# Syllogistic Logics with Verbs

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**Abstract**

This paper provides sound and complete logical systems for several fragments of English which go beyond syllogistic logic in that they use verbs as well as other limited syntactic material: universally and existentially quantified noun phrases, building on the work of Nishihara, Morita, and Iwata [7]; complemented noun phrases, following our [6]; and noun phrases which might contain relative clauses, recursively, based on McAllester and Givan [4]. The logics are all syllogistic in the sense that they do not make use of individual variables. Variables in our systems range over nouns, and in the last system, over verbs as well.

## 1 Introduction

This paper is an outgrowth of our work in the past few years on syllogistic logics [5, 6]. The basic idea was to look at extremely small fragments of natural language with their natural semantics, and then to axiomatize the resulting consequence relations in as simply a way as possible. For an example of what we mean, we mention the syllogistic logic of *All* and *Some* taken from [5]. We begin with *(noun) variables* $X$, $Y$, ..., and then take as sentences *All* $X$ *are* $Y$ and *Some* $X$ *are* $Y$ where $X$ and $Y$ are variables (possibly identical). We call this language $\mathcal{L}(\textit{all}, \textit{some})$. Then one gives a semantics by starting with a set $M$ and a subset $[\![X]\!]$ for each variable $X$. The resulting structure $\mathcal{M} = (M, [\![\ ]\!])$ is called a *model*. We say that $\mathcal{M} \models$ *All* $X$ *are* $Y$ iff $[\![X]\!] \subseteq [\![Y]\!]$, and $\mathcal{M} \models$ *Some* $X$ *are* $Y$ iff $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. We then write $\Gamma \models S$ to mean that every model of all sentences in the set $\Gamma$ is a model of the sentence $S$. The goal is then to present a sound and complete logical system for this fragment. We present such a system in Figure 1. The figure consists of *proof rules*, presented schematically. Then we define *proof trees* in the usual way. We write $\Gamma \vdash S$ if $\Gamma \cup \{S\}$ is a set of sentences in the fragment, and if there is a proof tree whose leaves are labeled with elements of $\Gamma$ (or with the axioms *All* $X$ *are* $X$) and whose root is labeled $S$, and such that at every interior point in the tree, some rule is matched. For example,

$$\{\textit{Some } A \textit{ are } B, \textit{All } A \textit{ are } X, \textit{All } B \textit{ are } Y\} \vdash \textit{Some } X \textit{ are } Y$$

as witnessed by tree below:

$$\dfrac{\textit{All B are Y} \qquad \dfrac{\dfrac{\textit{All A are X} \quad \dfrac{\textit{Some A are B}}{\textit{Some B are A}}}{\textit{Some B are X}}}{\textit{Some X are B}}}{\textit{Some X are Y}}$$

Then one can show the soundness and completeness of the proof theory with respect to the semantics: $\Gamma \vdash S$ iff $\Gamma \models S$. This is just one result of this type, and [5] presents others. Perhaps the first results of this type is the completeness of syllogistic logic on top of propositional logic, due to Łukasiewicz [3] and independently to Westerståhl [11]. Incidentally, completeness results are not limited to fragments of first-order logic. But all of the fragments in [5] and [6] do have a limitation: they use the copula ('is' or 'are') as their only verb. So it is of interest to go further.

The first paper in this direction is Nishihara, Morita, and Iwata [7]. We call their fragment the *NMI* fragment, and similarly for the logical system. It adds transitive verbs, so one can reason in the NMI system with sentences like *All students read a book*. Moreover, both scope readings are possible. The paper is the inspiration for what we do in Section 2 below. We reformulated the system and gave a completeness result for the boolean-free fragment. In a different direction, we can take syllogistic logic and enlarge the nouns by allowing $X'$ ("non-$X$") in addition to $X$, with semantics given by $[\![X']\!] = M \setminus [\![X]\!]$. A complete syllogistic logic for this fragment is presented in Section 3. That work connects to the origin of this volume, the Order, Algebra, and Logics conference held at Vanderbilt University in June 2007: the syllogistic logic of *All*, and the complement operation is analyzed by a very simple combination of algebra and order. The last section of our paper goes off in yet another direction, providing the first syllogistic completeness proofs for infinite fragments of language, ones proposed by McAllester and Givan in [4]; that paper proves the decidability, but not logical completeness.

We would like to comment on the particular choice of logics studied in this paper. The three are unrelated. What they have in common is that they go beyond what was known previously in the area, and at the same time they approach an "Aristotle boundary" of syllogistic systems. To make this precise, we mention a few results from Pratt-Hartmann and Moss [10]. First, consider adding negation to the verbs in the NMI fragment, obtaining sentences like *Some student does not read every book* or *Every student fails to read some book*. It is shown in [10] that there is a complete syllogistic system for this fragment, but *only* if one allows a rule of *reductio ad absurdum* (RAA). (More precisely, there are no pure syllogistic systems for the NMI fragment, and there is a system which uses (RAA) in a very minimal way: only as the final step of a deduction.) We take (RAA) to be a syllogistic rule, but not in the purest sense: using (RAA) allows assumptions to be discharged. On a technical level, (RAA) allows for more powerful systems than ones which lack it. Getting back to our results in Section 2 and 3, part of the interest is that the systems there do not employ (RAA). The system in Section 2 uses infinitely many rules, but this is unavoidable. Our work in Section 4 does not use (RAA) either: none of the systems in this paper admit contradictions. But it does use a rule allowing proof by cases, so again it is not syllogistic in the purest sense. Adding negation to the verbs would result in a system which cannot be axiomatized by any syllogistic system, even one with (RAA).

Our work here could be of interest to people working on natural logic (see van Benthem [1]), and to the complexity of various fragments of natural language (see Pratt-Hartmann [8, 9, 10]. It could conceivably be of some interest to people working on fragments of first-order logic such

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}$$

Figure 1: The logic of *All* $X$ *are* $Y$ and *Some* $X$ *are* $Y$.

as the two-variable fragment (see Grädel et al [2]). At one time I believed that this line of work could be pedagogically useful, because one obtains completeness theorems for various logics but works without the syntactically complicated features such as substitution, free and bound variables, etc. This might be true for the simpler completeness arguments in [5]. But the ones here are too complex for beginning students of logic.

## 2 An Explicitly Scoped Variant of the NMI Fragment

When one considers natural language noun phrases (NPs) in connection with logic, one of the first things to point out is *quantifier-scope ambiguity*. For example, *All dogs see a cat* could mean that each dog sees some cat or other, and it could also possibly mean that there is one cat which all dogs see. The standard line in linguistic semantics is that the semantic ambiguity reflects a syntactic one. The point of raising this is that a logical system for natural language has the same problem, and it must either respond to the problem by adopting one or another explicit syntax, or else work somehow with disambiguated representations.

The first paper to present a syllogistic logic with verbs was Nishihara, Morita, and Iwata [7]. Their work is important for the present paper because they showed that completeness was possible. We wish to reformulate the syntax of the NMI fragment, change the proof system, and then to obtain a completeness theorem. We should mention that their system also has proper nouns, which we ignore. More importantly, it also has boolean connectives. We wish to avoid these connectives because the complexity of the system is increased by adding them.

The system in [7] has the following syntax: The basic sentences are those of the following forms

$$NP\ V\ NP \qquad NP\ -V\ NP$$

where $NP$ is either *All* $X$, *Some* $X$, or a proper name, and $V$ is a verb which takes a direct object. The notation $-V$ indicates verb negation. In addition, we have boolean sentential connectives.

Their semantics is unusual in that they assume *all existential NPs have wide scope*. Here are some examples:

| Sentence | Semantics |
|---|---|
| *All* $X$ *love some* $Y$ | $(\exists y)(Y(y) \wedge (\forall x)(X(x) \rightarrow L(x,y))$ |
| *Some* $X$ *loves all* $Y$ | $(\exists x)(X(x) \wedge (\forall y)(Y(y) \rightarrow L(x,y))$ |
| $\sim$ (*Some* $X$ *not love all* $Y$) | $(\forall x)(X(x) \rightarrow (\exists y)(Y(y) \wedge L(x,y))$ |

### 2.1 The language $\mathcal{L}(\boldsymbol{all}, \boldsymbol{some}, \boldsymbol{see})$

The language in [7] is not so convenient to read and use, since the assumption that existentials have wide scope is unfamiliar. We are interested in reformulating it. At the same time, we like the fact that the language is unambiguous. One way to do keep this feature and yet have things more readable is to add explicit scope information to sentences. This is only needed in sentences with both universal and existential quantifiers. We can add explicit information either to the whole sentence or just to the verb; the choice at this level is immaterial.

The language is called $\mathcal{L}(all, some, see)$; here is a formal presentation of its syntax. We start with variables $X$, $Y$, ..., and also a single verb, *see*. (We write this verb in English rather than with the symbol $V$, since $V$ is close to the other letters already used in this paper.) Our syntax adds to the syllogistic logic of *All* and *Some* the sentences below:

$$\begin{array}{llll}
(\textit{All } X \textit{ see all } Y)_{\mathsf{sws}} & & (\textit{All } Y \textit{ see all } X)_{\mathsf{ows}} & \\
(\textit{All } X \textit{ see some } Y)_{\mathsf{sws}} & \forall x \exists y\ S(x,y) & (\textit{Some } Y \textit{ see all } X)_{\mathsf{ows}} & \forall x \exists y\ S(y,x) \\
(\textit{Some } X \textit{ see all } Y)_{\mathsf{sws}} & \exists x \forall y\ S(x,y) & (\textit{All } Y \textit{ see some } X)_{\mathsf{ows}} & \exists x \forall y\ S(y,x) \\
(\textit{Some } X \textit{ see some } Y)_{\mathsf{sws}} & & (\textit{Some } Y \textit{ see some } X)_{\mathsf{ows}} &
\end{array} \tag{1}$$

(We intend that $X$ and $Y$ be any variables, of course. Our choice of the letters $X$ and $Y$ in the sentences in (1) comes from a discussion at the end of this section.) The marking indicates whether the subject or object has the wide scope. For example,

$$(\textit{All } Y \textit{ see some } X)_{\mathsf{ows}} \tag{2}$$

indicates the object wide scope reading. So this is what the original NMI system writes as *All* $Y$ *see some* $X$. We can write this as or even as $(\exists x \in X)(\forall y \in Y)S(y,x)$. For the subject wide scope reading, we use $\mathsf{sws}$. We also adopt a convention that the scope markings are only used in sentences with two different quantifiers. In sentences with two universal quantifiers or two existential quantifiers, the different scope readings are logically equivalent, so we almost always drop the scope notation. However, occasionally it will be convenient to write our sentences in full, as we have done above.

We have added in (1) shorthand glosses next to some of the sentences; of course, the glosses are not part of the syntax.

Since we have the two scope readings explicitly present, there is no need to also include a sentential negation operation. This simplifies the system quite a bit. However, there are still eight different types of syntactic expressions, and so proofs about the system must involve lengthy case-by-case work.

In the semantics, we take a model to be a tuple $\mathcal{M} = (M, [\![\ ]\!], [\![see]\!])$, where $[\![X]\!] \subseteq M$ for all variables $X$, and $[\![see]\!] \subseteq M \times M$. That is, the verb is interpreted as a binary relation.

