Well-pointed Coalgebras

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Abstract. For set functors preserving intersections, a new description of the final coalgebra and the initial algebra is presented: the former consists of all well-pointed coalgebras. These are the pointed coalgebras having no proper subobject and no proper quotient. And the initial algebra consists of all well-pointed coalgebras that are well-founded in the sense of Osius [20] and Taylor [27]. Finally, the initial iterative algebra consists of all finite well-pointed coalgebras. Numerous examples are discussed e.g. automata, graphs, and labeled transition systems.

Keywords: Well-founded coalgebra, well-pointed coalgebra, initial algebra, final coalgebra, iterative algebra

1 Introduction

Initial algebras are known to be of primary interest in denotational semantics, where abstract data types are often presented as initial algebras for an endofunctor H expressing the type of the constructor operations of the data type. For example, binary trees are the initial algebra for the functor $HX = X \times X + 1$ on sets. Analogously, final coalgebras for an endofunctor H play an important role in the theory of systems developed by Rutten [21]: H expresses the system type, i. e., which kind of one-step reactions states can exhibit (input, output, state transitions etc.), and the elements of a final coalgebra represent the behavior of all states in all systems of type H (and the unique homomorphism from a system into the final one assign to every state its behavior). For example, deterministic automata with input alphabet I are coalgebras for $HX = X^I \times \{0, 1\}$, the final coalgebra is the set of all languages on I.

In this paper a unified description is presented for (a) initial algebras, (b) final coalgebras and (c) initial iterative algebras (in the automata example this is the set of all regular languages on I). We also demonstrate that this new description provides a unifying view of a number of other important examples.

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We work with set functors H preserving intersections. This is an extremely mild requirement that all "everyday" set functors satisfy. We prove that the final coalgebra of H can then be described as the set of all *well-pointed coalgebras*, i.e., pointed coalgebras not having any proper subobject and also not having any proper quotient. Moreover, the initial algebra can be described as the set of all well-pointed coalgebras which are well-founded in the sense of Osius [20] and Taylor [26, 27]. Before we mention the definition, recall that the notion of well-foundedness of relations has several alternative forms. Given a relation $R \subseteq X \times X$, we can study the following conditions:

1. Let $Y \subseteq X$ have the property that if all R-successors of a given point $x \in X$ lie in Y, then $x \in Y$ as well. Then $Y = X$.
2. There is no infinite sequence from X following R: $x_0 R x_1 R x_2 R \cdots$.
3. There is a map from $\mathsf{rk} : X \longrightarrow \mathbf{Ord}$ such that $\mathsf{rk}(x) > \mathsf{rk}(y)$ whenever xRy.

For sets and relations as usual, these are equivalent. The first of these is an *induction principle*, and this is closest to what we are calling well-foundedness in this paper, following Taylor. The equivalence of the first and the second requires Dependent Choice, a weak form of the Axiom of Choice; in any case, our work in this area does not use this at all. The last condition is close to a result which we will see, but note as well that even this requires something special about sets, namely the Replacement Axiom.

The notion of well-foundedness of a coalgebra (A, α) generalizes condition (1) above. It says that no proper subcoalgebra (A', α') of (A, α) forms a pullback

$$\begin{tikzcd} A' \arrow[r, "\alpha'"] \arrow[d, hook, "m"'] & HA' \arrow[d, hook, "Hm"] \\ A \arrow[r, "\alpha"'] & HA \end{tikzcd}$$

This concept was first studied by Osius [20] for graphs considered as coalgebras of the power-set functor $\mathscr{P}$: a graph is well-founded in the coalgebraic sense iff it is well-founded in any of the equivalent senses above. Taylor [26, 27] introduced well-founded coalgebras for general endofunctors, and he proved that for endofunctors preserving inverse images the concepts of initial algebra and final well-founded coalgebra coincide.

We must mention that our motivation differs from Taylor's. He is concerned with foundational matters connected to recursion and induction, while we are interested in studying initial algebras and final coalgebras in as wide a setting as possible.

Returning to our topic, we are going to prove that for every set functor H the concepts of initial algebra and final well-founded coalgebra coincide; the step towards making no assumptions on H is non-trivial. And if H preserves intersections, we describe its final coalgebra and initial algebra using well-pointed coalgebras as above. The first result will be proved in a much more general context, working with an endofunctor of a locally finitely presentable category preserving strong monomorphisms. We further assume that the functor preserves finite intersections, but later we prove that this extra assumption can be dropped in the case of set functors.

The last section takes a number of known important special cases: deterministic (Mealy and Moore) automata, trees, labeled transition systems, non-well-founded sets, etc., and demonstrates how well-pointed coalgebras work in each case. Here we describe, in every example, besides the initial algebra and the final coalgebra, the initial iterative algebra [6] (equivalently, final locally finite coalgebra, see [18, 9]) as the set of all finite well-pointed coalgebras.

2 Well-founded coalgebras

In this section we recall the concept of well-founded coalgebra of Osius [20] and Taylor [27]. Our main result is that

$$\text{initial algebra} = \text{final well-founded coalgebra}$$

holds for all endofunctors of **Set**. (In the case where the endofunctor preserves inverse images, this result can be found in [27].) For more general categories the above result holds whenever the endofunctor preserves finite intersections.

2A Well-founded coalgebras in locally finitely presentable categories

We make several assumptions on the base category $\mathscr{A}$ in our study.

Definition 2.1. *1. A category $\mathscr{A}$ is* locally finitely presentable *(LFP) if*

(a) *$\mathscr{A}$ is complete*

(b) *there is a set of finitely presentable objects whose closure under filtered colimits is all of $\mathscr{A}$*

(See [13] or [7] for more on LFP categories.)

2. An object A of (any category) $\mathscr{A}$ is called simple *if it has no proper quotients. That is, every epimorphism with domain A is invertible.*

Assumption 2.2. Throughout this section our base category $\mathscr{A}$ is locally finitely presentable and has a simple initial object 0.

Example 2.3. The categories of sets, graphs, posets, and semigroups are locally finitely presentable. The initial objects of these categories are empty, hence simple.

Definition 2.4. *For every endofunctor H denote by*

$$\mathbf{Coalg}\, H$$

the category of coalgebras $\alpha\colon A \longrightarrow HA$ and coalgebra homomorphisms.

Since subcoalgebras play a basic role in the whole paper, and quotients are important from Section 3 onwards, we need to make clear what we mean by those. Quotients are no problem: it is clear that the forgetful functor of the category of coalgebras preserves and reflects all colimits. Consequently, epimorphisms in $\mathbf{Coalg}\, H$ are precisely the homomorphisms carried by epimorphisms in the base category. And they represent the quotients of the domain coalgebra (up to isomorphism, as usual). What about subcoalgebras? If the base category is $\mathbf{Set}$, it turns out that the homomorphisms carried by monomorphisms are precisely the strong monomorphisms of $\mathbf{Coalg}\, H$. (Recall that a monomorphism is called *strong* if it has the diagonal fill-in property w.r.t. all epimorphisms. In "everyday" categories this is equivalent to being a regular monomorphism.) As shown in Lemma 2.6, for general base categories we have an analogous fact whenever the endofunctor H preserves strong monomorphisms: strong monomorphisms in $\mathbf{Coalg}\, H$ are precisely the homomorphisms $h\colon (A,\alpha) \longrightarrow (B,\beta)$ for which h is strongly monic in $\mathscr{A}$. For that reason we use the term *subcoalgebra* of a coalgebra (A,α) to mean a subobject represented by a strong monomorphisms $m\colon (A',\alpha') \longrightarrow (A,\alpha)$ in $\mathbf{Coalg}\, H$. But as we point out in Section 2C, one can obtain analogous results for more general factorization systems.

Remark 2.5. There are some consequences of the LFP assumption that play an important role in our development. These pertain to *strong monomorphisms.*

1. $\mathscr{A}$ has (epi, strong mono)-factorizations; see 1.16 in [7].
2. $\mathscr{A}$ is wellpowered with respect to strong monomorphisms; see 1.56 in [7]. This implies that for every object A the poset $\mathsf{Sub}(A)$ of all strong subobjects of A is a complete lattice.
3. strong monomorphisms are closed under wide intersections and inverse images (this is true for all factorizations systems: see Proposition 14.15 in [4]), and
4. strong monomorphisms are closed under filtered colimits: we prove this in Lemma 2.10.

Lemma 2.6. *Assume that H preserves strong monomorphisms. Strong monomorphisms in $\mathbf{Coalg}\, H$ are precisely the homomorphisms $h\colon (A,\alpha) \longrightarrow (B,\beta)$ for which h is strongly monic in $\mathscr{A}$.*

Proof. Since H preserves strong monomorphisms, the forgetful functor of $\mathbf{Coalg}\, H$ creates (epi, strong mono)-factorizations, and a coalgebra homomorphism h : $(A,\alpha) \longrightarrow (B,\beta)$ is a strong monomorphism in $\mathbf{Coalg}\, H$ iff it is one in $\mathscr{A}$. Indeed, let $h = m{\cdot}e$ be an (epi, strong mono)-factorization in $\mathscr{A}$, then the diagonal fill-in yields a coalgebra for which m and e are homomorphisms, and it is easy to see that m is a strong monomorphism in $\mathbf{Coalg}\, H$:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "e"'] & HA \arrow[d, "He"] \\
C \arrow[r, dashed, "\gamma"] \arrow[d, "m"'] & HC \arrow[d, "Hm"] \\
B \arrow[r, "\beta"] & HB
\end{tikzcd}$$

Now, if h is a strong monomorphism in $\mathbf{Coalg}\, H$, since $e\colon (A,\alpha) \longrightarrow (C,\gamma)$ is an epimorphism, it follows that it is invertible, thus, h is a strong monomorphism in $\mathscr{A}$. $\square$

Example 2.7. For a non-example which is still interesting for this paper, we consider the category $\mathbf{Set}_{0,1}$ of *bipointed sets*; these are sets with two distinguished points which morphisms must fix. $\mathbf{Set}_{0,1}$ is LFP. The initial object 0 is a set with two different elements, both distinguished. The final object 1 is a single point. The map $0 \longrightarrow 1$ is an epimorphism, so 0 is not simple. Observe that all monomorphisms in $\mathbf{Set}_{0,1}$ are strong.

Example 2.8. On the category $\mathbf{Gra} = \mathbf{Set}^{\rightrightarrows}$ of graphs define an endofunctor H by

$$HX = \begin{cases} X + \{t\} \text{ (no edges)} & \text{if } X \text{ has no edges} \\ 1, \text{terminal graph}, & \text{else.} \end{cases}$$

Observe that the initial algebra is carried by a countable set without edges.

