotheses and also that towards a contradiction that U and V were disjoint. We obviously have $|V| \geq |X \cap V|$, and the second hypothesis, together with the disjointness assumption, tells us that $|X \cap V| > |X \cap U|$. By the first hypothesis, we have $|X \cap U| = |U|$. So at this point we have $|V| > |U|$. But the last two hypotheses similarly give us the opposite inequality $|U| > |V|$. This is a contradiction.

At the time of this writing, I do not have a completeness result for $\mathcal{L}(\textit{all}, \textit{some}, \textit{most})$. The best that is known is for $\mathcal{L}(\textit{some}, \textit{most})$. The rules are are shown in Figure 9. We study these on top of the rules in Figure 3.

Proposition 8.3 *The following two axioms are complete for* Most.

$$\frac{\textit{Most X are Y}}{\textit{Most X are X}} \qquad \frac{\textit{Most X are Y}}{\textit{Most Y are Y}}$$

Moreover, if $\Gamma \subseteq \mathcal{L}(\text{most})$, $X \neq Y$, *and* $\Gamma \not\models \textit{Most X are Y}$, *then there is a model* $\mathcal{M}$ *of* Γ *which falsifies* Most X are Y *in which all sets of the form* $[\![U]\!] \cap [\![V]\!]$ *are nonempty, and* $|M| \leq 5$.

Proof Suppose that $\Gamma \not\vdash \textit{Most X are Y}$. We construct a model $\mathcal{M}$ which satisfies all sentences in Γ, but which falsifies *Most X are X*. There are two cases. If $X = Y$, then X does not occur in any sentence in Γ. We let $\mathcal{M} = \{*\}$, $[\![X]\!] = \emptyset$, and $[\![Y]\!] = \{*\}$ for $Y \neq X$.

The other case is when $X \neq Y$. Let $\mathcal{M} = \{1, 2, 3, 4, 5\}$, $[\![X]\!] = \{1, 2, 4, 5\}$, $[\![Y]\!] = \{1, 2, 3\}$, and for $Z \neq X, Y$, $[\![Z]\!] = \{1, 2, 3, 4, 5\}$. Then the only statement in *Most* which fails in the model $\mathcal{M}$ is *Most X are Y*. But this sentence does not belong to Γ. Thus $\mathcal{M} \models \Gamma$. $\dashv$

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$\dfrac{Most\ X\ are\ Y}{Some\ X\ are\ Y}$	$\dfrac{Some\ X\ are\ X}{Most\ X\ are\ X}$	$\dfrac{Most\ X\ are\ Y \quad Most\ X\ are\ Z}{Some\ Y\ are\ Z}$

Figure 9: Rules of *Most* to be used in conjunction with *Some*.

Theorem 8.4 *The rules in Figure 9 together with the first two rules in Figure 3 are complete for $\mathcal{L}(\mathrm{some}, \mathrm{most})$. Moreover, if $\Gamma \not\models S$, then there is a model $\mathcal{M} \models \Gamma$ with $\mathcal{M} \not\models S$, and $|M| \leq 6$.*

Proof Suppose $\Gamma \not\vdash S$, where S is *Some X are Y*. If $X = Y$, then Γ contains no sentence involving X. So we may satisfy Γ and falsify S in a one-point model, by setting $[\![X]\!] = \emptyset$ and $[\![Z]\!] = \{*\}$ for $Z \neq X$.

We next consider the case when $X \neq Y$. Then Γ does not contain S, *Some Y are X*, *Most X are Y*, or *Most Y are X*. And for all Z, Γ does not contain both *Most Z are X* and *Most Z are Y*. Let $M = \{1, 2, 3, 4, 5, 6\}$, and consider the subsets $a = \{1, 2, 3\}$, $b = \{1, 2, 3, 4, 5\}$, $c = \{2, 3, 4, 5, 6\}$, and $d = \{4, 5, 6\}$. Let $[\![X]\!] = a$ and $[\![Y]\!] = d$, so that $\mathcal{M} \not\models S$. For Z different from X and Y, if Γ does not contain *Most Z are X*, let $[\![Z]\!] = c$. Otherwise, Γ does not contain *Most Z are Y*, and so we let $[\![Z]\!] = b$. For all these Z, $\mathcal{M}$ satisfies whichever of the sentences *Most Z are X* and *Most Z are Y* (if either) which belong to Γ. $\mathcal{M}$ also satisfies all sentences *Most X are Z* and *Most Y are Z*, whether or not these belong to Γ. It also satisfies *Most U are U* for all U. Also, for Z, Z' each different from both X and Y, $\mathcal{M} \models$ *Most Z are Z'*. Finally, $\mathcal{M}$ satisfies all sentences *Some U are V* except for $U = X$ and $Y = V$ (or vice-versa). But those two sentences do not belong to Γ. The upshot is that $\mathcal{M} \models \Gamma$ but $\mathcal{M} \not\models S$.