At this point, we can say why our fragment has only one verb. We *could* formulate a language with more than one verb, of course. But each sentence in this fragment would only use one verb. (In more linguistic terms, since we have no relative clauses, there is no way to make an argument with multiple verbs.) Moreover, the whole system is a sub-logic of first-order logic. And if we have a semantic consequence $\Gamma \models S$, then the set $\Gamma_0$ of sentences in $\Gamma$ with the same verb as in $S$ would also semantically imply $S$. (This may be seen by applying the Craig Interpolation Lemma, but for this fragment it also can be shown by a simple direct semantic argument.) The upshot is that all of the interesting features come from a single verb.

$$
\begin{array}{ll}
\textit{All } X^{\downarrow} \textit{ are } Y^{\uparrow} & (\textit{All } X^{\downarrow} \textit{ see some } Y^{\uparrow})_{\mathsf{ows}} \\
\textit{Some } X^{\uparrow} \textit{ are } Y^{\uparrow} & (\textit{Some } X^{\uparrow} \textit{ see all } Y^{\downarrow})_{\mathsf{ows}} \\
\textit{All } X^{\downarrow} \textit{ see all } Y^{\downarrow} & (\textit{All } X^{\downarrow} \textit{ see some } Y^{\uparrow})_{\mathsf{sws}} \\
(\textit{Some } X^{\uparrow} \textit{ see all } Y^{\downarrow})_{\mathsf{sws}} & \textit{Some } X^{\uparrow} \textit{ see some } Y^{\uparrow}
\end{array}
$$

$$
\frac{(\textit{All } X \textit{ see } NP)_{\mathsf{sws}} \quad \textit{Some } X \textit{ are } Y}{(\textit{Some } Y \textit{ see } NP)_{\mathsf{sws}}}
\qquad
\frac{(NP \textit{ see all } X)_{\mathsf{ows}} \quad \textit{Some } X \textit{ are } Y}{(NP \textit{ see some } Y)_{\mathsf{ows}}}
$$

$$
\frac{(\textit{Some } X \textit{ see all } Y)_{\mathsf{sws}}}{(\textit{Some } X \textit{ see all } Y)_{\mathsf{ows}}}
\qquad
\frac{(\textit{All } X \textit{ see some } Y)_{\mathsf{ows}}}{(\textit{All } X \textit{ see some } Y)_{\mathsf{sws}}}
$$

$$
\frac{(\textit{Some } X \textit{ see } NP)_{\mathsf{sws}}}{\textit{Some } X \textit{ is an } X}
\qquad
\frac{(NP \textit{ see some } X)_{\mathsf{ows}}}{\textit{Some } X \textit{ is an } X}
$$

$$
\frac{X \rightsquigarrow Z \quad \textit{All } X \textit{ see all } Z \quad \textit{All } Z \textit{ are } Y}{(\textit{All } X \textit{ see some } Y)_{\mathsf{sws}}}\ (I)
$$

$$
\frac{Y \rightsquigarrow Z \quad \textit{All } Z \textit{ see all } Y \quad \textit{All } Z \textit{ are } X \quad \textit{Some } X \textit{ are } X}{(\textit{Some } X \textit{ see all } Y)_{\mathsf{sws}}}\ (J)
$$

$$
\frac{Y \rightsquigarrow Z \quad \textit{All } Z \textit{ see all } Y \quad \textit{All } Z \textit{ are } X}{(\textit{Some } X \textit{ see all } Y)_{\mathsf{ows}}}\ (K)
$$

$$
\frac{X \rightsquigarrow Z \quad \textit{All } X \textit{ see all } Z \quad \textit{All } Z \textit{ are } Y \quad \textit{Some } Y \textit{ are } Y}{(\textit{All } X \textit{ see some } Y)_{\mathsf{ows}}}\ (L)
$$

Figure 2: Rules for the scoped formulation of the NMI fragment, in addition to the rules of Figure 1. At the top are *monotonicity rules*, and the arrow notation is explained in the text. $NP$ denotes a noun phrase of the form *All* $X$ or one of the form *Some* $X$. The last four rules are actually schematic presentations of infinite families of rules; see the text for explanation of the $\rightsquigarrow$ notation.

We would have liked to provide a finite syllogistic system for this fragment. However, a result in Pratt-Hartmann and Moss [10] may be modified to show that *no finite syllogistic system exists* for this logic. One way to proceed would be to add verb-level negation and then to add a rule of *reductio ad absurdum.* This is the route taken by [10]. However, the fragment there does not have all the scope readings that we consider here. Ziegler [12] axiomatizes the closure of the fragment which we study here under the standard connectives. In this paper, we strike out on a different direction, employing infinite schemata of rules in addition to a finite list of more standard syllogistic rules.

**The $c$-operation** We define an operation $S \mapsto S^c$ on sentences of the language by interchanging the subject and object noun phrases, and then changing the scope. This corresponds to interchanging sentences across the rows in (1). For example,

$$
((\textit{All } W \textit{ see some } Z)_{\mathsf{sws}})^c \quad = \quad (\textit{Some } Z \textit{ see all } W)_{\mathsf{ows}}.
$$

For sentences $S$ in the syllogistic language with *All* and *Some* (and no verb), $S^c = S$.

The main fact about this operation is that

$$(M, [\![\ \ ]\!], [\![see]\!]) \models S \quad \text{iff} \quad (M, [\![\ \ ]\!], [\![see]\!]^c) \models S^c,$$

where $[\![see]\!]^c$ is the converse of the relation $[\![see]\!]$. As a result, if $\Gamma \models S$, then $\Gamma^c \models S^c$, where $\Gamma^c = \{T^c : T \in \Gamma\}$.

The main use of the $c$-operation in this paper is to shorten some of the arguments for the soundness and completeness of the proof system presented in the next section.

## 2.2 Proof system

Our proof system starts with the rules of Figure 1, and then we add the rules of Figure 2. We start with the *monotonicity rules* at the top of the figure. The up-arrow and down-arrow notation is taken from Johan van Benthem [1], see especially Chapter 6, Natural Logic. Our statements abbreviate 15 rules, in the following way. The upward arrow notation $X^\uparrow$ means that moving from $X$ to a superset preserves truth, while the notation $X^\downarrow$ means that moving from $X$ to a subset preserves truth. For example, the notation $(All\ X^\downarrow\ see\ some\ Y^\uparrow)_{\mathsf{ows}}$ represents the following two rules:

$$\frac{(All\ X\ \ see\ some\ Y)_{\mathsf{ows}} \quad All\ Y\ \ are\ Z}{(All\ X\ \ see\ some\ Z)_{\mathsf{ows}}} \qquad \frac{(All\ X\ \ see\ some\ Y)_{\mathsf{ows}} \quad All\ Z\ \ are\ X}{(All\ Z\ \ see\ some\ Y)_{\mathsf{ows}}}$$

The central rules of the system are those in the middle of Figure 2. Among these rules one can find the usual laws of scope: from $\exists x \forall y$ infer $\forall y \exists x$. Four of the rules in the figure are in a shorthand notation that actually covers eight rules. For example, the rule that infers *Some X is an X* from the premise $(Some\ X\ \ see\ NP)_{\mathsf{sws}}$ is to be understood as (1) an inference from the premise $(Some\ X\ \ see\ all\ Y)_{\mathsf{sws}}$, and (2) an inference from *Some X see some Y*.

The last rules are the infinite schemata mentioned earlier. To clarify this, we must define the $\rightsquigarrow$ notation. We write $X \rightsquigarrow Y$ as a shorthand for a sequence of sentences such as

$$\begin{array}{l} (All\ X\ \ see\ some\ C_1)_{\mathsf{sws}}, (All\ C_1\ \ see\ some\ C_2)_{\mathsf{sws}}, (Some\ C_3\ \ see\ all\ C_2)_{\mathsf{ows}}, \\ \ldots (Some\ C_n\ \ see\ all\ C_{n-1})_{\mathsf{ows}}, (All\ C_n\ \ see\ some\ Y)_{\mathsf{sws}} \end{array} \tag{3}$$

Each sentence in the sequence must be of the form $(All\ U\ \ see\ some\ W)_{\mathsf{sws}}$ or else of the form $(Some\ U\ \ see\ all\ W)_{\mathsf{ows}}$. Also, the narrow scope variable of one sentence must be the wide scope variable of the next. Finally, $X$ must be the wide scope variable of the first sentence in the sequence, and $Y$ the narrow scope variable of the last in the sequence.

We write $\Gamma \vdash X \rightsquigarrow Y$ to mean that there is a sequence of sentences as above, each of which is derivable from $\Gamma$. The idea is that if $\Gamma \vdash X \rightsquigarrow Y$ and *if* a model $\mathcal{M}$ has $X$s, then $\mathcal{M}$ must also have $Y$'s. So we could read $X \rightsquigarrow Y$ as "if $X$ is non-empty, so is $Y$."

In the definition of $\rightsquigarrow$ in (3), we allow $n = 0$. In this case, the sequence would be empty. So $X \rightsquigarrow X$ for all $X$. Taking $n = 0$ in $(I)$ corresponds to the case when $X$ and $Z$ are identical. This instance of $(I)$ would boil down to the inference of $(All\ X\ \ see\ some\ X)_{\mathsf{sws}}$ from $(All\ X\ \ see\ all\ X)_{\mathsf{sws}}$. In $(J)$, the sequence again may be empty; in this case $Y$ and $Z$ are identical, and we infer *Some X see all Y* from *All Y see all Y*, *All Y are X*, and $\exists X$.

**Soundness of the infinite rule schemata** We sketch the verification of the soundness of the infinite schemata $(I)$ and $(J)$. Usually we use the definition of $\rightsquigarrow$ in the contrapositive; so if $Y \rightsquigarrow Z$ and there are no $Z$ (in a particular model), then there are no $Y$'s (in that model).

For $(I)$, fix a sequence as in (3) of sentences showing that $X \rightsquigarrow Z$, and let $\mathcal{M}$ satisfy all sentences in this sequence. We argue by cases on $[\![Z]\!]$ in $\mathcal{M}$. If this set is empty, then so is $[\![X]\!]$, and so $\mathcal{M} \models (\textit{All } X \textit{ see some } Y)_{\mathsf{sws}}$ vacuously. Otherwise any $Z$ would be seen by all $X$ and at the same time a $Y$.

For $(J)$, let $\mathcal{M}$ satisfy the assumptions. We argue informally, by cases on $[\![Z]\!]$. If there are no $Z$, there are no $Y$; together with $\exists X$, we have the conclusion. If there are $Z$, then any such is an $X$, and so we have *Some* $X$ *see all* $Y$.

The other two schemata are obtained from $(I)$ and $(J)$ using the $c$-operation. This accounts for their soundness. In addition, the system has the property that $\Gamma \vdash S$ iff $\Gamma^c \vdash S^c$.