Example 2.9. Consider again the category $\mathbf{Set}_{0,1}$ of bipointed sets: put

$$H(X, x_0, x_1) = \begin{cases} 1 \quad \text{(final object)} & \text{if } x_0 = x_1 \\ (X+1, x_0, x_1) & \text{else} \end{cases}$$

This H preserves (strong) monomorphisms. However, we saw above that 0 is not simple. So H will re-appear in examples which show that the simplicity of 0 is necessary in most of our results below, as will the functor from Example 2.8.

Lemma 2.10. *Given a filtered colimit with a cocone $c_i\colon C_i \longrightarrow C$ $(i \in I)$, every morphism $f\colon C \longrightarrow D$ for which $f{\cdot}c_i$ are strong monomorphisms $(i \in I)$ is a strong monomorphism.*

Proof. It is our task, for every commutative square

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

where e is an epimorphism to find a diagonal. We can assume, without loss of generality, that X is finitely presentable: indeed, every epimorphism in a locally finitely presentable category is a filtered colimit of epimorphisms with finitely presentable domains.

Since $C = \operatorname{colim} C_i$ is a filtered colimit, there exists i such that u factorizes through c_i.

$$\begin{tikzcd}
 & X \arrow[r, "e"] \arrow[dl, "u'"'] \arrow[d, "u"] & Y \arrow[d, "v"] \arrow[dll, dashed, "d"'] \\
C_i \arrow[r, "c_i"'] & C \arrow[r, "f"'] & D
\end{tikzcd}$$

This yields, since $f{\cdot}c_i$ is a strong monomorphism, a diagonal $d\colon Y \longrightarrow C_i$ for the outward square. Then $c_i{\cdot}d$ is the desired diagonal for the original square.

To show that f is a monomorphism, assume that $f \cdot m = f \cdot n$. Take the coequalizer e of m and n, and let w be the unique mediating morphism with $w \cdot e = f$. Then the unique diagonal of the commutative square $f \cdot \mathrm{id} = w \cdot e$ satifies $d \cdot e = \mathrm{id}$, whence e is an isomorphism. Thus, $m = n$ as desired. $\square$

Definition 2.11. *A **cartesian subcoalgebra** of a coalgebra (A, α) is a subcoalgebra (A', α') forming a pullback*

$$\begin{tikzcd}
A' \arrow[r, "\alpha'"] \arrow[d, hook, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA' \arrow[d, hook, "Hm"] \\
A \arrow[r, "\alpha"'] & HA
\end{tikzcd}$$

*A coalgebra is called **well-founded** if it has no proper cartesian subcoalgebra.*

Example 2.12. (1) The concept of well-founded coalgebra was introduced originally by Osius [20] for the power set functor $\mathscr{P}$. A graph is a coalgebra (A, a) for $\mathscr{P}$, where $a(x)$ is the set of neighbors of A in the graph. Then a subcoalgebra of A is an (induced) subgraph A' with the property that every neighbor of a vertex of A' lies in A'. The subgraph A' is cartesian iff it contains every vertex all of whose neighbors lie in A'. The graph A is a well-founded coalgebra iff it has no infinite path.

(2) Let A be a deterministic automaton considered as a coalgebra for $HX = X^I \times \{0,1\}$. A subcoalgebra A' is cartesian iff it contains every state all whose successors (under the inputs from I) lie in A'. This holds, in particular, for $A' = \emptyset$. Thus, no nonempty automaton is well-founded.

(3) Coalgebras for $HX = X + 1$ are dynamical systems with deadlocks. A subcoalgebra A' of a dynamical system A is cartesian iff it contains all deadlocks and every state whose next state lies in A'.

A dynamical system is well-founded iff it has no infinite computation.

Definition 2.13. *Assume that H preserves strong monomorphisms. Then every coalgebra $\alpha\colon A \longrightarrow HA$ induces an endofunction of $\mathsf{Sub}(A)$ (see Remark 2.5.2) assigning to a strong subobject $m\colon A' \longrightarrow A$ the inverse image $\bigcirc m$ of Hm under α, i. e., we have a pullback square:*

$$
\begin{tikzcd}
\bigcirc A \arrow[r, "{\alpha[m]}"] \arrow[d, "\bigcirc m"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA' \arrow[d, "Hm"] \\
A \arrow[r, "\alpha"'] & HA
\end{tikzcd}
\tag{2.1}
$$

This function $m \longmapsto \bigcirc m$ is obviously order-preserving. By the Knaster-Tarski fixed point theorem, this function has a least fixed point.

Incidentally, the notation $\bigcirc m$ comes from modal logic, especially the areas of temporal logic where one reads $\bigcirc\phi$ as "ϕ is true in the next moment," or "next time ϕ for short.

Example 2.14. Recall our discussion of graphs from Example 2.12 (1). The pullback $\bigcirc A$ of a subgraph A' is the set of points in the overall graph all of whose neighbors belong to A'.

Remark 2.15. As we mentioned in the introduction, the concept of well-foundedness of a coalgebra was introduced by Taylor [26, 27]. Our formulation is a bit simpler. In [27, Definition 6.3.2] he calls a coalgebra (A, a) is well-founded if in every pullback of the form

$$
\begin{tikzcd}
C \arrow[r, "j"] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & B \arrow[r, "i"] & A \arrow[d, "a"] \\
HB \arrow[rr, "Hi"'] & & HA
\end{tikzcd}
$$

with i and j monomorphisms, i and j are, in fact, isomorphisms. Thus, in lieu of monomorphism we use strong ones and in lieu of pre-fixed points of $m \longmapsto \bigcirc m$ we use fixed points.

In addition, our overall work has a *methodological* difference from Taylor's that is worth mentioning at this point. Taylor is giving a general account of recursion and induction, and so he is concerned with general principles that underlie these phenomena. Indeed, he is interested in settings like non-boolean toposes where classical reasoning is not necessarily valid. On the other hand, in this paper we are studying initial algebras, final coalgebras, and similar concepts, using standard classical mathematical reasoning. In particular, we make free use of transfinite recursion. The definitions in Notation 2.17 just below would look out of place in Taylor's paper. But we believe they are an important step in our development.

Example 2.16. Here is an example showing that preservation of strong monomorphisms does not in general imply preservation of monomorphisms. On the category **Gra** of graphs, this time let

$$\begin{aligned} HA = {} & \text{all finite independent } a \subseteq A\text{, together with a new point } t \\ & \text{with } a \leftrightarrow t \text{ for all } a\text{, and also } t \to t \end{aligned}$$

For a graph morphism $f : A \longrightarrow B$, we take $Hf : HA \longrightarrow HB$ to be

$$Hf(a) = \begin{cases} f[a] & \text{if } f[a] \text{ is independent in } B \\ t & \text{otherwise} \end{cases}$$

This functor H preserves strong monomorphisms (they are the induced subgraphs), and indeed it preserves intersections of them as well. However, H does not preserve monomorphisms. So we expect some of the results which depend on preservation of $\mathscr{M}$ will fail with $\mathscr{M}$ = all monomorphisms.

Notation 2.17. (a) Assume that H preserves strong monomorphisms. For every coalgebra $\alpha\colon A \longrightarrow HA$ denote by

$$a^*\colon A^* \longrightarrow A \tag{2.2}$$

the least fixed point of the function $m \longmapsto \bigcirc m$ of Definition 2.13. (Thus, (A, a) is well-founded iff a^* is invertible.) Since a^* is a fixed point we have a coalgebra structure $\alpha^*\colon A^* \longrightarrow HA^*$ making a^* a coalgebra homomorphism.

(b) For every coalgebra $a\colon A \longrightarrow HA$ we define a chain of strong subobjects

$$a_i^*\colon A_i^* \longrightarrow A \qquad (i \in \mathbf{Ord})$$

of A on $\mathscr{A}$ by transfinite recursion:

$a_0^*\colon 0 \longrightarrow A$ unique;
given a_i^*, define a_{i+1}^* by the pullback

$$\begin{tikzcd} A_{i+1}^* \arrow[r] \arrow[d, "a_{i+1}^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA_i^* \arrow[d, "Ha_i^*"] \\ A \arrow[r, "\alpha"'] & HA \end{tikzcd}$$

and for limit ordinals i we take the colimit of the chain $(A_j)_{j<i}$ and then define a_i^* to be the colimit morphism. In other words,

$$a_i^* = \bigcup_{j<i} a_j^*.$$

Since 0 is simple, a_0^* is a strong monomorphism. By transfinite induction we see immediately that all a_i^* are strong monomorphisms (for the limit step use Lemma 2.10). Moreover, what we have is nothing else than the construction of the least fixed point of $m \longmapsto \bigcirc m$, see Remark 2.15, in the proof of the Knaster-Tarski Theorem in [25]. Thus, $a^* = \bigcup_{i \in \mathbf{Ord}} a_i^*$, where the union ranges over the class **Ord** of all ordinals. However, since A has only a set of subobjects,

$$a^* = a_{i_0}^* \qquad \text{for some ordinal } i_0. \tag{2.3}$$

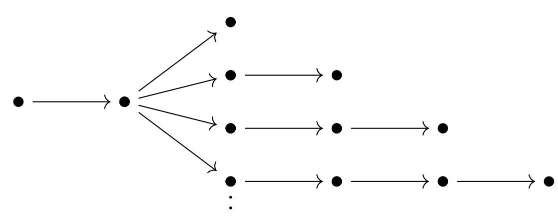
And for this ordinal i_0, an easy verification using the pullback property shows that

$$\alpha^* = \alpha[a^*] \tag{2.4}$$

Henceforth, we call A^* the *greatest cartesian subcoalgebra* of A.

From now on, whenever we use the notations $\bigcirc m$ and a^*, we only do so when H preserves strong monomorphisms.

Example 2.18. For every graph A considered as a coalgebra for $\mathscr{P}$, A^* is the subgraph on all vertices of A from which no infinite path starts. Since $m \mapsto \bigcirc m$ is not necessarily continuous, the ordinal i_0 above can be arbitrarily large. Here is an example with $i_0 = \omega + 1$:



Example 2.19. The endofunctor H of **Gra** in Example 2.8 has $1 = H1$ as its final coalgebra, and this coalgebra is well-founded. Of course, this functor H does not preserve strong monomorphisms, and so most of the foregoing results do not apply to it. In particular, with α as $\mathrm{id} : 1 \longrightarrow H1$ (the final coalgebra), there is no ordinal i such that $a_i^* = \mathrm{id}_1$. This shows that we must assume preservation of strong monomorphisms.