Up until now in this proof, we have considered the case when S is *Some X are Y*. We turn our attention to the case when S is *Most X are Y*. Suppose $\Gamma \not\vdash S$. If $X = Y$, then the second rule of Figure 9 shows that $\Gamma \not\vdash$ *Some X are X*. So we take $M = \{*\}$ and take $[\![X]\!] = \emptyset$ and for $Y \neq X$, $[\![Y]\!] = M$. It is easy to check that $\mathcal{M} \models \Gamma$.

Finally, if $X \neq Y$, we clearly have $\Gamma_{most} \not\vdash S$. Proposition 8.3 shows that there is a model $\mathcal{M} \models \Gamma_{most}$ which falsifies S in which all sets of the form $[\![U]\!] \cap [\![V]\!]$ are nonempty. So all *Some* sentences hold in $\mathcal{M}$. Hence $\mathcal{M} \models \Gamma$. $\dashv$

8.3 Adding $\exists^{\geq}$ to the boolean syllogistic fragment

We now put aside *Most* and return to the study of $\exists^{\geq}$ from earlier. We close this paper with the addition of $\exists^{\geq}$ to the fragment of Section 7.

Our logical system extends the axioms of Figure 7 by those in Figure 10. Note that the last new axiom expresses *cardinal comparison*. Axiom 4 in Figure 10 is just a transcription of the rule for *No* that we saw in Section 8.1. We do not need to also add the axiom

$$(Some\ Y\ are\ Y) \wedge \exists^{\geq}(X, Y) \rightarrow Some\ X\ are\ X$$

because it is derivable. Here is a sketch, in English. Assume that there are some Ys, and there are at least as many Xs as Ys, but (towards a contradiction) that there are no Xs. Then all X's are Ys. From our logic, all Ys are Xs as well. And since there are Y's, there are also X's: a contradiction.

1. *All X are Y* $\rightarrow \exists^{\geq}(Y, X)$

2. $\exists^{\geq}(X, Y) \wedge \exists^{\geq}(Y, Z) \rightarrow \exists^{\geq}(X, Z)$

3. *All Y are X* $\wedge \exists^{\geq}(Y, X) \rightarrow$ *All X are Y*

4. *No X are X* $\rightarrow \exists^{\geq}(Y, X)$

5. $\exists^{\geq}(X, Y) \vee \exists^{\geq}(Y, X)$

Figure 10: Additions to the system in Figure 7 for $\exists^{\geq}$ sentences.

Notice also that in the current fragment we can express *There are more X than Y*. It would be possible to add this directly to our previous systems.

Theorem 8.5 *The logic of Figures 7 and 10 is complete for assertions $\Delta \models \varphi$ in the language of boolean combinations of sentences in $\mathcal{L}(\text{all}, \text{some}, \text{no}, \exists^{\geq})$.*

Proof   We need only build a model for a maximal consistent set Δ in the language of this section. We take the *basic* sentences to be those of the form *All X are Y*, *Some X and Y*, *J is M*, *J is an X*, $\exists^{\geq}(X, Y)$, or their negations. Let

$$\Gamma \quad = \quad \{S : \Delta \models S \text{ and } S \text{ is basic}\}.$$

As in Claim 7.3, we need only build a model $\mathcal{M} \models \Gamma$. We construct $\mathcal{M}$ such that for all A and B,

(α) $[\![A]\!] \subseteq [\![B]\!]$ iff $A \leq B$.

(β) $A \leq_c B$ iff $|[\![A]\!]| \leq |[\![B]\!]|$.

(γ) For $A \leq_c B$, $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$.

Let $\mathcal{V}$ be the set of variables in Γ. Let $\leq_c$ and $\equiv_c$ be as in Section 8. Proposition 8.1 again holds, and now the quotient $\mathcal{V}/\equiv_c$ is a linear order due to the last axiom in Figure 10. We write it as

$$[U_0] \quad <_c \quad [U_2] \quad <_c \quad \cdots \quad <_c \quad [U_k]$$

We define by recursion on $i \leq k$ the interpretation $[\![V]\!]$ of all $V \in [U_i]$. The case of $i = 0$ is special. If $\Gamma \models$ *No U_0 is a U_0*, then the same holds for all $W \equiv_c U_0$. In this case, we set $[\![W]\!] = \emptyset$ for all these W. Note that by our fourth axiom in Figure 10, all of the other variables W are such that $\Gamma \vdash \exists W$. In any case, we must interpret the variables in $[U_0]$ even when $\Gamma \vdash (\exists U_0)$. In this case, we may take each $[\![W]\!]$ to be a singleton, with the added condition that $V \equiv W$ iff $[\![V]\!] = [\![W]\!]$.