## 2.3 Completeness

To simplify our notation, we abbreviate our sentences as $\sigma_1$, …, $\sigma_6$ as defined in Figure 3. Actually, we should write these as $\sigma_{i,U,W}$; we only do this when we need to. We also note the diagram of implications in the figure. It is important to note that the arrows $\sigma_1 \rightarrow \sigma_2$ and $\sigma_5 \rightarrow \sigma_6$ require $\Gamma \vdash \exists U$; $\sigma_1 \rightarrow \sigma_3$ and $\sigma_4 \rightarrow \sigma_6$ requires $\Gamma \vdash \exists W$.

We consider subsets of $\{\sigma_1, \ldots, \sigma_6\}$. There are eleven subsets $s$ which are *implication-closed*: if $s \cup \{\exists U, \exists W\} \vdash S$, then $S \in s$. In pictures, these are the *down-closed* subsets of the hexagonal diagram in Figure 3. For example, for all $\Gamma \subseteq \mathcal{L}(\textit{all}, \textit{some}, \textit{see})$ and all variables $U$ and $W$ such that $\Gamma \vdash \exists U, \exists W$, we have a down-closed set

$$Th_\Gamma(U,W) \quad = \quad \{\sigma_{i,U,W} : \Gamma \vdash \sigma_{i,U,W}\}. \tag{4}$$

But if $\Gamma \not\vdash \exists U$ or if $\Gamma \not\vdash \exists W$, then $Th_\Gamma(U,W)$ might not be down-closed: $Th_\Gamma(U,W)$ might contain *All* $U$ *see all* $W$ without containing the sentences below it in Figure 3. However, even in this case, we can take the *downward closure* of $Th_\Gamma(U,W)$ in the order shown in Figure 3, and we write this as $\downarrow Th_\Gamma(U,W)$.

Our intention is to take a set $\Gamma$ of sentences in this fragment and a set $\mathcal{S}$ of variables and to build a model $\mathcal{M}(\Gamma, \mathcal{S})$ from the sets $\downarrow Th_\Gamma(U,W)$. We shall return to this model construction shortly, but first we need a more basic construction.

Let $U, W$ be any variables. Let $U_1$, $U_2$, $U_3$, $W_1$, $W_2$, and $W_3$ be three copies of $U$ and $W$. For each down-closed subset $s \subseteq \{\sigma_{1,U,W}, \ldots, \sigma_{6,U,W}\}$, we specify a fixed relation

$$R_{U,W,s} \quad \subseteq \quad \{U_1, U_2, U_3\} \times \{W_1, W_2, W_3\}.$$

These relations $R_{U,W,s}$ are shown in Figure 3.

**Proposition 2.1** *Let $U$ and $W$ be any variables.*

1. *The down-closed subsets of $\{\sigma_{1,U,W}, \ldots, \sigma_{6,U,W}\}$ are the* 11 *sets listed in Figure 3.*
2. *For all down-closed $s$, and all $1 \leq i \leq 6$, $\sigma_i \in s$ iff $(A_{U,W}, R_{U,W,s}) \models \sigma_i$.*
3. *If $(U_2, W_j) \in R_{U,W,s}$ for some $j$, then $(A_{U,W}, R_{U,W,s}) \models \sigma_5$.*

**Proof** These points are verified by direct inspection. $\dashv$

| | |
|---|---|
| $\sigma_1$ | *All U see all W* |
| $\sigma_2$ | (*Some U see all W*)$_{\mathsf{sws}}$ |
| $\sigma_3$ | (*All U see some W*)$_{\mathsf{ows}}$ |
| $\sigma_4$ | (*Some U see all W*)$_{\mathsf{ows}}$ |
| $\sigma_5$ | (*All U see some W*)$_{\mathsf{sws}}$ |
| $\sigma_6$ | *Some U see some W* |

$$\begin{tikzcd}
& \sigma_1 \arrow[dl] \arrow[dr] & \\
\sigma_2 \arrow[d] & & \sigma_3 \arrow[d] \\
\sigma_4 \arrow[dr] & & \sigma_5 \arrow[dl] \\
& \sigma_6 &
\end{tikzcd}$$

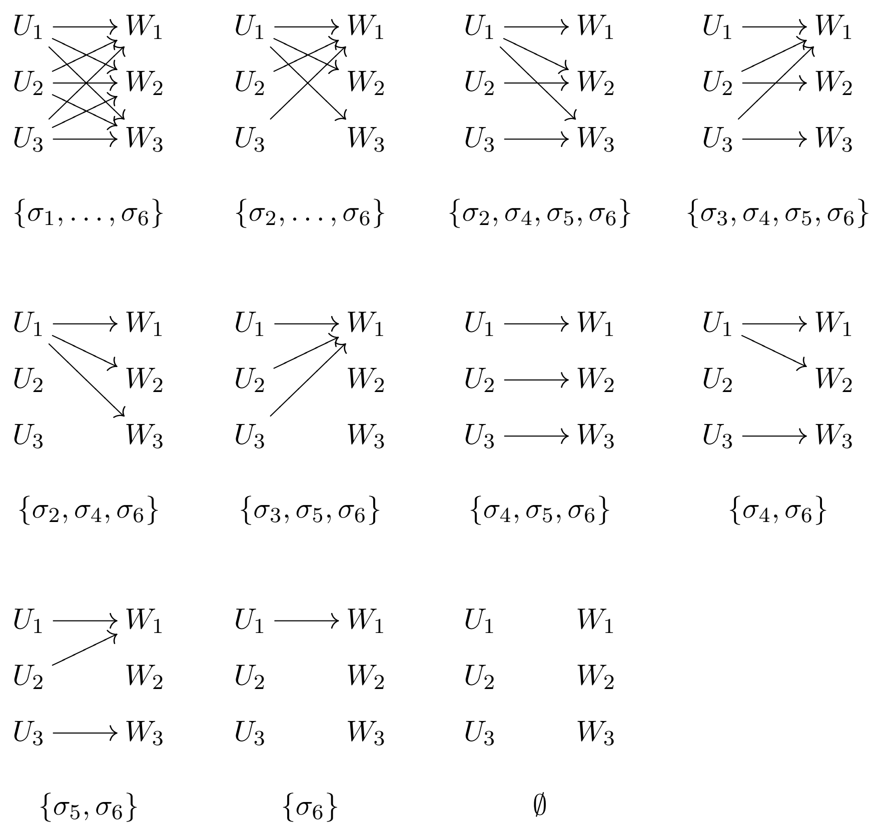


Figure 3: At the top are the six sentences in fixed variables $U$ and $W$. The hexagon shows the semantic implication relations among these sentences, assuming $\exists U$ and $\exists W$. Below these are the down-closed sets and the definition of special relations used in the completeness proof.

**A model construction** Let $\Gamma$ be a set of sentences in $\mathcal{L}(\textit{all}, \textit{some}, \textit{see})$. We write $\mathcal{E}(\Gamma)$ for $\{U : \Gamma \vdash \textit{Some } U \textit{ are } U\}$. We also say that a set $\mathcal{S}$ of variables is $\Gamma$-*existentially closed* (EC) if whenever $U \in \mathcal{S}$ and $\Gamma \vdash U \rightsquigarrow W$, then $W \in \mathcal{S}$. For example, $\mathcal{E}(\Gamma)$ is $\Gamma$-EC, and the union of two $\Gamma$-EC sets is again $\Gamma$-EC.

Let $\Gamma$ be a set of sentences in our fragment, and let $\mathcal{S}$ be $\Gamma$-EC and include $\mathcal{E}(\Gamma)$. We turn to a description of $\mathcal{M} = \mathcal{M}(\Gamma, \mathcal{S})$. First, let

$$M \quad = \quad \{U_1, U_2, U_3 : U \in \mathcal{S}\} \cup \{\{A, B\} : A \neq B \text{ and } \Gamma \vdash \textit{Some } A \textit{ are } B\}$$

So we have three copies of each variable $U$ from $\mathcal{S}$, together with some other unordered pairs. We assume that the union above is disjoint, so no variable is itself a pair of others. The only use of subscripts in the rest of this proof is to refer to the copied variables. The purpose of $\{A, B\}$ will be to insure that *Some A are B* will hold in our model $\mathcal{M}$.

The rest of the structure of $\mathcal{M}$ deals with the interpretation of nouns $U$ and of the verb *see*:

| | |
|---|---|
| $W_i \in [\![U]\!]$ | iff $\Gamma \vdash \textit{All } W \textit{ are } U$ |
| $\{A, B\} \in [\![U]\!]$ | iff $\Gamma \vdash \textit{All } A \textit{ are } U$, or $\Gamma \vdash \textit{All } B \textit{ are } U$ |
| $U_i [\![\textit{see}]\!] W_j$ | iff $(U_i, W_j) \in R_{U,W,\downarrow Th_\Gamma(U,W)}$ |
| $\{U, Z\} [\![\textit{see}]\!] W_2$ | iff $\Gamma \vdash \sigma_{1,U,W}$ or $\Gamma \vdash \sigma_{1,Z,W}$ |
| $\{U, Z\} [\![\textit{see}]\!] W_1$ | iff $\{U, Z\} [\![\textit{see}]\!] W_2$, or $\Gamma \vdash \sigma_{3,U,W}$, or $\Gamma \vdash \sigma_{3,Z,W}$ |
| $\{U, Z\} [\![\textit{see}]\!] W_3$ | iff $\{U, Z\} [\![\textit{see}]\!] W_2$, or $\Gamma \vdash \sigma_{5,U,W}$ or $\Gamma \vdash \sigma_{5,Z,W}$ |
| $U_2 [\![\textit{see}]\!] \{W, Z\}$ | iff $\Gamma \vdash \sigma_{1,U,W}$ or $\Gamma \vdash \sigma_{1,U,Z}$ |
| $U_1 [\![\textit{see}]\!] \{W, Z\}$ | iff $U_2 [\![\textit{see}]\!] \{W, Z\}$, or $\Gamma \vdash \sigma_{3,U,W}$, or $\Gamma \vdash \sigma_{2,U,Z}$ |
| $U_3 [\![\textit{see}]\!] \{W, Z\}$ | iff $U_2 [\![\textit{see}]\!] \{W, Z\}$, or $\Gamma \vdash \sigma_{4,U,W}$, or $\Gamma \vdash \sigma_{4,Z,W}$ |
| $\{A, B\} [\![\textit{see}]\!] \{C, D\}$ | iff $\Gamma \vdash \sigma_{1,A,C}$, or $\Gamma \vdash \sigma_{1,A,D}$, or $\Gamma \vdash \sigma_{1,B,C}$, or $\Gamma \vdash \sigma_{1,B,D}$ |

($Th_\Gamma(U, W)$ is defined in (4), $\downarrow Th(U, W)$ is its downward closure, and the relation $R_{U,W,s}$ is presented in Figure 3.)

**Lemma 2.2** *Assume that $\mathcal{S}$ is $\Gamma$-EC and that $\mathcal{E}(\Gamma) \subseteq \mathcal{S}$. Then $\mathcal{M}(\Gamma, \mathcal{S}) \models \Gamma$.*

**Proof** It is easy to check that for sentences $S$ in $\mathcal{L}(\textit{all}, \textit{some})$ which are belong to $\Gamma$, $\mathcal{M}(\Gamma, \mathcal{S}) \models S$. ($\mathcal{L}(\textit{all}, \textit{some})$ was mentioned in the Introduction.) For *All* sentences, this is a routine monotonicity point. Here is the reasoning for a sentence of the form *Some A are B*. Note that $\{A, B\}$ belongs to $M$, and it also belongs to $[\![A]\!] \cap [\![B]\!]$.