Proposition 2.20. *Assume that H preserves strong monomorphisms. For every coalgebra (A, α), the greatest cartesian subcoalgebra (A^*, α^*) is its coreflection in the full subcategory of well-founded coalgebras.*

Remark. We thus prove that (A^*, α^*) is well-founded, and for every homomorphism $f\colon (B, \beta) \longrightarrow (A, \alpha)$ with (B, β) well-founded there exists a unique homomorphism

$$\bar{f}\colon (B, \beta) \longrightarrow (A^*, \alpha^*) \qquad \text{with} \qquad f = a^* \cdot \bar{f}.$$

Proof. (i) We first observe that for all ordinals $i \leq j$ the connecting maps

$$\begin{tikzcd} A_i^* \arrow[rr, "a_{ij}^*"] \arrow[dr, "a_i^*"'] & & A_j^* \arrow[dl, "a_j^*"] \\ & A & \end{tikzcd}$$

of the chain of Notation 2.17 form the following commutative diagram which can be used as a definition of a_{ij}^*'s (via the universal property of pullbacks):

$$\begin{tikzcd} A_{i+1}^* \arrow[r, "\alpha[a_i^*]"] \arrow[d, "a_{i+1,j+1}^*"] \arrow[dd, bend right, "a_{i+1}^*"'] & HA_i^* \arrow[d, "Ha_{ij}^*"'] \arrow[dd, bend left, "Ha_i^*"] \\ A_{j+1}^* \arrow[r, "\alpha[a_j^*]"'] \arrow[d, "a_{j+1}^*"] & HA_j^* \arrow[d, "Ha_j^*"'] \\ A \arrow[r, "\alpha"'] & HA \end{tikzcd} \tag{2.5}$$

(ii) A^* is a well-founded coalgebra: for every ordinal number the outside square of the diagram

$$\begin{tikzcd} A_{i+1}^* \arrow[r, "\alpha[a_i^*]"] \arrow[d] \arrow[dd, bend right, "a_{i+1}^*"'] & HA_i^* \arrow[d] \arrow[dd, bend left, "Ha_i^*"] \\ A^* \arrow[r, "\alpha[a^*]"] \arrow[d, "a^*"] & HA^* \arrow[d, "Ha^*"'] \\ A \arrow[r, "\alpha"'] & HA \end{tikzcd}$$

is a pullback, thus so is the upper square. This shows that $(A^*)_i^* = A_i^*$. Therefore $(A^*)^* = A^*$ by (2.3).

(iii) Suppose we are given a well-founded coalgebra $\beta : B \longrightarrow HB$ and a coalgebra homomorphism $f\colon B \longrightarrow A$. Since a^* is a monomorphism there is at most one coalgebra homomorphism $\bar{f}\colon B \longrightarrow A^*$ with $a^* \cdot \bar{f} = f$. Thus, we are finished if we show that $\bar{f}$ exists. To this end denote by $b_{i,j}^*\colon B_i^* \longrightarrow B_j^*$ the chain whose colimit is $B^* = B$ with the colimit injections $b_i^*\colon B_i^* \longrightarrow B^*$. We define the components of a natural transformation $\bar{f}_i\colon B_i^* \longrightarrow A_i^*$ by transfinite recursion on ordinals i, satisfying

$$\begin{tikzcd} B_i^* \arrow[r, "\bar{f}_i"] \arrow[d, "b_i^*"'] & A_i^* \arrow[d, "a_i^*"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \tag{2.6}$$

The naturality follows due to the commutativity from the diagram below for $i \leq j$:

$$
\begin{tikzcd}
B_i^* \arrow[r, "\bar{f}_i"] \arrow[d, "b_{i,j}^*"] \arrow[dd, bend right=60, "b_i^*"'] & A_i^* \arrow[d, "a_{i,j}^*"'] \arrow[dd, bend left=60, "a_i^*"] \\
B_j^* \arrow[r, "\bar{f}_j"] \arrow[d, "b_j^*"] & A_j^* \arrow[d, "a_j^*"'] \\
B \arrow[r, "f"'] & A
\end{tikzcd}
\tag{2.7}
$$

The desired upper square commutes because all other parts and the outside square do and since a_j^* is a monomorphism.

We define $\bar{f}_i$ by transfinite recursion. Let $\bar{f}_0 = \mathrm{id}\colon \emptyset \longrightarrow \emptyset$. Then (2.6) clearly commutes for $i = 0$. For isolated steps consider the diagram below:

$$
\begin{tikzcd}
B_{i+1}^* \arrow[rrr, "\beta[b_i^*]"] \arrow[dr, dashed, "\bar{f}_{i+1}"] \arrow[ddd, "b_{i+1}^*"'] & & & HB_i^* \arrow[dl, "H\bar{f}_i"'] \arrow[ddd, "Hb_i^*"] \\
& A_{i+1}^* \arrow[r, "\alpha[a_i^*]"] \arrow[d, "a_{i+1}^*"'] & HA_i^* \arrow[d, "Ha_i^*"] & \\
& A \arrow[r, "\alpha"'] & HA & \\
B \arrow[ur, "f"] \arrow[rrr, "\beta"'] & & & HB \arrow[ul, "Hf"']
\end{tikzcd}
\tag{2.8}
$$

The inner and outside squares commute by the definition of A_{i+1}^* and B_{i+1}^*, respectively. For the lower square we use that f is a coalgebra homomorphism, and the right-hand one commutes by the induction hypothesis. The inner pullback induces the desired morphism $\bar{f}_{i+1}$ and the commutativity of the left-hand square is that of (2.6) for $i+1$. Finally, for a limit ordinal j let $\bar{f}_j = \mathrm{colim}_{i<j}\, \bar{f}_i$, in other words, $\bar{f}_j$ is the unique morphism such that the squares

$$
\begin{tikzcd}
B_j^* \arrow[r, "\bar{f}_j"] & A_j^* \\
B_i^* \arrow[u, "b_{i,j}^*"] \arrow[r, "\bar{f}_i"'] & A_i^* \arrow[u, "a_{i,j}^*"']
\end{tikzcd}
$$

commute for all $i < j$. We need to verify that (2.6) commutes for $\bar{f}_j$. This is the commutativity of the lower square in (2.7), which follows since all other parts and the outside square commute for all $i < j$.

To complete the proof consider any ordinal i such that $B_i^* = B^* = B$ and $A_i^* = A^*$ hold. Then $\bar{f} = \bar{f}_i \colon B \longrightarrow A^*$ is a coalgebra homomorphism with $a_i^* \cdot \bar{f} = f$ by the commutativity of the upper and left-hand part of Diagram (2.8). □

For endofunctors preserving inverse images the following lemma is in [26] and Exercise VI.16 in [27]:

Lemma 2.21. *Assuming that H preserves strong monomorphisms, the subcategory of* **Coalg** H *consisting of the well-founded coalgebras is closed under quotients and colimits in* **Coalg** H.

This follows from a general result on coreflective subcategories: the category **Coalg** H has an (epi, strong mono)-factorization system (see Remark 2.5), and its full subcategory of well-founded coalgebras is coreflective with strong monomorphic coreflections (see Proposition 2.20). Consequently, it is closed under quotients and colimits. For more general results, see Theorem 16.8 and Corollary 13.20 of [4].

Example 2.22. The initial coalgebra $0 \longrightarrow H0$ is well-founded.

2B Initial algebras are well-founded

Our next result is that the initial H-algebra (if it exists) is well-founded. That is, suppose that H has an initial algebra $(I, \varphi : HI \longrightarrow I)$. Then by Lambek's Lemma, φ is invertible. We prove that (I, φ^{-1}) is a well-founded coalgebra.

In the proof we use the *initial chain* defined in [3]. This is the chain

$$W_i \quad (i \in \mathbf{Ord}) \qquad \text{and} \qquad w_{ij} \colon W_i \longrightarrow W_j \quad (i \leq j) \tag{2.9}$$

defined uniquely up to natural isomorphism by

$$\begin{aligned} W_0 &= 0 \qquad \text{(initial object of } \mathscr{A}) \\ W_{i+1} &= HW_i \quad \text{and} \quad w_{i+1,j+1} = Hw_{i,j} \end{aligned}$$

and for limit ordinals i

$$W_i = \operatorname*{colim}_{j<i} W_j \qquad \text{with colimit cocone } w_{ij} \quad (i < j).$$

The chain is said to *converge* at i if the connecting map $w_{i,i+1} \colon W_i \longrightarrow HW_i$ is invertible. The inverse then makes W_i an initial algebra.

Proposition 2.23 ([30]).

(1) Whenever there exists an object $X \cong HX$, then H has an initial algebra.
(2) If H preserves strong monomorphisms and has an initial algebra, then the initial chain converges.

This result was shown in Theorem II.4 of [30]. The proof uses the LFP-property of $\mathscr{A}$, especially one of its consequence, the wellpoweredness of $\mathscr{A}$.

Theorem 2.24. *Let H preserve strong monomorphisms. Initial algebras are, as coalgebras, well-founded.*

This was proved in Taylor [27] under the additional assumption that H preserves inverse images.

Proof. If H has an initial algebra, then, by Proposition 2.23, it has the form $(w_{j,j+1})^{-1}\colon HW_j \longrightarrow W_j$ for some ordinal j. We prove that for $i \leq j$, the morphisms a_i^* of Notation 2.17 are the same as the morphisms w_{ij}. Consequently, $a^* = \mathrm{id}_{W_j}$, as requested. For $i = 0$ the equality $a_0^* = w_{0j}$ is clear. For the isolated step we need to prove that the square

$$\begin{tikzcd} W_{i+1} \arrow[r, "\mathrm{id}"] \arrow[d, "w_{i+1,j}"'] & HW_i \arrow[d, "Hw_{ij}"] \\ W_j \arrow[r, "w_{j,j+1}"'] & HW_j \end{tikzcd}$$

is a pullback. Indeed, the square commutes since $Hw_{ij} = w_{i+1,j+1}$, and it is a pullback since both horizontal arrows are invertible. Limit steps follow automatically. The coalgebra W_j is thus well-founded by Proposition 2.20. □

2C $\mathscr{M}$-Well-Founded Coalgebras

Although we have worked above with strong monomorphisms only, the whole theory can be developed for a general class $\mathscr{M}$ of monomorphisms in the base category $\mathscr{A}$. We need to assume that

(a) $\mathscr{A}$ is $\mathscr{M}$-wellpowered
(b) $\mathscr{M}$ is closed under inverse images
(c) $\mathscr{M}$ is constructive in the sense of [30].