Suppose we have $[\![W]\!]$ for all $j \leq i$ and all $W \equiv_c U_j$. Let

$$\mathcal{X}_{i+1} \quad = \quad \bigcup_{j \leq i, W \equiv_c U_j} [\![W]\!],$$

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and note that this is the set of all points used in the semantics of any variable so far. Let $m = |\Gamma_{some}|$, and let

$$n \quad = \quad 1 + m + |\mathcal{X}_{i+1}| \tag{15}$$

For all $V \equiv_c U_{i+1}$, we shall arrange that $[\![V]\!]$ be a set of size n.

Now $[U_{i+1}]$ splits into equivalence classes of the finer relation $\equiv$. For a moment, consider one of those finer classes, say $[A]_{\equiv}$. We must interpret each variable in this class by the same set. For this A, let

$$\mathcal{Y}_A \quad = \quad \bigcup \{ [\![B]\!] : (\exists j \le i) \; V_j \equiv_c B \le A \}.$$

Note that $\mathcal{Y}_A \subseteq \mathcal{X}_{i+1}$ so that $|\mathcal{Y}_A| \le |\mathcal{X}_{i+1}|$ for all $A \equiv_c U_{i+1}$. We shall set $[\![A]\!]$ to be $\mathcal{Y}_A$ plus other points. Let $\mathcal{Z}_A$ be the set of pairs $\{A, B\}$ with $B \equiv_c U_{i+1}$ and $A \uparrow_\Gamma B$. (This is the same as saying that *Some A are B* in Γ_{some}.) Notice that if both A and B are $\equiv_c U_{i+1}$ and $A \uparrow_\Gamma B$, then $\{A, B\} \in \mathcal{Z}_A \cap \mathcal{Z}_B$.) We shall set $[\![A]\!]$ to be $\mathcal{Y}_A \cup \mathcal{Z}_A$ plus one last group of points. If $C <_c U_{i+1}$ and $A \uparrow_\Gamma C$, then we must pick some element of $[\![C]\!]$ and put it into $[\![A]\!]$. Note that the number of points selected like this plus $|\mathcal{Z}_A|$ is still $\le |\Gamma_{some}|$. So the number of points so far in $[\![A]\!]$ is $\le |\Gamma_{some}| + m$. We finally add fresh elements to $[\![A]\!]$ so that the total is n.

We do all of this for all of the other $\equiv$-classes which partition the $\equiv_c$-class of U_{i+1}. We must insure that for $A \not\equiv A'$, the fresh elements added into $[\![A']\!]$ are disjoint from the fresh elements added into $[\![A]\!]$. This is needed to arrange that neither $[\![A]\!]$ nor $[\![A']\!]$ will be a subset of the other.

This completes the definition of the model. We say a few words about why requirements (α)–(γ) are met. First, and easy induction on i shows that if $j < i$, then $|[\![U_j]\!]| < |[\![U_i]\!]|$. The point is that $|[\![U_j]\!]| \le |\mathcal{X}_i| < |[\![U_i]\!]|$. The argument for (β) is the same as in the proof of Theorem 8.2. For that matter, the proof of (α) is also essentially the same. The point is that when $A \equiv_c B$ and $A \not\equiv B$, then $[\![A]\!]$ and $[\![B]\!]$ each contain a point not in the other.

For (γ), suppose that $A \le_c B$. Let $i \le j$ be such that $A \equiv_c U_i$ and $B \equiv U_j$. The construction arranged that $[\![A]\!]$ and $[\![B]\!]$ be disjoint except for the case that $A \uparrow B$.

So this verifies that (α)–(γ) hold. We would like to conclude that $\mathcal{M} \models \Gamma$, but there is one last point: (γ) appears to be a touch too weak. We need to know that $[\![A]\!] \cap [\![B]\!] \ne \emptyset$ iff $A \uparrow B$ (without assuming $A \le_c B$). But either $A \le_c B$ or $B \le_c A$ by our last axiom. So we see that indeed $[\![A]\!] \cap [\![B]\!] \ne \emptyset$ iff $A \uparrow B$. $\dashv$

The next step in this direction would be to consider *At least as many X as Y are Z*.

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