Moving on, consider a sentence in $\Gamma$ such as $\sigma_{3,U,W}$ from above, $(\textit{Some } U \textit{ see all } W)_{\mathsf{sws}}$. In this case, $\Gamma \vdash \exists U$, and so $U \in \mathcal{E}(\Gamma)$. We claim that $U_1$ is related to all elements of $[\![W]\!]$. For if $Z_i \in [\![W]\!]$ by virtue of $Z \in \mathcal{S}$ and $\Gamma \vdash \textit{All } Z \textit{ are } W$, then $\Gamma \vdash \sigma_{3,U,Z}$. And so in $\mathcal{M}$, $U_1 [\![\textit{see}]\!] Z_i$ construction of the model. For the elements of $[\![W]\!]$ of the form $\{A, B\}$, the structure of $\mathcal{M}$ also ensures that $U_1 [\![\textit{see}]\!] \{A, B\}$.

We next consider $\sigma_{5,U,W}$, $(\textit{All } U \textit{ see some } W)_{\mathsf{sws}}$. Assuming that $\sigma_{5,U,W}$ belongs to $\Gamma$, we check that this holds in $\mathcal{M}(\Gamma, \mathcal{S})$. If $[\![U]\!] = \emptyset$, then trivially $\mathcal{M}(\Gamma, \mathcal{S}) \models \sigma_{5,U,W}$. If $[\![U]\!] \neq \emptyset$, then $U \in \mathcal{S}$. Since $\mathcal{S}$ is $\Gamma$-EC and $\sigma_{5,U,W} \in \Gamma$, we have $W \in \mathcal{S}$ as well. Turning to the verification, we must consider elements of $[\![U]\!]$ of the form $A_i$, and also the elements of the form $\{A, B\}$. First, we consider the elements $A_i$. For this, $\Gamma \vdash \textit{All } A \textit{ are } U$. By monotonicity, $\Gamma \vdash \sigma_{5,A,W}$. Thus $\sigma_{5,A,W} \in Th_\Gamma(A, W) \subseteq \downarrow Th_\Gamma(A, W)$. Hence by construction $\mathcal{M}(\Gamma, \mathcal{S}) \models \sigma_{5,A,W}$. In particular, for the element $A_i$ we are considering, there is some $W_j$ such that $A_i [\![\textit{see}]\!] W_j$. We also must consider elements of $[\![U]\!]$ such as $\{A, B\}$. For this, $\Gamma \vdash$ *Some A are B*. We may assume that $\Gamma \vdash$ *All A are U*. Thus $\Gamma \vdash \sigma_{5,A,W}$. By construction, $\{A, B\}[\![see]\!]W_2$. In this way, we have verified that all elements of $[\![U]\!]$ are related to some element of $[\![W]\!]$ or other.

We omit the rest of the similar verifications showing that $\mathcal{M}(\Gamma, \mathcal{S}) \models \Gamma$. $\dashv$

**Theorem 2.3** *The logical system determined by the rules in Figures 1 and 2 is sound and complete for* $\mathcal{L}(\text{all}, \text{some}, \text{see})$.

**Proof** Suppose $\Gamma \models S$. We show that $\Gamma \vdash S$. The argument splits into cases according to the syntax of $S$.

**The first case:** $S \in \mathcal{L}(\textbf{\textit{all}}, \textbf{\textit{some}})$**.** Let

$$\Delta \quad = \quad \{T \in \mathcal{L}(\textit{all}, \textit{some}) : \Gamma \vdash T\}.$$

We show that $\Delta \models S$ in $\mathcal{L}(\textit{all}, \textit{some})$. For this, let $\mathcal{M} \models \Delta$. We may assume that if $[\![Z]\!] \neq \emptyset$, then $\Gamma \vdash \exists Z$. (Otherwise, re-set $[\![Z]\!]$ to be empty, and check that this does not change the truth values in $\mathcal{L}(\textit{all}, \textit{some})$.) Turn $\mathcal{M}$ into a structure $\mathcal{M}^+$ for $\mathcal{L}(\textit{all}, \textit{some}, \textit{see})$ by $[\![\textit{see}]\!] = M \times M$; i.e., by relating every point to every point. We shall check that $\mathcal{M}^+ \models \Gamma$. It then follows that $\mathcal{M}^+ \models S$. And since the set variables are interpreted the same way on the two models, we see that $\mathcal{M} \models S$. Now to check that $\mathcal{M}^+ \models \Gamma$, we argue by cases. For example, suppose that $\Gamma$ contains the sentence in (2). One of our axioms implies directly that $\Gamma \vdash T$, where $T$ is *Some Y is a Y*. Thus $T \in \Delta$. So $[\![Y]\!] \neq \emptyset$ in $\mathcal{M}$. Then the structure of $\mathcal{M}^+$ tells us that this model indeed satisfies (2). Similar arguments apply to sentences of forms different than that of (2). We shall look at one more case, the subject wide scope version of (2). We may assume that $[\![X]\!] \neq \emptyset$. Recall from above that we may assume that $\Gamma \vdash$ *Some X are X*. And now we have the following $\Gamma$-deduction:

$$\dfrac{\dfrac{(\textit{All } X \textit{ see some } Y)_{\mathsf{sws}} \quad \begin{array}{c}\vdots \\ \textit{Some } X \textit{ are } X\end{array}}{\textit{Some } X \textit{ see some } Y}}{\textit{Some } Y \textit{ are } Y}$$

The rest of the argument is similar. We now know that $\Delta \models S$. We use the completeness of the syllogistic logic of *All* and *Some* (cf. [5], Theorem 3.4) to see that $\Delta \vdash S$. Since the system here includes this, $\Gamma \vdash S$.

$S$ **is** $\sigma_{6,X,Y}$**,** ***Some X see some Y*****.** Here we let $\mathcal{S} = \mathcal{E}(\Gamma)$, and consider $\mathcal{M}(\Gamma, \mathcal{S})$. By Lemma 2.2, $\mathcal{M} \models \Gamma$. Hence $\mathcal{M} \models S$. There are several cases here, depending on the data witnessing $S$. One case is when there are $U_j \in [\![X]\!]$ and $W_j \in [\![Y]\!]$ related by $[\![\textit{see}]\!]$. We argue from $\Gamma$. We have *All U are X*, *All W are Y*, $\exists U$, $\exists W$, and finally, $\sigma_{k,U,W}$ for some $1 \leq k \leq 6$. Using $\exists U$ and $\exists W$, we easily see that $\sigma_{6,U,W}$ holds. And then by monotonicity, $\sigma_{6,X,Y}$, as desired.

We must also consider the case $[\![\textit{see}]\!]$ contains a pair such as $(\{A, B\}, \{C, D\})$. There would be four subcases here, and we go into details on only one of them: suppose $\{A, B\}[\![\textit{see}]\!]\{C, D\}$. Without loss of generality, $\Gamma \vdash \sigma_{1,A,C}$. Since $\Gamma$ also derives *Some A exists*, *All A are X*, *Some B exists*, *All B are Y*, we easily get the desired $\sigma_{6,X,Y}$. The reasoning is similar when $[\![\textit{see}]\!]$ contains a pair such as $(\{A, B\}, Y_j)$ or one such as $(X_i, \{A, B\})$.

$S$ **is** $\sigma_{1,X,Y}$**,** ***All*** $X$ ***see all*** $Y$**.** This time, we take

$$\mathcal{S} \quad = \quad \mathcal{E}(\Gamma) \cup \{U : \Gamma \vdash X \rightsquigarrow U \text{ or } \Gamma \vdash Y \rightsquigarrow U\}.$$

This $\mathcal{S}$ is easily seen to be $\Gamma$-EC. As a result, $\mathcal{M}(\Gamma, \mathcal{S}) \models \Gamma$. Hence also $\mathcal{M}(\Gamma, \mathcal{S}) \models S$. Note that $X$ and $Y$ belong to $\mathcal{S}$. Being a universal sentence, $S$ is preserved under submodels. $(A_{X,Y}, R_{X,Y,\downarrow Th(X,Y)})$ is a submodel of $\mathcal{M}$, and so it satisfies $\sigma_{1,X,Y}$. By Proposition 2.1, part 2, $\sigma_{1,X,Y} \in \downarrow Th(X,Y)$. But this means that indeed $\sigma_{1,X,Y}$ belongs to $Th(X,Y)$.

$S$ **is** $\sigma_{5,X,Y}$**,** $($***All*** $X$ ***see some*** $Y)_{\mathsf{sws}}$**.** Here we take

$$\mathcal{S} \quad = \quad \mathcal{E}(\Gamma) \cup \{U : \Gamma \vdash X \rightsquigarrow U\}. \tag{5}$$

As in our previous cases, $\mathcal{M}(\Gamma, \mathcal{S}) \models S$. We have $X \in \mathcal{S}$, by definition. In particular, $X_2 \in M$. The witness to $\sigma_{5,X,Y}$ in $[\![Y]\!]$ for $X_2$ is either of the form $Z_i$ or a pair $\{A, B\}$. This paragraph only presents the details for a witness of the first form. Let $Z_i$ be such that $X_2[\![\mathit{see}]\!]Z_i$ and $\Gamma \vdash$ *All* $Z$ *are* $Y$. By Proposition 2.1, part 3, $\sigma_{5,X,Z} \in \downarrow Th_\Gamma(X,Z)$. We would like to know that $\sigma_{5,X,Z} \in Th_\Gamma(X,Z)$. Looking at the hexagon in Figure 3, we three possibilities: $Th_\Gamma(X,Z)$ must contain either $\sigma_{5,X,Z}$, $\sigma_{3,X,Z}$, or $\sigma_{1,X,Z}$. If $\sigma_{5,X,Z} \in Th_\Gamma(X,Z)$, then by monotonicity, $\Gamma \vdash \sigma_{5,X,Y}$; so we are done. If $\sigma_{3,X,Z} \in Th_\Gamma(X,Z)$, then by our logic, $\Gamma \vdash \sigma_{5,X,Z}$. Then just as above, $\Gamma \vdash \sigma_{5,X,Y}$. Finally, we have the case that $\sigma_{1,X,Z} \in Th_\Gamma(X,Z)$. Either $Z \in \mathcal{E}(\Gamma)$, or else $\Gamma \vdash X \rightsquigarrow Z$. If $Z \in \mathcal{E}(\Gamma)$, then by monotonicity, $Y \in \mathcal{E}(\Gamma)$. In this case, $\Gamma \vdash S$ as follows:

$$\frac{\begin{array}{c}\vdots \\ (\mathit{All}\ X\ \mathit{see\ all}\ Y)_{\mathsf{sws}}\end{array} \qquad \begin{array}{c}\vdots \\ \mathit{Some}\ Y\ \mathit{are}\ Y\end{array}}{(\mathit{All}\ X\ \mathit{see\ some}\ Y)_{\mathsf{sws}}} \tag{6}$$

(We used the first rule in Figure 2.) When $\Gamma \vdash X \rightsquigarrow Z$, we use $(I)$ to immediately derive the desired conclusion $\sigma_{5,X,Y}$.