The last point means that $\mathscr{M}$ is closed under composition, and for every chain of monomorphisms in $\mathscr{M}$, (i) a colimit exists and is formed by monomorphisms in $\mathscr{M}$, and (ii) the factorization morphism of every cocone of monomorphisms in $\mathscr{M}$ is again a monomorphism in $\mathscr{M}$. For strong monomorphisms in a locally finitely presentable category this in particular states that the initial object is simple.

Examples 2.25. For the categories of sets, graphs, posets, and semigroups, we can take $\mathscr{M}$ to be the constructive class of all monomorphisms.

Example 2.26. (a) In Example 2.8, the endofunctor preserves strong monomorphisms, but it does not preserve monomorphisms. In the case where we take $\mathscr{M}$ to be all monomorphisms, then the initial algebra is easily seen to be the final $\mathscr{M}$-well-founded coalgebra. Indeed, every $\mathscr{M}$-well-founded coalgebra is carried by a graph without edges. In contrast, for $\mathscr{M}$ consisting of strong monomorphisms, the final well-founded coalgebra is carried by 1, the final graph, see Example 2.34.

(b) The endofunctor of **Gra** defined by

$$HX = \begin{cases} X & \text{if } X \text{ has a loop} \\ \text{the discrete graph on } X & \text{otherwise} \end{cases}$$

preserves monomorphisms, but not strong monomorphisms.

We then can define *$\mathscr{M}$-well-founded coalgebra* as one that has no proper cartesian subcoalgebra carried by an $\mathscr{M}$-monomorphism.

All results above hold in this generality. In Theorem 2.24 we must assume that H preserves $\mathscr{M}$, that is, if m lies in $\mathscr{M}$ then so does Hm. In Theorem 2.35 we need to assume that H preserves $\mathscr{M}$ and finite intersections of $\mathscr{M}$-monomorphisms.

We defined well-foundedness in Definition 2.11 implicitly using choice of $\mathscr{M}$ = strong monomorphisms. The reason we did so is that, as shown in Lemma 2.6, the assumption that H preserves strong monomorphisms implies that strong monomorphisms in the category of coalgebras are the same as strong monomorphisms in the base category, and vice-versa. In contrast, no characterization of monomorphisms in the category of coalgebras is known.

The concept of $\mathscr{M}$-well-founded coalgebra introduced in this section depends on the choice of $\mathscr{M}$ as Example 2.16 below demonstrates. In particular, assume that H preserves strong monomorphisms but not all monomorphisms. Then for every coalgebra (A, α) we can still form the subcoalgebra (A^*, α^*) as in Notation 2.17, but there is no reason why the latter should be mono-wellpowered: on the class of all monomorphisms we cannot define the function of Definition 2.13 that made the argument of Proposition 2.20 work. Example 2.16 provides such a case.

Example 2.27. We return to the functor $H : \mathbf{Gra} \longrightarrow \mathbf{Gra}$ of Example 2.16 By Theorem 2.35, the initial algebra of H is the same as its final well-founded coalgebra. This is

$$I = HF \cup \{t\}$$

where $HF = \mathscr{P}_f HF$ is the initial algebra of the finite power set functor on **Set**, taken as a discrete graph, and t is connected in both directions to all $x \in HF$, and to itself.

In contrast, I is not $\mathscr{M}$-well-founded, where $\mathscr{M}$ is the class of *all monomorphisms.* Here is the reason. Let J be the same as I, except that we drop all edges between t and the elements of HF. (We keep the loop at t.) Then $HJ = HI = I$. The inclusion $i : J \longrightarrow I$ is of course monic, and $Hi = \mathrm{id}_{HI}$. It is easy to check that this inclusion is a coalgebra morphism, and indeed that it gives a pullback. This verifies that I is not well-founded, for the class $\mathscr{M}$ of all monomorphisms.

A different point: H shows that we can have an $\mathscr{M}$-well-founded coalgebra (A, a) such that (HA, Ha) is not well-founded. For this, take $\mathscr{M}$ to be all monomorphisms, and take (A, a) to be $0 \longrightarrow H0$. This is well-founded, since 0 is the empty graph and thus has no subobjects. Moreover $H0$ is a graph $t \leftrightarrow x$, with a loop on t. Similarly, $HH0$ adds another point y, and also has $t \leftrightarrow x$. We claim that the coalgebra $H0 \longrightarrow HH0$ is not cartesian. Let J be two points, t and x, with a loop on t, and no other edges. Then $HJ = HH0$, and the inclusion of J in $H0$ gives a cartesian subcoalgebra which is not the identity.

2D Recursive coalgebras

Definition 2.28. *A coalgebra $\alpha\colon A \longrightarrow HA$ is* ***recursive*** *if for every algebra $\beta\colon HB \longrightarrow B$ there exists a unique coalgebra-to-algebra homomorphism*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & HA \arrow[d, "Hh"] \\ B & HB \arrow[l, "\beta"] \end{tikzcd}$$

This concept was, for introduced by Taylor under the name "coalgebra obeying the recusion scheme", the name recursive coalgebra stems from [10].

Examples 2.29 (see [10]).

(1) $0 \longrightarrow H0$ is a recursive coalgebra.
(2) If $\alpha\colon A \longrightarrow HA$ is recursive, then so is $H\alpha\colon HA \longrightarrow HHA$.
(3) A colimit of recursive coalgebras is recursive. Combining these results we see that in the initial chain (2.9) all the coalgebras

$$w_{i,i+1}\colon W_i \longrightarrow HW_i$$

are recursive.

We are going to prove that for set functors, well-founded coalgebras are recursive. Before we do this, let us discuss the converse. In general, recursive coalgebras need not be well-founded, even for set functors. However for all set functors preserving inverse images recursiveness does imply well-foundedness, as shown by Taylor [26, 27]. We typically have a slightly stronger result; see Example 2.31. In contrast, here is an example of the phenomenon mentioned above, a coalgebra for a set functor that is recursive but not well-founded.

Example 2.30 (see [5]). A recursive coalgebra need not be well-founded. Let $H\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be defined on objects by

$$HX = (X \times X)/{\sim}$$

where $\sim$ merges the diagonal to a single element, d. For morphisms $f\colon X \longrightarrow Y$ we take $Hf(d) = d$ and

$$Hf(x_1, x_2) = \begin{cases} d & \text{if } f(x_1) = f(x_2) \\ (fx_1, fx_2) & \text{else} \end{cases}$$

This functor H preserves (strong) monomorphisms. The coalgebra $A = \{0,1\}$ with the structure α constant to $(0,1)$ is recursive: given an algebra $\beta\colon HB \longrightarrow B$, the unique coalgebra-to-algebra homomorphism $h\colon \{0,1\} \longrightarrow B$ is

$$h(0) = h(1) = \beta(d).$$

But A is not well-founded: $\emptyset$ is a cartesian subcoalgebra.

$$\begin{tikzcd} \emptyset \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ \{0,1\} \arrow[r, "\alpha"'] & H\{0,1\} \end{tikzcd}$$

Example 2.31. There is a $\mathscr{P}$-algebra (B,b) such that for all $\mathscr{P}$-coalgebras (A,a), if (A,a) is not well-founded, then there are at least two coalgebra-to-algebra homomorphisms $h : A \longrightarrow B$.

We take $B = \{0,1,2\}$, with $b : \mathscr{P}B \longrightarrow B$ defined as follows:

$$b(x) = \begin{cases} 0 & \text{if } x = \emptyset \text{ or } x = \{0\} \\ 1 & \text{else if } 1 \in x \\ 2 & \text{if } 2 \in x \text{ and } 1 \notin x \end{cases}$$

If (A,a) is any coalgebra which is not well-founded, we show that there are at least two coalgebra-to-algebra homomorphisms $h : A \longrightarrow B$. We can take

$$h_1(x) = \begin{cases} 0 & \text{if there are no infinite sequences } x = x_0 \to x_1 \to x_2 \cdots \\ 1 & \text{if there is an infinite sequence } x = x_0 \to x_1 \to x_2 \cdots \end{cases}$$

and also h_2 defined the same way, but using 2 as a value instead of 1. The verification that h_1 and h_2 are coalgebra-to-algebra homomorphisms hinges on two facts: first, $h(x) = 0$ iff there is no infinite sequence starting from x; and second, if $h_i(x) \neq 0$, then there is some $y \in a(x)$ such that $h_i(y) \neq 0$ as well.

Theorem 2.32. *If H preserves strong monomorphisms, then every well-founded coalgebra is recursive.*

For functors preserving inverse images this follows from [26, Theorem 6.3.13].

Proof. Let $\alpha\colon A \longrightarrow HA$ be well-founded. For every algebra $e\colon HX \longrightarrow X$ we prove the existence and uniqueness of a coalgebra-to-algebra homomorphism $A \longrightarrow X$. We use the initial chain (W_i) of (2.9) and also the chain (A_i^*) from Notation 2.17.

(1) Existence. We prove first that there is a unique natural transformation

$$f_i\colon A_i^* \longrightarrow W_i \qquad (i \in \mathbf{Ord})$$

such that for all ordinals i we have

$$f_{i+1} = \left(A_{i+1}^* \xrightarrow{\alpha[a_i^*]} HA_i^* \xrightarrow{Hf_i} HW_i = W_{i+1} \right). \tag{2.10}$$

In fact, since both of the transfinite chains (A_i^*) and (W_i) are defined by colimits on all limit ordinals i, we only need to show how f_0 is specified and, given f_i, how f_{i+1} is specified so that the naturality square

$$\begin{tikzcd} A_i^* \arrow[r, "f_i"] \arrow[d, "a^*_{i,i+1}"'] & W_i \arrow[d, "w_{i,i+1}"] \\ A_{i+1}^* \arrow[r, "f_{i+1}"'] & W_{i+1} \end{tikzcd} \tag{2.11}$$

commutes for every non-limit ordinal i. The first step is trivial since $A_0^* = \emptyset$; we take $f_0 = \mathrm{id} \colon \emptyset \longrightarrow \emptyset$. In the induction step, f_{i+1} is defined by the above formula (2.10) and we need to prove the commutativity of the above square (2.11). For this, the diagram below commutes by the the induction hypothesis (2.11) and by the commutativity of the upper inner square of (2.5):

$$\begin{tikzcd} A_{i+1}^* \arrow[r, "{\alpha[a_i^*]}"] \arrow[d, "a^*_{i+1,i+2}"'] & HA_i^* \arrow[r, "Hf_i"] \arrow[d, "Ha^*_{i,i+1}"] & HW_i \arrow[d, "Hw_{i,i+1}"] \\ A_{i+2}^* \arrow[r, "{\alpha[a_{i+1}^*]}"'] & HA_{i+1}^* \arrow[r, "Hf_{i+1}"'] & HW_{i+1} \end{tikzcd}$$

Next, since the W_i are recursive coalgebras, see Example 2.29, we have unique coalgebra-to-algebra homomorphisms into X. These form a natural transformation into the constant functor with value X:

$$r_i \colon W_i \longrightarrow X \qquad (i \in \mathbf{Ord}) \ .$$

Consequently, we obtain a natural transformation $r_i f_i \colon A_i^* \longrightarrow X$ which, for i such that $A = A_i^*$, yields

$$h = r_i f_i \colon A \longrightarrow X \ .$$

Now consider the diagram below.

$$\begin{tikzcd} A_i^* = A_{i+1}^* \arrow[r, "{\alpha[a_i^*]}"] \arrow[d, "f_i"] \arrow[dr, "f_{i+1}"] \arrow[dd, bend right, "h"'] & HA_i^* \arrow[d, "Hf_i"'] \arrow[dd, bend left, "Hh"] \\ W_i \arrow[r, "w_{i,i+1}"] \arrow[d, "r_i"] & HW_i \arrow[d, "Hr_i"] \\ X & HX \arrow[l, "e"] \end{tikzcd}$$

The morphism at the top is α, by (2.4). The sides are the definition of h, the bottom square is the definition of r_i, and the top triangle is the definition of f_{i+1}. The bottom triangle is (2.11); note that $a^*_{i,i+1} = \mathrm{id}$. The overall outside of the figure shows that h is a coalgebra-to-algebra homomorphism as desired.

(2) Uniqueness. If $h_1, h_2 \colon A \longrightarrow X$ are coalgebra-to-algebra homomorphisms, then we prove $h_1 = h_2$ by showing that

$$h_1 \cdot a_i^* = h_2 \cdot a_i^* \qquad \text{for all } i \in \mathbf{Ord}.$$

The case $i = 0$ is clear, in the isolated step use the commutative diagrams (with $t = 1, 2$):

$$\begin{tikzcd}
A_{i+1}^* \arrow[r, "{\alpha[a_i^*]}"] \arrow[d, "a_{i+1}^*"'] & HA_i^* \arrow[d, "Ha_i^*"] \\
A \arrow[r, "\alpha"] \arrow[d, "h_t"'] & HA \arrow[d, "Hh_t"] \\
X & HX \arrow[l, "e"]
\end{tikzcd}$$

and the limit steps follow from $A_j^* = \operatorname{colim}_{i<j} A_i^*$ for limit ordinals j. ◻

Remark 2.33. The concepts "initial algebra" and "final recursive coalgebra" coincide for all endofunctors, as proved by Capretta et al [10]. This is not true in general for well-foundedness in lieu of recursiveness:

Example 2.34. We present two examples that show that our overall assumptions are needed in Theorem 2.35 just below.

(a) The endofunctor H of **Gra** in Example 2.8 has 1 as its final well-founded coalgebra, and its initial algebra is a countably infinite set (the initial chain converges in the ω-th step).
(b) In Example 2.9, we saw an example of $H : \mathbf{Set}_{0,1} \longrightarrow \mathbf{Set}_{0,1}$ which preserved strong monomorphisms, but where 0 is not simple. The initial algebra for H is clearly infinite. But the final well-founded coalgebra is $1 = H1$, since 1 has no proper subobject.
    This latter example demonstrates that the simplicity of 0 is needed in Theorem 2.35 just below. Furthermore, our endofunctor H even preserves finite intersections (pullbacks of (strong) monomorphisms)! Compare this with the following

For endofunctors preserving inverse images the following theorem is Corollary 9.9 of [26]. As we mentioned in the introduction, it is non-trivial to relax the assumption on the endofunctor, and so our proof is different from Taylor's. As a result we obtain in Theorem 2.38 below that for a set endofunctor no assumptions are needed.

Theorem 2.35. *If H preserves strong monomorphisms and finite intersections, then*

$$\textit{initial algebra} = \textit{final well-founded coalgebra}$$

That is, an algebra $\varphi \colon HI \longrightarrow I$ is initial iff $\varphi^{-1} \colon I \longrightarrow HI$ is the final well-founded coalgebra.

Proof. (a) Let I be an initial algebra. By a result of Capretta et al. [10] which we mentioned in Remark 2.33, I is a final recursive coalgebra. Since I is well-founded by Theorem 2.24, it is a final well-founded coalgebra due to Theorem 2.32.

(b) Let $\psi : I \longrightarrow HI$ be a final well-founded coalgebra.

(b1) Factorize $\psi = m\cdot e$ where e is an epimorphism and m a strong monomorphism (Remark 2.5). By diagonal fill-in

$$\begin{tikzcd}
I \arrow[r, "\psi"] \arrow[d, "e"'] & HI \arrow[d, "He"] \\
I' \arrow[r, dashed, "\psi'"] \arrow[d, "m"'] & HI' \arrow[d, "Hm"] \\
HI \arrow[r, "H\psi"'] & HHI
\end{tikzcd}$$

we obtain a quotient (I', ψ') which, by Lemma 2.21 and Theorem 2.32, is recursive. Consequently, a coalgebra homomorphism $f\colon (I', \psi') \longrightarrow (I, \psi)$ exists. Then fe is an endomorphism of the final well-founded coalgebra, hence, $fe = \mathrm{id}_I$. This proves that e is an isomorphism, and from the commutativity of the lower square above we see that ψ' is a strong monomorphism, in other words

$$\psi \text{ is a strong monomorphism.}$$

(b2) The coalgebra $(HI, H\psi)$ is well-founded. Indeed, consider a cartesian subcoalgebra (A', a')

$$\begin{tikzcd}
J \arrow[r, dashed, "\psi'"] \arrow[d, dashed, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA' \arrow[d, "Hm"] \\
I \arrow[r, dashed, "\psi"'] & HI \arrow[r, "H\psi"'] & HHI
\end{tikzcd}$$

Form the intersection J of m and ψ. Since H preserves this intersection, it follows that m and Hm' represent the same subobject of HI, thus, we have

$$u\colon A' \longrightarrow HJ, \quad \text{an isomorphism,} \qquad \text{with} \quad m = Hm'\cdot u.$$

This yields a cartesian subcoalgebra

$$\begin{tikzcd}
J \arrow[r, "\psi'"] \arrow[d, "m'"'] \arrow[drr, phantom, "\lrcorner", very near start] & A' \arrow[r, "u"] & HJ \arrow[d, "Hm'"] \\
I \arrow[rr, "\psi"'] & & HI
\end{tikzcd}$$

and since (I, ψ) is well-founded, we conclude that m' is invertible. Consequently, $m = Hm'\cdot u$ is invertible.

(b3) ψ is invertible. Indeed, we have, by (2), a homomorphism $h\colon (HI, H\psi) \longrightarrow (I,\psi)$:

$$\begin{tikzcd} I \arrow[r, "\psi"] \arrow[d, "\psi"'] & HI \arrow[d, "H\psi"] \\ HI \arrow[r, "H\psi"] \arrow[d, "h"'] & HHI \arrow[d, "Hh"] \\ I \arrow[r, "\psi"'] & HI \end{tikzcd}$$

Then $h{\cdot}\psi$ is an endomorphism of (I,ψ), thus, $h{\cdot}\psi = \mathrm{id}$. And the lower square yields $\psi{\cdot}h = H(h{\cdot}\psi) = \mathrm{id}$, whence $I \cong HI$,

(b4) By Proposition 2.23, the initial chain converges, and for some ordinal i, $w_{i,i+1}^{-1}\colon HW_i \longrightarrow W_i$ is an initial algebra. Moreover, $w_{i,i+1}\colon W_i \longrightarrow HW_i$ is by (a) a final well-founded coalgebra, thus, isomorphic to $\psi\colon I \longrightarrow HI$. Thus (I,ψ^{-1}) is isomorphic to the initial algebra above. $\square$

2E Initial algebras of set functors

The main result of this section is that for all endofunctors H of **Set** the equality

$$\text{initial algebra} = \text{final well-founded coalgebra} \tag{2.12}$$

holds, i. e., for the particular case of our given LFP category being $\mathscr{A} = \mathbf{Set}$ one can lift the assumption that H preserves (strong) monomorphisms and intersections in Theorem 2.35. We have already remarked that if H preserves inverse images, this result can be found in [27].

Proposition 2.36 (Trnková [29]). *For every endofunctor H of* **Set** *there exists an endofunctor $\bar{H}$ preserving monomorphisms and finite intersections and identical with H on all nonempty sets (and nonempty functions).*

Now $\bar{H}$ fulfils the desired equality (2.12) by Theorem 2.35. And the proof of the next result is the transfer of (2.12) from $\bar{H}$ to H. To do so, let us recall how Trnková defined $\bar{H}$:

Denote by C_{01} the functor $\emptyset \longmapsto \emptyset$ and $X \longmapsto 1$ for all $X \neq \emptyset$. Define $\bar{H}$ as H on all nonempty sets, and put

$$\bar{H}\emptyset = \{\tau; \tau\colon C_{01} \longrightarrow H \text{ a natural transformation}\}.$$

(To check that we have a set here and not a proper class, note that each $\tau : C_{0,1} \longrightarrow H$ is determined by $\tau_1 : 1 \longrightarrow H1$. For a nonempty set A, if $k : 1 \longrightarrow A$ is arbitrary, $\tau_A = Hk \circ \tau_1$.) Given a nonempty set X, $\bar{H}$ assigns to the empty map $q_X : \emptyset \longrightarrow X$ the map

$$\bar{H}q_X\colon \tau \longmapsto \tau_X \qquad \text{for every} \quad \tau\colon C_{01} \longrightarrow H,$$

where that $\tau_X : 1 \longrightarrow HX$ is simply an element of HX.