We have another overall case, going back to $X_2 \in M$. Suppose that $\{A, B\} \in [\![Y]\!]$ (so that *Some* $A$ *are* $B$) and $X_2[\![\mathit{see}]\!]\{A, B\}$. There are four cases here, perhaps one representative one is when $\Gamma \vdash$ *All* $A$ *are* $Y$ and $\Gamma \vdash \sigma_{1,X,B}$. Then from $\Gamma$ we have *All* $X$ *see all* $B$, hence $(\mathit{All}\ X\ \mathit{see\ some}\ A)_{\mathsf{sws}}$, Finally, by monotonicity we have $(\mathit{All}\ X\ \mathit{see\ some}\ Y)_{\mathsf{sws}}$.

$S$ **is** $\sigma_{2,X,Y}$**,** $($***Some*** $X$ ***see all*** $Y)_{\mathsf{sws}}$**.** On our assumption that $\Gamma \models S$, we have $\Gamma \models \exists X$. Thus by the first case on $S$ in this overall proof of this theorem, we know that $X \in \mathcal{E}(\Gamma)$. In the model construction, we take $\mathcal{S} = \mathcal{E}(\Gamma) \cup \{U : \Gamma \vdash Y \rightsquigarrow U\}$. The resulting model $\mathcal{M}(\Gamma, \mathcal{S})$ then satisfies $S$. Suppose that the witness to the subject quantifier in $S$ is $Z_i$. (In the next paragraph, we see what happens when it is of the form $\{A, B\}$.) Then $\Gamma \vdash$ *All* $Z$ *are* $X$. By the structure of the model, we either have $\Gamma \vdash S$ (and then we are easily done by monotonicity) or $\Gamma \vdash$ *All* $Z$ *see all* $Y$. In the latter case, if $Z \in \mathcal{E}(\Gamma)$, we easily see that $\Gamma \vdash S$. So we are left with the case that $\Gamma \vdash Y \rightsquigarrow Z$. And here we use $(J)$.

We also need to consider the situation when the witness to the subject quantifier in $S$ is of the form $\{A, B\}$. Then $\{A, B\} \in [\![X]\!]$ is related to $Y_2$. By the structure of the model, we have $\Gamma \vdash \sigma_{1,A,Y}$ (for example), and also $\Gamma \vdash$ *All* $B$ *are* $X$ and $\Gamma \vdash$ *Some* $A$ *are* $B$. We would then have in several steps that $\Gamma \vdash (\mathit{Some}\ X\ \mathit{see\ all}\ Y)_{\mathsf{sws}}$.

$S$ **is** $\sigma_{3,X,Y}$**,** ($\textbf{\textit{All}}\ X\ \textbf{\textit{see some}}\ Y)_{\mathsf{ows}}$**, or** $\sigma_{4,X,Y}$**,** ($\textbf{\textit{Some}}\ X\ \textbf{\textit{see all}}\ Y)_{\mathsf{ows}}$**.** These cases are entirely parallel to the previous two. In fact, completeness in these cases may also be proved using the $c$-operation as follows. If $\Gamma \models \sigma_{3,X,Y}$ (for example), then $\Gamma^c \models (\sigma_{3,X,Y})^c = \sigma_{4,Y,X}$, and then $\Gamma^c \vdash \sigma_{4,Y,X}$, by what we have already seen. Hence $\Gamma \vdash \sigma_{3,X,Y}$.

This concludes the proof of Theorem 2.3. $\dashv$

# 3 Syllogistic Logic with *All*, a Verb, and Noun-level Complements

## 3.1 *All*, a verb, and noun-level complements

We consider the fragment with *All*, a single verb, and noun-level complements. For example, here is an extended syllogistic argument in the fragment:

$$\frac{\begin{array}{l}\textit{All xenophobics hate all actors}\\ \textit{All yodelers hate all zookeepers}\\ \textit{All non-yodelers hate all non-actors}\\ \textit{All wardens are xenophobics}\end{array}}{\textit{All wardens hate all zookeepers}}$$

Why does the conclusion follow? Take a warden. He or she will be a xenophobic, and hence hate all actors. If also a yodeler, he or she will certainly hate all zookeepers; if not, he or she will hate all non-actors and hence hate everyone whatsoever, a fortiori all zookeepers.

We formalize this fragment in the obvious way. We add a *complemented variable* $X'$ for each $X$. We insist in the semantics that $[\![X']\!]$ be the complement of $[\![X]\!]$. We also identify each $X''$ with the corresponding variable $X$. Finally, we continue to use "see" as the verb, our example argument above notwithstanding. The resulting language is called $\mathcal{L}(\mathit{all}, \mathit{see}, ')$. A logic for it is shown in Figure 4. 'LEM" stands for "law of the excluded middle." $VP$ means "verb phrase." So the rule (Zero) allows one to conclude from *All* $Y$ *are* $Y'$ that *All* $Y$ *are* $Z$, *All* $Y$ *see all* $A$, etc. (Zero-VP) allows one to conclude from the same premise that *All* $X$ *see all* $Y$. The rule (3pr) is so-named because it has three premises; I am not aware of a standard name for it. Most of the soundness details are variations on what we have seen above. The informal argument above corresponds to the formal proof below:

$$\dfrac{\dfrac{\textit{All } X \textit{ hate all } A \quad \textit{All } Y \textit{ hate all } Z \quad \textit{All } Y' \textit{ hate all } A'}{\textit{All } X \textit{ hate all } Z}\ \textit{3pr} \qquad \textit{All } W \textit{ are } X}{\textit{All } W \textit{ hate all } Z}\ \textit{monotonicity}$$

**Theorem 3.1** *Let* $\Gamma \cup \{S\} \subseteq \mathcal{L}(\mathrm{all}, \mathrm{see}, ')$. *Then* $\Gamma \vdash S$ *iff* $\Gamma \models S$.

As we start in on the proof, we first claim that the case when $S$ is a sentence without the verb is handled already by the completeness of the syllogistic logic $\mathcal{L}(\mathit{all}, ')$ of *All* and complement studied in our earlier paper [6]. Specifically, if $\Gamma \models S$ in our current language $\mathcal{L}(\mathit{all}, \mathit{see}, ')$, we claim that $\Gamma \cap \mathcal{L}(\mathit{all}, ') \models S$. [For this: take a model of $\Gamma \cap \mathcal{L}(\mathit{all}, ')$, and relate all points to all other points. This gives interpretation of the verbs, hence a model of $S$.] Then $\Gamma \vdash S$ in the smaller language $\mathcal{L}(\mathit{all}, ')$.

**Definition** Fix a set $\Gamma$. We write $A \leq B$ if $\Gamma \vdash$ *All* $A$ *are* $B$. A set $\mathcal{S}$ of variables is a *point* if

$$\frac{\textit{All } Y \textit{ are } Y'}{\textit{All } Y\ VP}\ \textit{Zero} \qquad \frac{\textit{All } Y' \textit{ are } Y}{\textit{All } X \textit{ are } Y}\ \textit{One}$$

$$\frac{\textit{All } Y \textit{ are } X'}{\textit{All } X \textit{ are } Y'}\ \textit{Antitone} \qquad \frac{\textit{All } Y \textit{ are } Y'}{\textit{All } X \textit{ see all } Y}\ \textit{Zero-VP}$$

$$\frac{\textit{All } X \textit{ see all } Y \quad \textit{All } X' \textit{ see all } Y}{\textit{All } Z \textit{ see all } Y}\ \textit{LEM} \qquad \frac{\textit{All } X \textit{ see all } Y \quad \textit{All } X \textit{ see all } Y'}{\textit{All } X \textit{ see all } Z}\ \textit{LEM}'$$

$$\frac{\textit{All } X \textit{ see all } A \quad \textit{All } Y \textit{ see all } Z \quad \textit{All } Y' \textit{ see all } A'}{\textit{All } X \textit{ see all } Z}\ \textit{3pr}$$

Figure 4: The *All* syllogistic logic with verbs and noun-level complements, leaving off the axioms *All* $X$ *are* $X$ and the monotonicity rules *All* $X^{\downarrow}$ *are* $Y^{\uparrow}$ and *All* $X^{\downarrow}$ *see all* $Y^{\downarrow}$.

$\mathcal{S}$ is up-closed in this order and for each $A$, $\mathcal{S}$ contains either $A$ or $A'$, but not both.

This notion of a point comes from our paper [6], and it also has roots in work on quantum logic. It would be called a *quantum state* there. In a sense, it plays the role for this logic analogous to that of an ultrafilter in modal logic: the canonical model of the logic is constructed from points. However, there are quite a few differences, and this analogy is not terribly useful.

**Lemma 3.2** *Fix* $\Gamma$, *and also fix* $X$ *and* $Y$ *such that*

$$\Gamma \not\vdash \text{All } X \text{ see all } Y.$$

*Then there are points* $\mathcal{S}$ *and* $\mathcal{T}$ *such that* $X \in \mathcal{S}$, $Y \in \mathcal{T}$, *and for all* $A \in \mathcal{S}$ *and* $B \in \mathcal{T}$, $\Gamma \not\vdash \text{All } A \text{ see all } B$.

**Proof** We use Zorn's Lemma. We consider the set of pairs $(\mathcal{S}, \mathcal{T})$ of sets of variables with the properties below:

1. $X \in \mathcal{S}$ and $Y \in \mathcal{T}$.

2. $\mathcal{S}$ is *pairwise compatible*: for all $A$ and $B$ in $\mathcal{S}$, $A \not\leq B'$ and $B \not\leq A'$; and so is $\mathcal{T}$.

3. For all $A \in \mathcal{S}$ and $B \in \mathcal{T}$, $\Gamma \not\vdash$ *All* $A$ *see all* $B$.

4. For all $B, C \in \mathcal{T}$ and all $D$, either $\Gamma \not\vdash$ *All* $D$ *see all* $B$ or $\Gamma \not\vdash$ *All* $D'$ *see all* $C$.

5. For all $B, C \in \mathcal{S}$ and all $D$, either $\Gamma \not\vdash$ *All* $B$ *see all* $D$ or $\Gamma \not\vdash$ *All* $C$ *see all* $D'$.

The collection $P$ of such pairs of sets is ordered in the natural way, by inclusion in both components. Call the resulting poset $\mathbb{P} = (P, \subseteq)$. We intend to apply Zorn's Lemma to $\mathbb{P}$. This poset is obviously closed under unions of chains. We also claim that $(\{X\}, \{Y\})$ belongs to $P$. For this, (1) and (3) are obvious. For (2), if $X \leq X'$, then we use the (zero) rule to get a contradiction; if $Y \leq Y'$, we use (Zero-VP). (4) uses (LEM), and (5) uses (LEM$'$).