Continuing, observe that there exists a map $u \colon H\emptyset \longrightarrow \bar{H}\emptyset$ such that for every set $A \neq \emptyset$ the triangle

$$\begin{tikzcd} H\emptyset \arrow[rr, "u"] \arrow[dr, "Hq_A"'] & & \bar{H}\emptyset \arrow[dl, "\bar{H}q_A"] \\ & HA = \bar{H}A & \end{tikzcd} \tag{2.13}$$

commutes. For each element $x \in H\emptyset$, let the natural transformation $u(x) \colon C_{01} \longrightarrow H$ have components $u(x)_A = Hq_A(x)$ for all $A \neq \emptyset$. Then

$$\bar{H}q_A(u(x)) = (u(x))_A = Hq_A(x).$$

Lemma 2.37. *Let (A, a) be a well-founded H-coalgebra, with $A \neq \emptyset$, so that (A, a) is also an $\bar{H}$-coalgebra. Then $\emptyset$ is not the carrier of any cartesian $\bar{H}$-subcoalgebra of (A, a).*

Proof. Assume towards a contradiction that $q_{\bar{H}\emptyset} : \emptyset \longrightarrow \bar{H}\emptyset$ were a cartesian subcoalgebra of (A, a). We claim that the square below is a pullback:

$$\begin{tikzcd} \emptyset \arrow[r, "q_{H\emptyset}"] \arrow[d, "q_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & H\emptyset \arrow[d, "Hq_A"] \\ A \arrow[r, "a"'] & HA \end{tikzcd} \tag{2.14}$$

We show that there are no $y \in A$ and $x \in H\emptyset$ such that that $a(y) = Hq_A(x)$. For assume that y and x exist with these properties. Then by (2.13), $\bar{H}q_A(u(x)) = a(y)$. This contradicts our assumption that $(\emptyset, q_{\bar{H}\emptyset})$ is a cartesian subcoalgebra of (A, a). Thus, y and x do not exist as assumed, and hence, the square in (2.14) is indeed a pullback. Therefore q_A is an isomorphism. But $A \neq \emptyset$, and this is a contradiction. ☐

Theorem 2.38. *For every endofunctor of* **Set** *we have:*

initial algebra = final well-founded coalgebra.

Proof. Given H, we know from Theorem 2.35 that the statement holds for $\bar{H}$. From this we are going to prove it for H.

(a) If $\varphi \colon HI \longrightarrow I$ is an initial algebra, we prove that $\varphi^{-1} \colon I \longrightarrow HI$ is a final well-founded coalgebra.

The first case is when $H\emptyset = \emptyset$. In this case $I = \emptyset$. And the only (hence, the final) well-founded coalgebra is the empty one. Indeed, if $a \colon A \longrightarrow HA$ is well-founded, then the following cartesian subcoalgebra

$$\begin{tikzcd} \emptyset \arrow[r, "\mathrm{id}"] \arrow[d, "q_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d, "Hq_A"] \\ A \arrow[r, "a"'] & HA \end{tikzcd} \tag{2.15}$$

demonstrates that q_A is an isomorphism, so $A = \emptyset$.

The second case is when $H\emptyset \neq \emptyset$. Then $\bar{H}\emptyset \neq \emptyset$ via u in (2.13) above. The $\bar{H}$-algebra $\varphi \colon \bar{H}I \longrightarrow I$ is initial because every $\bar{H}$-algebra is nonempty, hence, it also is an H-algebra. And the unique homomorphism from I w.r.t. H is also a homomorphism w.r.t. $\bar{H}$. By Theorem 2.35, $\varphi^{-1} \colon I \longrightarrow \bar{H}I$ is a final well-founded $\bar{H}$-coalgebra. Let us now verify that it is also well-founded w.r.t. H. Consider a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA' \arrow[d, "Hm"] \\ I \arrow[r, "\varphi^{-1}"'] & HI \end{tikzcd} \tag{2.16}$$

We claim that A' cannot be empty. For if it were, then since $HA' = H\emptyset \neq \emptyset$, we take any $x \in HA'$ and consider x and $(\varphi \cdot Hm)x$. By the pullback property, there is some $y \in A'$ so that $a'(y) = x$. In particular, this contradicts $A' = \emptyset$.

As a result, $HA' = \bar{H}A'$, and $Hm = \bar{H}m$. So (2.16) is a cartesian subcoalgebra for $\bar{H}$. Thus m is invertible, as desired.

At this point we know that $\varphi^{-1} \colon I \longrightarrow \bar{H}I$ is a well-founded H coalgebra; we conclude with the verification that φ^{-1} is *final* among these. This follows from the observation that every nonempty well-founded H-coalgebra $a \colon A \longrightarrow HA$ is also well-founded w.r.t. $\bar{H}$. Indeed, consider a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{H}A' \arrow[d, "\bar{H}m"] \\ A \arrow[r, "a"'] & \bar{H}A \end{tikzcd} \tag{2.17}$$

By Lemma 2.37, $A' \neq \emptyset$. Thus $\bar{H}m = Hm$ and we conclude that m is invertible.

(b) If $\psi \colon I \longrightarrow HI$ is a final well-founded coalgebra, we prove that ψ is invertible and $\psi^{-1} \colon HI \longrightarrow I$ is an initial algebra. Unfortunately, we cannot use the converse implication of what we have just proved (every nonempty well-founded $\bar{H}$-coalgebra is also well-founded w.r.t. H) since this is false in general (see Example 2.39 below). We can assume $H\emptyset \neq \emptyset$, since the case $H\emptyset = \emptyset$ is trivial.

Consider first the coalgebra

$$b \colon \bar{H}\emptyset \longrightarrow H\bar{H}\emptyset$$

defined by

$$b(\tau) = \tau_{\bar{H}\emptyset} \qquad \text{for all } \tau \colon C_{01} \longrightarrow H.$$

Let us show that this coalgebra is well-founded for H. Consider a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & HA' \arrow[d, "Hm"] \\ \bar{H}\emptyset \arrow[r, "b"'] & H\bar{H}\emptyset \end{tikzcd} \tag{2.18}$$

It is our task to prove that m is surjective (thus, invertible). First, assume that $A' \neq \emptyset$. Given $\tau\colon C_{01} \longrightarrow H$ in $\bar{H}\emptyset$, the element $\tau_{A'}$ of HA' (recall that $H\emptyset \neq \emptyset$, thus $HA' \neq \emptyset$ for all sets A') fulfils

$$b(\tau) = \tau_{\bar{H}\emptyset} = Hm(\tau_{A'}).$$

(The last equality uses the naturality of τ and the fact that $C_{01}m = \mathrm{id}_1$.) Thus, there exists an element of A' that m maps to τ. Our second case is when $A' = \emptyset$. We show that this case leads to a contradiction. Observe that $m = q_{\bar{H}\emptyset}\colon \emptyset \longrightarrow \bar{H}\emptyset$, and let $x \in H\emptyset$, so that $u(x) \in \bar{H}\emptyset$. By (2.13),

$$b(u(x)) = (u(x))_{\bar{H}\emptyset} = Hq_{\bar{H}\emptyset}(x)$$

Thus x and $u(x)$ are mapped to the same element of $H\bar{H}\emptyset$ by Hm and b, respectively, contradicting the assumption that $\emptyset$ is a pullback in (2.18) above.

The first point of this coalgebra $(\bar{H}\emptyset, b)$ is that its well-foundedness and non-emptiness implies that the final well-founded H-coalgebra I must also be nonempty. Thus I is also a coalgebra for $\bar{H}$. Let us prove that it is well-founded w.r.t. $\bar{H}$. Given a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{H}A' \arrow[d, "\bar{H}m"] \\ I \arrow[r, "\psi"'] & \bar{H}I \end{tikzcd}$$

by Lemma 2.37, $A' \neq \emptyset$. So $\bar{H}m = Hm$, hence m is invertible.

We next prove that (I, ψ) is the final well-founded $\bar{H}$-coalgebra. Let $a\colon A \longrightarrow \bar{H}A$ be a nonempty well-founded $\bar{H}$-coalgebra. We prove that the coproduct

$$(A, a) + (\bar{H}\emptyset, b) \qquad \text{in } \mathbf{Coalg}\, H$$

is a well-founded H-coalgebra. This will conclude the proof: we have a unique homomorphism from that coproduct into (I, ψ) in $\mathbf{Coalg}\, H$, hence, a unique homomorphism from (A, a) to (I, ψ). We know that every nonempty well-founded coalgebra for H is also well-founded for $\bar{H}$, thus, both of the above summands are well-founded $\bar{H}$-coalgebras. Since coproducts of coalgebras are formed on the level of sets, the two categories $\mathbf{Coalg}\, H$ and $\mathbf{Coalg}\, \bar{H}$ have the same formation of coproduct of nonempty coalgebras. Let

$$(A, a) + (\bar{H}\emptyset, b) = (A + \bar{H}\emptyset, c)$$

be a coproduct in $\mathbf{Coalg}\,\bar{H}$, then this coalgebra is well-founded w.r.t. $\bar{H}$ by Lemma 2.21. To prove that it is also well-founded w.r.t. H, we only need to consider the empty subcoalgebra: we must prove that the square

$$\begin{tikzcd}
\emptyset \arrow[r, "a'"] \arrow[d, "m"'] & H\emptyset \arrow[d, "Hm"] \\
A + \bar{H}\emptyset \arrow[r, "c"'] & H(A + \bar{H}\emptyset)
\end{tikzcd}$$

is not a pullback. Indeed, choose an element $x \in H\emptyset$ and put $\tau = u(x)$ (see (2.13)). Then $m = q_{A+\bar{H}\emptyset}$ implies

$$Hm(x) = \tau_{A+\bar{H}\emptyset}.$$

We also have $\tau \in \bar{H}\emptyset$ and the coproduct injection $v \colon \bar{H}\emptyset \longrightarrow A + \bar{H}\emptyset$ fulfils $c{\cdot}v = Hv{\cdot}b$ (due to the formation of coproducts in $\mathbf{Coalg}\,\bar{H}$). Therefore

$$c\big(v(\tau)\big) = Hv\big(b(\tau)\big) = Hv(\tau_{\bar{H}\emptyset}) = \tau_{A+\bar{H}\emptyset}.$$

Since we presented elements of $A + \bar{H}\emptyset$ and $H\emptyset$ that are mapped to the same element by c and Hm, respectively, the above square is not a pullback. This finishes the proof that (I, ψ) is a final well-founded $\bar{H}$-coalgebra.

By Theorem 2.35 we conclude that ψ is invertible and (I, ψ^{-1}) is an initial $\bar{H}$-algebra. It is also an initial H-algebra: due to $H\emptyset \neq \emptyset \neq \bar{H}\emptyset$, the two functors have the same categories of algebras. ☐

Example 2.39. Let $H = C_{01} + C_1$ be the constant functor of value 2 except $\emptyset \mapsto 1$. The functor $\bar{H}$ in the above proof is the constant functor with value $1 + 1$, expressed, say as $\{a, b\}$. Here

$$H\emptyset = \{b\} \qquad \text{and} \qquad HA = \{a, b\} \quad \text{for } A \neq \emptyset.$$

The coalgebra

$$\{a\} \hookrightarrow \{a, b\}$$

is obviously well-founded w.r.t. $\bar{H}$ but not w.r.t. H since we have the pullback:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "q_{\{a\}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & H\emptyset = \{b\} \arrow[d, "Hq_{\{a\}}"] \\
\{a\} \arrow[r, hook] & \{a, b\}
\end{tikzcd}$$

2F Initial algebras for functors on vector spaces

For every field K, the category Vec_K of vector spaces over K also has the property that the equality (2.12) holds for all endofunctors. This follows from the next lemma whose proof is a variation of Trnková's proof of Proposition 2.36 (cf. [29]):

Lemma 2.40. *All monomorphisms in* Vec_K *are strong, and their finite intersections are absolute, i.e., preserved by every functor with domain* Vec_K*.*

Corollary 2.41. *For every endofunctor of* Vec_K *we have*

$$\text{initial algebra} = \text{final well-founded coalgebra.}$$

Remark 2.42. The existence of an initial algebra is equivalent to the existence of a space $X \cong HX$, see Proposition 2.23.

3 Well-pointed coalgebras

We arrive at the centerpiece of this paper, characterizations of the initial algebra, final coalgebra, and initial iterative algebra for set functors.

Throughout this section H denotes an endofunctor of **Set** which preserves (wide) intersections. This is an extremely mild condition: examples include

(a) the power-set functor, all polynomial functors, the finite distribution functor,
(b) products, coproducts, quotients, and subfunctors of functors preserving intersections, and
(c) "almost" all finitary functors, see Lemma 3.33 below.

Recall from Remark 2.5 that a *subcoalgebra* of a coalgebra (A, a) is represented by a monic homomorphism with codomain (A, a), and a *quotient* is represented by an epic homomorphism with domain (A, a). When speaking about morphisms between pointed coalgebras we mean those preserving the points. In particular, given a pointed coalgebra $1 \xrightarrow{x} A \longrightarrow HA$ by a *subobject* is meant a subcoalgebra containing the *initial state* x.

3A Canonical graphs of coalgebras of set functors preserving wide intersections

At this point, we associate to each H-coalgebra (A, a) a certain *canonical graph*. The idea is that the neighbors of a point x in this graph are the elements of A "most immediately used" in $a(x)$. Here is the formal definition.

Definition 3.1. *For every coalgebra* $a\colon A \longrightarrow HA$ *define the* ***canonical graph*** *on* A*: the neighbors of* $x \in A$ *are precisely those elements of* A *which lie in the least subset* $m\colon M \hookrightarrow A$ *with* $a(x) \in Hm[HM]$*.*

Remark 3.2. (a) Gumm observed in [15] that (under our standing assumption that H preserves intersections) we obtain a "subnatural" transformation from it to the power-set functor $\mathscr{P}$ by defining functions

$$\tau_A\colon HA \longrightarrow \mathscr{P}A, \quad \tau_A(x) = \text{the least subset } m\colon M \hookrightarrow A \text{ with } x \in Hm[HM].$$

The naturality squares do not commute in general, but for every monomorphism $m\colon A' \hookrightarrow A$ we have a commutative square

$$\begin{tikzcd} HA' \arrow[r, "\tau_{A'}"] \arrow[d, "Hm"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}A' \arrow[d, "\mathscr{P}m"] \\ HA \arrow[r, "\tau_A"'] & \mathscr{P}A \end{tikzcd} \tag{3.1}$$

which even is a pullback. The canonical graph of a coalgebra $a\colon A \longrightarrow HA$ is simply the graph $\tau_A{\cdot}a\colon A \longrightarrow \mathscr{P}A$.

(b) Recall that a graph is well-founded iff it has no infinite directed paths. This also fully characterizes well-foundedness of H-coalgebras:

Proposition 3.3. *If a set functor H preserves intersections, then a coalgebra for H is well-founded iff its canonical graph is well-founded.*

Remark. For functors H preserving inverse images this fact is proved by Taylor, see 6.3.4 in [27]. Our proof is the same and we present it for completeness only.

Proof. Let $a\colon A \longrightarrow HA$ be a well-founded coalgebra. Given a subgraph (A', a') of the associated graph $(A, \tau_A{\cdot}a)$ forming a pullback

$$\begin{tikzcd} A' \arrow[rr, "a'"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathscr{P}A' \arrow[d, "\mathscr{P}m"] \\ A \arrow[r, "a"'] & HA \arrow[r, "\tau_A"'] & \mathscr{P}A \end{tikzcd}$$

we are to prove that m is invertible. Use the pullback of (3.1):

$$\begin{tikzcd} A' \arrow[r, "a''"] \arrow[d, "m"'] & HA' \arrow[r, "\tau_{A'}"] \arrow[d, "Hm"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}A' \arrow[d, "\mathscr{P}m"] \\ A \arrow[r, "a"'] & HA \arrow[r, "\tau_A"'] & \mathscr{P}A \end{tikzcd} \tag{3.2}$$

We get a unique $a''\colon A' \longrightarrow HA'$ with $a' = \tau_{A'}{\cdot}a$, and (A', a'') is a subcoalgebra of (A, a). Moreover, in the above diagram the outside square and the right-hand one are both pullbacks, thus, the left-hand square is also a pullback. Consequently, m is invertible since (A, a) is well-founded.

Conversely, assume that the graph $(A, \tau_A{\cdot}a)$ is well-founded. We are to prove that if the left-hand square of (3.2) is a pullback then m is invertible. Indeed, in that case, by composition, the outside square is a pullback for the subcoalgebra $(A', \tau_{A'}{\cdot}a')$ of $(A, \tau_A{\cdot}a)$. Thus, since the last coalgebra is well-founded, m is invertible. □

Corollary 3.4. *Subcoalgebras of a well-founded coalgebra are well-founded, whenever H is a set functor preserving intersections.*

Remark 3.5. If H preserve inverse images, a much stronger result holds, as Taylor proved in [27]: every coalgebra from which a homomorphism into a well-founded coalgebra exists is well-founded.

3B Well-pointedness

Definition 3.6. *A **well-pointed coalgebra** is a pointed coalgebra which has no proper subobjects and no proper quotients.*

Remark 3.7. Recall the concept of a *simple coalgebra* (called *minimal coalgebra* by Gumm [14]): it is a coalgebra (A, a) with no nontrivial quotient. That is, a coalgebra such that every homomorphism $h\colon (A, a) \longrightarrow (B, b)$ has h monic (in **Set**). Gumm observed that

(a) the full subcategory of $\mathbf{Coalg}\, H$ given by all simple coalgebras is reflective: the reflection of a coalgebra (A, a) is the simple quotient

$$e_{(A,a)}\colon (A, a) \longrightarrow (\bar{A}, \bar{a})$$

obtained as the wide pushout of all quotients of (A, a).

(b) Every subcoalgebra of a simple coalgebra is simple.

(c) The coalgebra map $a\colon A \longrightarrow HA$ of a simple coalgebra is monic.

Remark 3.8. Thus, $1 \xrightarrow{x} A \xrightarrow{a} HA$ is a well-pointed coalgebra iff (A, a) is simple and is generated by x. We call the latter condition reachability. That is, a pointed coalgebra is reachable if it has no proper pointed subcoalgebra. This can be translated to reachability of its canonical graph, see Definition 3.1:

Lemma 3.9. *A pointed coalgebra (A, a, x) is reachable iff its pointed canonical graph is, i.e., every vertex can be reached from x by a directed path.*

Proof. Recall $\tau_A\colon HA \longrightarrow \mathscr{P}A$ from Remark 3.2. For an arbitrary subcoalgebra (A', a') containing x we see that A' is a subcoalgebra of the canonical graph $(A, \tau_A{\cdot}a)$ (as a pointed coalgebra for $\mathscr{P}$):

$$\begin{tikzcd}
 & A' \arrow[r, "a'"] \arrow[d, "m"'] & HA' \arrow[r, "\tau_{A'}"] \arrow[d, "Hm"'] & \mathscr{P}A' \arrow[d, "\mathscr{P}m"] \\
1 \arrow[r, "x"'] \arrow[ur, dashed, "x'"] & A \arrow[r, "a"'] & HA \arrow[r, "\tau_A"'] & \mathscr{P}A
\end{tikzcd}$$

Conversely, if $m\colon A' \longrightarrow A$ is a subobject of the pointed canonical graph then, since the square in Remark 3.2 is a pullback, we have a unique structure $a'\colon A' \longrightarrow HA'$ of a subobject of (A, a, x). Therefore, (A, a, x) is reachable w.r.t. H iff $(A, \tau_A{\cdot}a, x)$ is reachable w.r.t. $\mathscr{P}$. It is easy to see that the latter means that every element can be reached from x by a directed path. $\square$

Examples 3.10. (a) A deterministic automaton with a given initial state is a pointed coalgebra for $HX = X^I \times \{0,1\}$. Reachability means that every state can be reached (in finitely many steps) from the initial state. Simplicity means that the automaton is *observable*, i.e., for every pair of different states there exists an input word leading one of them to an accepting state and the other to a non-accepting state. See Section 4 for more details.
The usual terminology is that reachability and observability together are called minimality.

(b) For the power-set functor the pointed coalgebras are the pointed graphs. Well-pointed means reachable and simple, where simplicity states that no pair of different vertices is bisimilar. See Section 4 for more details.

(c) Whenever a final coalgebra A exists, it is clearly simple. Every element $x \in A$ generates a subcoalgebra A_x (since H preserves intersections) which is reachable and, by Remark 3.7(b), simple. Thus, A_x is a well-pointed coalgebra. We will prove below that every well-pointed coalgebra is isomorphic to A_x for a unique x in A.

Notation 3.11. Since H preserves intersections, there is a canonical process of turning an arbitrary pointed coalgebra (A, a, x) into a well-pointed one: form the simple quotient, see Remark 3.7(a) pointed by $e_{(A,a)}{\cdot}x\colon 1 \longrightarrow \bar{A}$, then form the least subcoalgebra containing that point:

$$\begin{tikzcd}
 & & \bar{A}_0 \arrow[r, "\bar{a}_0"] \arrow[d, "m"'] & H\bar{A}_0 \arrow[d, "Hm"] \\
1 \arrow[rru, "x_0"] \arrow[r, "x"'] & A \arrow[r, "e_{(A,a)}"'] & \bar{A} \arrow[r, "\bar{a}"'] & H\bar{A}
\end{tikzcd}$$

That is, $m = m_{(A,a)}$ is the intersection of all subcoalgebras of $(\bar{A}, \bar{a})$ through which $e_{(A,a)}{\cdot}x$ factorizes. Then $(\bar{A}_0, \bar{a}_0, x_0)$ is well-pointed due to Remark 3.7(b).