By Zorn's Lemma, $\mathbb{P}$ contains a maximal pair $(\mathcal{S}, \mathcal{T})$. The central claim is that any such $(\mathcal{S}, \mathcal{T})$ must have the property that for all $D$, $\mathcal{S}$ contains either $D$ or $D'$, and $\mathcal{T}$ also contains either $D$ or $D'$. From this and pairwise compatibility it follows that both $\mathcal{S}$ and $\mathcal{T}$ must be up-closed in the order. Hence $\mathcal{S}$ and $\mathcal{T}$ would both be points, as desired. We shall prove this claim for $\mathcal{S}$; the arguments for $\mathcal{T}$ are parallel. Suppose towards a contradiction that neither $D$ nor $D'$ belongs to $\mathcal{S}$. Consider $\mathcal{S} \cup \{D\}$ and $\mathcal{S} \cup \{D'\}$. For each of these sets, one of conditions (2), (3), or (5) must fail. Moreover, $D$ and $D'$ must be implicated in the failures. There are six overall cases. But also, a failure of (5) for one of our sets splits into subcases depending on whether $D$ (or $D'$) is the witness to the failure of only one quantifier in (5) or of both.

<u>Case 1</u> Condition (2) fails for $\mathcal{S}\cup\{D\}$ and $\mathcal{S}\cup\{D'\}$ because there are $B, C \in \mathcal{S}$ such that $B \leq D'$ and $C \leq D$. Then $D' \leq C'$ using (antitone). Hence $B \leq C'$ by transitivity. This contradicts condition (2) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 2</u> Condition (2) fails for $\mathcal{S} \cup \{D\}$, and condition (3) fails for $\mathcal{S} \cup \{D'\}$ because there are $C \in \mathcal{S}$ and $B \in \mathcal{T}$ such that $C \leq D'$ and $\Gamma \vdash$ *All* $D'$ *see all* $B$. Then by monotonicity, we have $\Gamma \vdash$ *All* $C$ *see all* $B$, and this contradicts (3) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 3</u> Condition (2) fails for $\mathcal{S} \cup \{D\}$, and condition (5) fails for $\mathcal{S} \cup \{D'\}$ because there are $B, C \in \mathcal{S}$ and also $E$ such that $C \leq D'$, $\Gamma \vdash$ *All* $B$ *see all* $E$, and $\Gamma \vdash$ *All* $D'$ *see all* $E'$. In this case, $\Gamma \vdash$ *All* $C$ *see all* $E'$ by monotonicity. And so we violate (5) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 3a</u> Condition (2) fails for $\mathcal{S} \cup \{D\}$, and condition (5) fails for $\mathcal{S} \cup \{D'\}$ because there is some $C \in \mathcal{S}$ and also $E$ such that $C \leq D'$, $\Gamma \vdash$ *All* $D'$ *see all* $E$, and $\Gamma \vdash$ *All* $D'$ *see all* $E'$. Now we have $\Gamma \vdash$ *All* $D'$ *see all* $Y$ by (*LEM*); $Y$ here is from the statement of our lemma. And since $C \leq D'$, we see by monotonicity that $\Gamma \vdash$ *All* $C$ *see all* $Y$. This contradicts condition (3) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 4</u> Condition (3) fails for both $\mathcal{S} \cup \{D\}$ and $\mathcal{S} \cup \{D'\}$ because there are $B, C \in \mathcal{T}$ such that $\Gamma \vdash$ *All* $D$ *see all* $B$, and $\Gamma \vdash$ *All* $D'$ *see all* $C$. This contradicts (4) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 5</u> Condition (3) fails for $\mathcal{S} \cup \{D\}$, and condition (5) fails for $\mathcal{S} \cup \{D'\}$ because there are $A \in \mathcal{S}$, $B \in \mathcal{T}$, and also $E$ such that $\Gamma \vdash$ *All* $D$ *see all* $B$, $\Gamma \vdash$ *All* $A$ *see all* $E$, and $\Gamma \vdash$ *All* $D'$ *see all* $E'$. By the rule (*3pr*), $\Gamma \vdash$ *All* $A$ *see all* $B$. This contradicts (3) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 5a</u> Condition (3) fails for $\mathcal{S}\cup\{D\}$, and condition (5) fails for $\mathcal{S}\cup\{D'\}$ because there are $B \in \mathcal{T}$, and also $E$ such that $\Gamma \vdash$ *All* $D$ *see all* $B$, $\Gamma \vdash$ *All* $D'$ *see all* $E$, and $\Gamma \vdash$ *All* $D'$ *see all* $E'$. As in case 3a, $\Gamma \vdash$ *All* $D'$ *see all* $B$. And for the same reason, we see that $\Gamma \vdash$ *All* $X$ *see all* $B$. This contradicts (3) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 6</u> Condition (5) fails for both $\mathcal{S} \cup \{D\}$ and $\mathcal{S} \cup \{D'\}$ because there are $B, C \in \mathcal{S}$ and also $E$ and $F$ such that

(i) $\Gamma \vdash$ *All* $B$ *see all* $E$

(ii) $\Gamma \vdash$ *All* $D$ *see all* $E'$

(iii) $\Gamma \vdash$ *All* $C$ *see all* $F$

(iv) $\Gamma \vdash$ *All* $D'$ *see all* $F'$

Using (i), (ii), (iv), and the rule (*3pr*), we see that $\Gamma \vdash$ *All* $B$ *see all* $F'$. And then this and (iii) contradicts condition (5) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 6a</u> Condition (5) fails for both $\mathcal{S} \cup \{D\}$ and $\mathcal{S} \cup \{D'\}$ because there is some $B \in \mathcal{S}$ and also $E$ and $F$ such that

(i) $\Gamma \vdash$ *All* $B$ *see all* $E$

(ii) $\Gamma \vdash$ *All* $D$ *see all* $E'$

(iii) $\Gamma \vdash$ *All* $D'$ *see all* $F$

(iv) $\Gamma \vdash$ *All* $D'$ *see all* $F'$

Then (iii) and (iv) show that $\Gamma \vdash$ *All* $D'$ *see all* $Y$, using (*LEM*$'$). This with (i), (ii), and (*3pr*) then shows that $\Gamma \vdash$ *All* $B$ *see all* $Y$; with $Y$ as in the statement of this lemma. Again we contradict (3) for $(\mathcal{S}, \mathcal{T})$.

<u>Case 6b</u> Condition (5) fails for both $\mathcal{S} \cup \{D\}$ and $\mathcal{S} \cup \{D'\}$ because

(i) $\Gamma \vdash$ *All* $D$ *see all* $E$

(ii) $\Gamma \vdash$ *All* $D$ *see all* $E'$

(iii) $\Gamma \vdash$ *All* $D'$ *see all* $F$

(iv) $\Gamma \vdash$ *All* $D'$ *see all* $F'$

This time the logic implies that $\Gamma \vdash$ *All* $X$ *see all* $Y$, with $X$ and $Y$ as in the statement of this lemma. This is contrary to the hypothesis in our lemma.

This shows that a maximal pair $(\mathcal{S}, \mathcal{T})$ consists of a pair of points, thereby completing the proof of this lemma. $\dashv$

We now prove Theorem 3.1, the completeness result for the logical system of this section. We restate it in the following way.

**Lemma 3.3** *Let* $\Gamma \subseteq \mathcal{L}(\text{all}, \text{see}, ')$. *Let* $S$ *be All* $X$ *see all* $Y$. *Assume that* $\Gamma \not\vdash S$. *There is a two-point model* $\mathcal{M} \models \Gamma$ *such that* $\mathcal{M} \not\models S$.

**Proof** Let $\mathcal{S}$ and $\mathcal{T}$ be from Lemma 3.2. Let $M = \{\mathcal{S}, \mathcal{T}\}$. Let $[\![A]\!] = \{\mathcal{Q} \in M : A \in \mathcal{Q}\}$, and let

$$[\![\mathit{see}]\!] \quad = \quad \{(\mathcal{Q}, \mathcal{R}) \in M \times M : (\exists A \in \mathcal{Q})(\exists B \in \mathcal{R}) \ \Gamma \vdash \textit{All } A \textit{ see all } B\}.$$

We have a proper interpretation: $[\![A']\!] = M \setminus [\![A]\!]$ for all $A$. This comes from the fact that every point contains either $A$ or $A'$. The fact that points are up-closed means that the semantics is monotone; hence sentences such as *All* $A$ *are* $B$ in $\Gamma$ hold in $\mathcal{M}$. To conclude the verification that $\mathcal{M} \models \Gamma$, we need only consider sentences *All* $A$ *see all* $B$. Let $\mathcal{Q}$ and $\mathcal{R}$ be any points in $\mathcal{M}$ such that $A \in \mathcal{Q}$ and $B \in \mathcal{R}$. (There might not be any such points, but this is not a problem.) Then $\mathcal{Q} \, [\![\mathit{see}]\!] \, \mathcal{R}$.

To complete the proof, note that $[\![\mathit{see}]\!]$ does not contain $(\mathcal{S}, \mathcal{T})$. $\dashv$

## 4 Fragments with Class Expressions

McAllester and Givan in [4] study a fragment which we shall call $\mathcal{L}_{MG}(all, some)$. It begins with variables $X$, $Y$, etc., and also *verbs* $V$, $W$, etc. The fragment then has *class expressions* $c$, $d$, etc., of the following forms:

1. $X, Y, Z, \ldots$
2. $V$ *all* $c$
3. $V$ *some* $c$

Note that we have recursion, so we have class expressions like

$$R\, all\, (S\, some\, (T\, all\, c))$$

We might use this in the symbolization of a predicate like

recognizes everyone who sees someone who treasures all chrysanthemums.

$\mathcal{L}_{MG}(all, some)$ is our first infinite fragment. We also have formulas of the form *all c d* and *some c d*. In these, $c$ and $d$ are class expressions. The original paper also uses boolean combinations and proper names; we shall not do so.

The semantics interprets variables by subsets of an underlying model $\mathcal{M}$, just as we have been doing. It also interprets a verb $V$ by a binary relation $[\![V]\!] \subseteq M^2$. Then

$$\begin{array}{lcl} [\![V\ \ all\ c]\!] & = & \{x \in M : \text{for all } y \in [\![c]\!],\ x\, [\![V]\!]\, y\} \\ [\![V\ \ some\ c]\!] & = & \{x \in M : \text{for some } y \in [\![c]\!],\ x\, [\![V]\!]\, y\} \end{array}$$

Incidentally, the logic in this section uses more than one verb (unlike our work in Section 2). This is because sentences in general have more than one, and there are non-trivial deductions which use more than one verb. To get a hint of this, and of our logic, the reader might try to see that from *all watches are gold objects*, it follows that *everyone who likes everyone who has stolen all watches likes everyone who has has stolen all gold objects*.

We write $\exists c$ for *some c c*. We axiomatize $\mathcal{L}_{MG}(all, some)$ and also a smaller fragment $\mathcal{L}_{MG}(all)$ which only uses *all*. We study $\mathcal{L}_{MG}(all)$ to foreshadow the more complicated work for the larger fragment.

The main technical result in [4] is that the satisfiability problem for $\mathcal{L}_{MG}(all, some)$ is NP-complete. We are not concerned in this paper with complexity results but rather with logical completeness results. However, some of the steps are the same, and our treatment was influenced by McAllester and Givan [4].

Our logic is presented in Figure 5. The soundness of the first four rules is easy. We delay a discussion of the Cases rule until Section 4.2; it is not needed in our work in Section 4.1 just below.

$$\frac{\textit{all } c\ d}{\textit{all } (V\ \textit{all } d)\ (V\ \textit{all } c)} \qquad \frac{\textit{all } c\ d}{\textit{all } (V\ \textit{some } c)\ (V\ \textit{some } d)}$$

$$\frac{\textit{some } c\ d}{\textit{all } (V\ \textit{all } c)\ (V\ \textit{some } d)} \qquad \frac{\exists (V\ \textit{some } c)}{\exists c}$$

$$\frac{\Gamma \cup \{\exists c\} \vdash \varphi \quad \Gamma \cup \{\textit{all } c\ d : d \in \mathcal{L}\} \cup \{\textit{all } d\ (V\ \textit{all } c) : d, V \in \mathcal{L}\} \vdash \varphi}{\Gamma \vdash \varphi}\ (\text{Cases on } c)$$

Figure 5: Rules for $\mathcal{L}_{MG}(\textit{all}, \textit{some})$, our version of the McAllester-Givan fragment. We also use rules for *all* and *some* in Figure 1. The inference rule at the bottom discussed in Section 4.2.

### 4.1 Completeness for $\mathcal{L}_{MG}(\textbf{\textit{all}})$

In this section, we study the fragment $\mathcal{L}_{MG}(\textit{all})$ obtained from the variables and verbs by the constructions $V\ \textit{all } c$ for the class expressions, and $\textit{all } c\ d$ for the sentences. Our logical system uses the axioms $\textit{all } c\ c$, the monotonicity rule *Barbara* (transitivity of *All X are Y*), and the first rule in Figure 5, which we might call *antitonicity*. It is repeated below:

$$\frac{\textit{all } c\ d}{\textit{all } (V\ \textit{all } d)\ (V\ \textit{all } c)}$$

The rest of the system in Figure 5 is not needed for this fragment.

Let $\Gamma \subseteq \mathcal{L}_{MG}(\textit{all})$. Construct a model $\mathcal{M} = \mathcal{M}(\Gamma)$ as follows:

$$\begin{array}{lcl} M & = & \{c : c \text{ is a class expression of } \mathcal{L}_{MG}(\textit{all})\} \\ [\![X]\!] & = & \{c \in M : \Gamma \vdash \textit{all } c\ X\} \\ [\![V]\!] & = & \{(d, c) \in M \times M : \Gamma \vdash \textit{all } d\ (V\ \textit{all } c)\} \end{array}$$

**Lemma 4.1** *For all $c$, $[\![c]\!] = \{d \in M : \Gamma \vdash \textit{all } d\ c\}$.*

**Proof** By induction on $c$. The base case being immediate, we assume our lemma for $c$ and then consider a class expression of the form $V\ \textit{all } c$.

Let $d \in [\![V\ \textit{all } c]\!]$. The induction hypothesis and reflexivity axioms imply that $c \in [\![c]\!]$. So we have $d[\![V]\!]c$, and thus $\Gamma \vdash \textit{all } d\ (V\ \textit{all } c)$.

Conversely, suppose that $\Gamma \vdash \textit{all } d\ (V\ \textit{all } c)$. We claim that $d \in [\![V\ \textit{all } c]\!]$. For this, let $c' \in [\![c]\!]$ so that $\Gamma \vdash \textit{all } c'\ c$ . We have the following derivation from $\Gamma$:

$$\frac{\begin{array}{c}\vdots\\ \textit{all } d\ (V\ \textit{all } c)\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \textit{all } c'\ c\end{array}}{\textit{all } (V\ \textit{all } c)\ (V\ \textit{all } c')}}{\textit{all } d\ (V\ \textit{all } c')}$$

We see that $d[\![V]\!]c'$. This for all $c' \in [\![c]\!]$ shows that $d \in [\![V\ \textit{all } c]\!]$. $\dashv$

**Lemma 4.2** *Then $\mathcal{M}(\Gamma) \models \textit{all } c\ d$ iff $\Gamma \vdash \textit{all } c\ d$.*

**Proof** Suppose that $\Gamma \vdash \textit{all } c\ d$. Consider $\mathcal{M}(\Gamma)$. By transitivity and Lemma 4.1, $[\![c]\!] \subseteq [\![d]\!]$.

So $\mathcal{M} \models$ *all c d*. For the converse, assume that $[\![c]\!] \subseteq [\![d]\!]$. Then $c \in [\![c]\!] \subseteq [\![d]\!]$. So by Lemma 4.1, $\Gamma \vdash$ *all c d*, just as desired. $\dashv$

**Theorem 4.3** *The logical system determined by the all-rules in Figures 1 and 5 is sound and complete for $\mathcal{L}_{MG}(\text{all})$: $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.*

**Proof** Assume that $\Gamma \models \varphi$, and consider $\mathcal{M} = \mathcal{M}(\Gamma)$. By Lemma 4.2, $\mathcal{M} \models \Gamma$. So $\mathcal{M} \models \varphi$. And thus by Lemma 4.2 again, $\Gamma \vdash \varphi$. $\dashv$

## 4.2 The Cases rule

We turn back to our logical system for $\mathcal{L}_{MG}(\textit{all}, \textit{some})$, as presented in Figure 5. For the first time in this paper, we study a system which actually is a natural deduction calculus. One should read the first four rules as saying, for example,

$$\frac{\Gamma \vdash \exists c}{\Gamma \vdash \textit{all}\ (V\ \ \textit{all}\ c)\ (V\ \ \textit{some}\ c)}$$

The last rule allows for a case-by-case analysis on whether the interpretation of a class expression is empty or not. The idea is that we would like to take cases as to whether (a) there is some $c$ or (b) there are no $c$. However, we cannot directly say (b) in this fragment, so we do the next best thing: we use consequences of (b). To derive a sentence $\varphi$ from $\Gamma$ by cases on $c$, first derive $\varphi$ from $\Gamma$ with the sentence *some c are c*; and second, derive $\varphi$ from $\Gamma \cup \Delta$, where $\Delta$ is the set of sentences *all c are d* and *all d* ($V$ *all c*), where $V$ is any of our verbs.

Our statement of the Cases rule in Figure 5 will look less forbidding when recast as a natural deduction rule. Here is an example of how it would work. We show that $\Gamma \vdash \exists c$, where $\Gamma$ is $\{\textit{all}\ (V\ \ \textit{all}\ c)\ c, \exists d\}$. Before we give the formal derivation, here is the informal semantic argument. Take any model $\mathcal{M}$ of the hypotheses. If $[\![c]\!]$ is non-empty, we are done. Otherwise, let $x \in [\![d]\!]$ by the first hypothesis. Then (vacuously) we have $x\,[\![V]\!]\,y$ for all $y \in [\![c]\!]$. So by our second hypothesis, $x \in [\![c]\!]$. Here is a formal derivation which corresponds to this explanation:

$$\dfrac{[\exists c] \qquad \dfrac{\dfrac{\textit{all}\ (V\ \ \textit{all}\ c)\ c \qquad \dfrac{[\textit{all}\ d\ (V\ \ \textit{all}\ c)] \quad \exists d}{\textit{some}\ d\ (V\ \ \textit{all}\ c)}}{\textit{some}\ d\ c}}{\exists c}}{\exists c}\ (\text{Cases on } c)$$

For the soundness of the Cases rule, we argue by induction on the number $n$ of uses of rule in derivations. Suppose that $\Gamma \vdash \varphi$ via a derivation with $n+1$ uses of the rule, and suppose that the derivation is by cases on $c$. Thus $\Gamma \cup \{\exists c\} \vdash \varphi$ via a derivation with $n$ uses, and the same for $\Gamma \cup \Delta \vdash \varphi$. Finally, fix a model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$. We wish to show that $\mathcal{M} \models \varphi$, and we argue by cases on $[\![c]\!]$. If $[\![c]\!] \neq \emptyset$, then our first assumption and the induction hypothesis on $n$ imply that $\mathcal{M} \models \varphi$. If $[\![c]\!] = \emptyset$, then vacuously all sentences in $\Delta$ hold. So again $\mathcal{M} \models \varphi$.

### 4.3 Completeness for $\mathcal{L}_{MG}(\boldsymbol{all}, \boldsymbol{some})$

Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{MG}(\mathit{all}, \mathit{some})$. Let $\mathit{occ}(\Gamma, \varphi)$ be the set of class expressions $c$ which occur in $\Gamma \cup \{\varphi\}$. We say that a set $\Delta \subseteq \mathcal{L}_{MG}(\mathit{all}, \mathit{some})$ *determines existentials for* $\Gamma$ *and* $\varphi$ if for all class expressions $c \in \mathit{occ}(\Gamma, \varphi)$, either

1. $\Delta \vdash \exists c$, or else
2. For all $d \in \mathit{occ}(\Gamma, \varphi)$,
   (a) $\Delta \vdash \mathit{all}\ c\ d$;
   (b) If $(V\ \mathit{all}\ c) \in \mathit{occ}(\Gamma, \varphi)$ then $\Delta \vdash \mathit{all}\ d\ (V\ \mathit{all}\ c)$.

We make a model $\mathcal{M} = \mathcal{M}(\Delta, \Gamma, \varphi)$ as follows.

$$
\begin{array}{lcl}
M & = & \{(c_1, c_2, Q) : c_1, c_2 \in \mathit{occ}(\Gamma, \varphi), Q \in \{\forall, \exists\}, \Delta \vdash \mathit{some}\ c_1\ c_2\} \\
[\![X]\!] & = & \{(c_1, c_2, Q) \in M : \Delta \vdash \mathit{all}\ c_1\ X \text{ or } \Delta \vdash \mathit{all}\ c_2\ X\} \\
(c_1, c_2, Q)[\![V]\!](d_1, d_2, \forall) & \text{iff} & \text{for some } i \text{ and } j,\ \Delta \vdash \mathit{all}\ c_i\ (V\ \mathit{all}\ d_j) \\
(c_1, c_2, Q)[\![V]\!](d_1, d_2, \exists) & \text{iff} & \text{either } (c_1, c_2, Q)[\![V]\!](d_1, d_2, \forall); \text{or else for some } i \text{ and } j, \\
 & & \Delta \vdash \mathit{all}\ c_i\ (V\ \mathit{some}\ d_j) \text{ and also } \Delta \vdash \mathit{all}\ d_j\ d_{3-j}
\end{array}
$$

**Lemma 4.4** *Assume that $\Delta$ determines existentials for $\Gamma$ and $\varphi$. For all $c \in \mathit{occ}(\Gamma, \varphi)$,*

$$[\![c]\!] \quad = \quad \{(d_1, d_2, Q) \in M : \text{either } \Delta \vdash \mathit{all}\ d_1\ c, \text{ or } \Delta \vdash \mathit{all}\ d_2\ c\}.$$

**Proof** By induction on $c \in \mathit{occ}(\Gamma, \varphi)$. The base case being immediate, we assume our lemma for $c$ and then consider class expressions belonging to $\mathit{occ}(\Gamma, \varphi)$ of the form $V\ \mathit{all}\ c$ and $V\ \mathit{some}\ c$. The induction hypothesis implies that provided $\Delta \vdash \exists c$, both $(c, c, \forall)$ and $(c, c, \exists)$ belong to $[\![c]\!]$.

Let $(d_1, d_2, Q) \in [\![V\ \mathit{all}\ c]\!]$. If $\Delta \vdash \exists c$, then $(c, c, \forall) \in [\![c]\!]$. By the overall semantics of $\mathcal{L}_{MG}(\mathit{all}, \mathit{some})$, $(d_1, d_2, Q)[\![V]\!](c, c, \forall)$. And we see that $\Delta \vdash \mathit{all}\ d_i\ (V\ \mathit{all}\ c)$ for some $i$, as desired. If $\Delta \nvdash \exists c$, we trivially have the same conclusion $\Delta \vdash \mathit{all}\ d_i\ (V\ \mathit{all}\ c)$, this time for both $i$. (This is where the assumption that $\Delta$ determines existentials for $\Gamma$ and $\varphi$ is used; it also is used in Lemma 4.5 below.) Hence we are done then also.

Conversely, fix $i$ and suppose that $\Delta \vdash \mathit{all}\ d_i\ (V\ \mathit{all}\ c)$. Fix $Q$; we claim that $(d_1, d_2, Q) \in [\![V\ \mathit{all}\ c]\!]$. For this, let $(e_1, e_2, Q') \in [\![c]\!]$ so that $\Delta \vdash \mathit{all}\ e_j\ c$ for some $j$. As in the proof of Theorem 4.3, we see that $\Delta \vdash \mathit{all}\ d_i\ (V\ \mathit{all}\ e_j)$. We conclude that $(d_1, d_2, Q)[\![V]\!](e_1, e_2, \forall)$, and also $(d_1, d_2, Q)[\![V]\!](e_1, e_2, \exists)$. This for all elements of $[\![c]\!]$ shows that $(d_1, d_2, Q) \in [\![V\ \mathit{all}\ c]\!]$.

Here is the induction step for $V\ \mathit{some}\ c$. Let $(d_1, d_2, Q) \in [\![V\ \mathit{some}\ c]\!]$. Thus we have $(d_1, d_2, Q)[\![V]\!](e_1, e_2, Q')$ for some $(e_1, e_2, Q') \in [\![c]\!]$. We first consider the case that $Q' = \forall$. Here there are four subcases. One representative subcase is $\Delta \vdash \mathit{all}\ d_1\ (V\ \mathit{all}\ e_1)$. We have $\Delta \vdash \mathit{some}\ e_1\ e_2$, since $(e_1, e_2, Q') \in M$. By induction hypothesis, either $\Delta \vdash \mathit{all}\ e_1\ c$ or else $\Delta \vdash \mathit{all}\ e_2\ c$. Without loss of generality, $\Delta \vdash \mathit{all}\ e_1\ c$. The derivation from $\Delta$ below shows that

$\Delta \vdash \mathit{all}\ d_1\ (V\ \mathit{some}\ c)$, as desired:

$$\dfrac{\begin{matrix}\vdots\\ \mathit{all}\ d_1\ (V\ \mathit{all}\ e_1)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \mathit{all}\ e_1\ c\end{matrix}\quad \dfrac{\begin{matrix}\vdots\\ \mathit{some}\ e_1\ e_2\end{matrix}}{\mathit{some}\ e_1\ e_1}}{\mathit{some}\ e_1\ c}}{\mathit{all}\ (V\ \mathit{all}\ e_1)\ (V\ \mathit{some}\ c)}}{\mathit{all}\ d_1\ (V\ \mathit{some}\ c)}$$

This concludes the work when $Q' = \forall$. In the other case, $Q' = \exists$. We again have a number of subcases; one is that $\Delta \vdash \mathit{all}\ d_1\ (V\ \mathit{some}\ e_1)$ and $\Delta \vdash \mathit{all}\ e_1\ e_2$. And by induction hypothesis, $\Delta \vdash \mathit{all}\ e_1\ c$ or else $\Delta \vdash \mathit{all}\ e_2\ c$. Either way, we get $\Delta \vdash \mathit{all}\ e_1\ c$. And further we get the desired conclusion, $\Delta \vdash \mathit{all}\ d_1\ (V\ \mathit{some}\ c)$:

$$\dfrac{\begin{matrix}\vdots\\ \mathit{all}\ d_1\ (V\ \mathit{some}\ e)\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \mathit{all}\ e\ c\end{matrix}}{\mathit{all}\ (V\ \mathit{some}\ e)\ (V\ \mathit{some}\ c)}}{\mathit{all}\ d_1\ (V\ \mathit{some}\ c)}$$

This concludes half of the induction step for $V\ \mathit{some}\ c$.

For the other half, let $(d_1, d_2, Q) \in M$, and fix $i$ such that $\Delta \vdash \mathit{all}\ d_i\ (V\ \mathit{some}\ c)$. Then $\Delta \vdash \exists d_i$, and in a few steps we also have $\Delta \vdash \exists c$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists d_i\end{matrix} \quad \begin{matrix}\vdots\\ \mathit{all}\ d_i\ (V\ \mathit{some}\ c)\end{matrix}}{\exists (V\ \mathit{some}\ c)}}{\exists c}$$

Therefore $(c, c, \exists) \in M$. By induction hypothesis, $(c, c, \exists) \in [\![c]\!]$. We have $(d_1, d_2, Q)[\![V]\!](c, c, \exists)$ because $\Delta \vdash \mathit{all}\ \ c\ c$. So $(d_1, d_2, Q) \in [\![V\ \mathit{some}\ c]\!]$. ⊣

**Lemma 4.5** *Assume that $\Delta$ determines existentials for $\Gamma$ and $\varphi$, and also that $\Gamma \models \varphi$.*

1. $\mathcal{M}(\Delta, \Gamma, \varphi) \models \Gamma$.
2. $\Delta \vdash \varphi$.

**Proof** First, consider a sentence in $\Gamma$ of the form $\mathit{all}\ c\ d$, so that $c$ and $d$ belong to $\mathit{occ}(\Gamma, \varphi)$. By a routine monotonicity calculation and Lemma 4.4, $[\![c]\!] \subseteq [\![d]\!]$. We also consider a sentence in $\Gamma$ of the form $\mathit{some}\ c\ d$. Notice that $(c, d, \exists) \in M$. Indeed, $(c, d, \exists) \in [\![c]\!] \cap [\![d]\!]$, by Lemma 4.4.

At this point, we know that $\mathcal{M}(\Delta, \Gamma, \varphi) \models \Gamma$. By our assumption that $\Gamma \models \varphi$, we see that $\mathcal{M}(\Delta, \Gamma, \varphi) \models \varphi$.

For the second part, we again consider cases on $\varphi$. If $\varphi$ is $\mathit{all}\ c\ d$, we therefore have $[\![c]\!] \subseteq [\![d]\!]$. If $\Delta \vdash \exists c$, then $(c, c, \forall) \in [\![c]\!] \subseteq [\![d]\!]$. So by Lemma 4.4, $\Delta \vdash \mathit{all}\ c\ d$, just as desired. But if $\Delta \nvdash \exists c$ the assumption that $\Delta$ determines existentials tells us directly that $\Delta \vdash \mathit{all}\ c\ d$. If $\varphi$ is $\mathit{some}\ c\ d$, let $(c', d', Q) \in [\![c]\!] \cap [\![d]\!]$. We use Lemma 4.4 again and reduce to four cases; one of them is $\Delta \vdash \mathit{all}\ c'\ c$, and $\Delta \vdash \mathit{all}\ d'\ d$. And as we also have $\Delta \vdash \mathit{some}\ c'\ d'$, we also have $\Delta \vdash \varphi$. (This is indicated by the proof tree in our Introduction.) The other cases are easier. ⊣

**Theorem 4.6** *The logical system determined by the rules in Figures 1 and 5 is sound and complete for* $\mathcal{L}_{MG}(\text{all}, \text{some})$*:* $\Gamma \models \varphi$ *iff* $\Gamma \vdash \varphi$.

**Proof** The soundness half is trivial. Suppose that $\Gamma \models \varphi$. Since the fragment is a sublanguage of first-order logic, if is compact. So we may assume that $\Gamma$ is finite. In particular, $occ(\Gamma, \varphi)$ is a finite set of class expressions.

For any finite set $\Delta \supseteq \Gamma$, let $n(\Delta, \Gamma, \varphi)$ be the number of class expressions $c \in occ(\Gamma, \varphi)$ such that (1) $\Delta \nvdash \exists c$; and (2) for some $d \in occ(\Gamma, \varphi)$, either $\Delta \nvdash$ *all c d*, or else $(V \ \textit{all}\ c) \in occ(\Gamma, \varphi)$ and $\Delta \nvdash$ *all d* $(V \ \textit{all}\ c)$. This number $n(\Delta, \Gamma, \varphi)$ measures how far $\Delta$ is from determining existentials for $\Gamma$ and $\varphi$.

We show by induction on the number $k$ that for all finite $\Delta \supseteq \Gamma$ with $n(\Delta, \Gamma, \varphi) = k$, $\Delta \vdash \varphi$. Applying this to the original $\Gamma$ with $k = n(\Gamma, \Gamma, k)$, we see that $\Gamma \vdash \varphi$, as required.

If $n(\Delta, \Gamma, \varphi) = 0$, then $\Delta$ determines existentials for $\Gamma$ and $\varphi$. By Lemma 4.5, $\Delta \vdash \varphi$.

Now assume our result for $k$, and suppose that $n(\Delta, \Gamma, \varphi) = k + 1$. Fix a class expression $c$ with (1) and (2) above. Consider

$$\begin{array}{lcl} \Delta_1 & = & \Delta \cup \{\exists c\} \\ \Delta_2 & = & \Delta \cup \{\textit{all}\ c\ d : d \in occ(\Gamma, \varphi)\} \\ & & \cup \{\textit{all}\ d\ (V\ \textit{all}\ c) : d, (V\ \text{all}\ c) \in occ(\Gamma, \varphi), V \text{ a verb}\} \end{array}$$

A fortiori, $\Delta_1 \models \varphi$ and also $\Delta_2 \models \varphi$. Further, $n(\Delta_1) \leq k$, and similarly for $\Delta_2$. By induction hypothesis, $\Delta_1 \vdash \varphi$, and $\Delta_2 \vdash \varphi$. So using Cases on $c$, $\Delta \vdash \varphi$. $\dashv$

The proof shows that if $\Gamma \vdash S$, then there is a proof tree which uses the Cases rule "at the bottom": below any use of the Cases rule are found only other uses of the same rule.

# 5 Conclusion

This paper has shown several completeness results for logical systems which are syllogistic in the sense that they avoid individual variables and yet are more expressive than the classical syllogistic logic. (However, the system in the first section uses infinitely many rules, and the one in the last section uses a Cases rule. So they are not 'syllogistic' in the strictest sense.) Results on different but related systems may be found in [10]. These include completeness proofs, and also negative results, showing that some naturally-defined logics either do not have a syllogistic proof system at all, or that they do not have one without *reductio ad absurdum*. We would like to continue the study of the boundary between logics which admit syllogistic systems and ones which do not. In addition, the overarching goal of all of this research is to find complete fragments of natural language, using whatever kinds of proof systems are necessary. We hope that the techniques in this paper will help with other results in the area.


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