Example 3.12. For deterministic automata our process $A \longmapsto \bar{A}_0$ above means that we first merge the states that are observably equivalent and then discard the states that are not reachable. A more efficient way is first discarding the unreachable states and then merging observably equivalent pairs. Both ways are possible since our functor preserves inverse images: this implies that a quotient of a reachable pointed coalgebra is reachable.

Remark 3.13. Let H preserve inverse images. Then a quotient of a reachable pointed coalgebra is reachable. Indeed, given such a quotient e and its subcoalgebra m containing the given point x, form the inverse image of m along e in **Set**:

$$\begin{tikzcd}
A' \arrow[rr, "m'"] \arrow[dd, "e'"'] & & A \arrow[dd, "e"] \\
 & 1 \arrow[ul, dashed] \arrow[ur, "x"'] \arrow[dl, "x_0"] & \\
\bar{A}_0 \arrow[rr, "m"'] & & \bar{A}
\end{tikzcd}$$

Since H preserves inverse images, $m'\colon A' \longrightarrow A$ is a subcoalgebra of A, and the universal property of pullbacks implies that A' contains the given point x. Consequently, m' is invertible, thus, $m{\cdot}e'$ is epic, therefore m is invertible.

Thus, we have an alternative procedure of forming well-pointed coalgebras from pointed ones, (A, a, x): first form the least pointed subcoalgebra (A_0, a_0, x). Then form the simple quotient of (A_0, a_0).

3C Final coalgebras

Notation 3.14. The collection of all well-pointed coalgebras up to isomorphism is denoted by

$$\nu H.$$

For every coalgebra $a\colon A \longrightarrow HA$ we have a function

$$a^+\colon A \longrightarrow \nu H$$

assigning to every element $x\colon 1 \longrightarrow A$ the well-pointed coalgebra of Notation 3.11:

$$a^+(x) = (\bar{A}_0, \bar{a}_0, x_0).$$

Theorem 3.15. *H has a final coalgebra iff it has only a set of well-pointed coalgebras up to isomorphism. And, if it is the case, νH is a final coalgebra.*

Remark. Whenever νH is a set, it carries a canonical coalgebra structure

$$\psi\colon \nu H \longrightarrow H(\nu H).$$

It assigns to every member (A, a, x) of νH the following element of $H(\nu H)$:

$$1 \xrightarrow{x} A \xrightarrow{a} HA \xrightarrow{Ha^+} H(\nu H). \tag{3.3}$$

We prove below that this is a final coalgebra.

Proof. (1) If H has a final coalgebra, then due to Remark 3.7 every simple coalgebra is its subcoalgebra, since the unique homomorphism is monic. The final coalgebra has only a set of subcoalgebras, consequently, there exists up to isomorphism only a set of simple coalgebras. Consequently, only a set of well-pointed coalgebras.

(2) Let H have a set νH of representative well-pointed coalgebras. We prove that νH with the coalgebra structure ψ from (3.3) is final.

(2a) We first prove that every coalgebra homomorphism $h\colon (A, a) \longrightarrow (B, b)$ makes the triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "a^+"'] & & B \arrow[dl, "b^+"] \\ & \nu H & \end{tikzcd}$$

commutative. Given $x\colon 1 \longrightarrow A$, then $b^+ \cdot h$ assigns to it the well-pointed coalgebra $(\bar{B}_0, \bar{b}_0, y_0)$ obtained from (B, b, y), where $y = h \cdot x$, as in Notation 3.11. It is our task to prove that this well-pointed coalgebra is isomorphic to $(\bar{A}_0, \bar{a}_0, x)$. Due to Remark 3.7(a) we have a homomorphism

$$\bar{h}\colon (\bar{A}, \bar{a}) \longrightarrow (\bar{B}, \bar{b})$$

such that the square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e_{(A,a)}"'] & B \arrow[d, "e_{(B,b)}"] \\ \bar{A} \arrow[r, "\bar{h}"'] & \bar{B} \end{tikzcd}$$

commutes. The image of $\bar{h}$ is a subcoalgebra of $(\bar{B}, \bar{b})$ containing $y = e_{(B,b)} \cdot h \cdot x$, thus, we have a homomorphism $\bar{h}_0$ as a domain-codomain restriction of $\bar{h}$:

$$\begin{tikzcd} \bar{A}_0 \arrow[r, "\bar{h}_0"] \arrow[d, "m_{(A,a)}"'] & \bar{B}_0 \arrow[d, "m_{(B,b)}"] \\ \bar{A} \arrow[r, "\bar{h}"'] & \bar{B} \end{tikzcd}$$

Moreover $\bar{h}_0$ is monic by Remark 3.7(b) and epic since the image of $\bar{h}_0$ is a subcoalgebra of $(\bar{B}, \bar{b})$ containing y. Thus, $\bar{h}_0$ is an isomorphism, as requested.

(2b) νH is a weakly final coalgebra because for every coalgebra (A, a) we have a coalgebra homomorphism

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "a^+"'] & HA \arrow[d, "Ha^+"] \\ \nu H \arrow[r, "\psi"'] & H(\nu H) \end{tikzcd}$$

Indeed, this square commutes: for every $x\colon 1 \longrightarrow A$ the lower passage yields

$$1 \xrightarrow{x_0} \bar{A}_0 \xrightarrow{\bar{a}_0} H\bar{A}_0 \xrightarrow{H\bar{a}_0^+} H(\nu H).$$

And this is precisely what the upper passage assigns to x: see the commutative diagram

$$\begin{tikzcd}
1 \arrow[r, "x_0"] \arrow[d, "x"'] & \bar{A}_0 \arrow[r, "\bar{a}_0"] \arrow[d, "m"'] & H\bar{A}_0 \arrow[r, "H\bar{a}_0^+"] \arrow[d, "Hm"'] & H(\nu H) \arrow[dddlll, bend left, "\mathrm{id}"] \\
A \arrow[r, "e"] \arrow[d, "a"'] & \bar{A} \arrow[r, "\bar{a}"] \arrow[d, "\bar{a}"] & H\bar{A} \arrow[ur, "H\bar{a}^+"] \arrow[dl, "\mathrm{id}"] & \\
HA \arrow[r, "He"] \arrow[d, "Ha^+"'] & H\bar{A} \arrow[dl, "H\bar{a}^+"'] & & \\
H(\nu H) & & &
\end{tikzcd}$$

Indeed, the upper and lower triangles commute since m and e are homomorphisms, see (2a).

(2c) We next prove that for the coalgebra $\psi\colon \nu H \longrightarrow H(\nu H)$ we have

$$\psi^+ = \mathrm{id}_{\nu H} \,. \tag{3.4}$$

Indeed, given a well-pointed coalgebra $(A, a, x) \in \nu H$, consider the triangle of (2a) with $h = a^+$ and $b = \psi$. Since, by (2b), a^+ is a homomorphism, the triangle commutes. Of course, $a^+(x) = (A, a, x)$, since (A, a) is simple and (A, a, x) is reachable. Then $\psi^+(A, a, x) = (A, a, x)$.

(2d) Finally, to prove that, for every coalgebra (A, a), a^+ is a unique homomorphism to νH, combine (2b), (2a) and the equality (3.4). $\square$

Examples 3.16. (a) For deterministic automata the final coalgebra (for $HX = X^I \times \{0, 1\}$) consists of all minimal (i. e., reachable and simple) automata. The more usual description is: the set $\mathscr{P} I^*$ of all formal languages. However, this is isomorphic: every formal language is accepted by a minimal automaton, unique up to isomorphism.

(b) The final coalgebra for the finite power-set functor is the coalgebra of all finitely branching well-pointed graphs. See Section 4 for more details.

Remark 3.17. Not every set functor H has a final coalgebra, but, as proved by Peter Aczel and Nax Mendler [2], it does have a large final coalgebra. And this, again, can be described as

$$\nu H = \text{all well-pointed coalgebras}$$

(up to isomorphism). More precisely, H has an extension $\bigcirc H$ to the category of classes (unique up to natural isomorphism). And $\bigcirc H$ was proved to have a final coalgebra in [2]. Now the above class νH has a coalgebra structure

$$\psi\colon \nu H \longrightarrow \bigcirc H(\nu H)$$

completely analogous to that of Theorem 3.15: to every well-pointed coalgebra (A, a, x) it assigns the element

$$1 \xrightarrow{x} A \xrightarrow{a} HA \xrightarrow{\bigcirc H a^+} \bigcirc H(\nu H).$$

Corollary 3.18. *For every (intersection preserving) set functor H the coalgebra νH is final. More precisely, $(\nu H, \psi)$ is a final coalgebra for $\bigcirc H$.*

Indeed, for every small coalgebra for $\bigcirc H$ (or for H) a unique homomorphism exists to $(\nu H, \psi)$, this is proved as in Theorem 3.15. And for every large coalgebra A of $\bigcirc H$ we use the fact (called Small Subcoalgebra Lemma in [2]) that A is a colimit of the diagram of all of its small subcoalgebras.

Example 3.19. The final coalgebra for $\mathscr{P}$ is the class of all well-pointed graphs. See Section 4 for the details.

Example 3.20. We present an example of a set functor H with a final coalgebra that cannot be described as the set of all well-pointed coalgebras (since H fails to preserve intersections).

Put

$$HX = X^\omega/{\sim} + 1$$

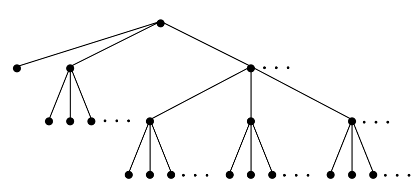
where $\sim$ merges two sequences in X iff they agree in all but finitely many coordinates. This functor does not preserve intersections: although the intersection of all the subsets $A_n = \{n, n+1, n+2, \dots\}$ of $\mathbb{N}$ is empty, H maps $A_n \hookrightarrow \mathbb{N}$ to an isomorphism for every n.

The final coalgebra for H can be described as $T/{\equiv}$, where T is the set of all countably branching trees (up to isomorphism) and $\equiv$ is the smallest equivalence merging a tree X with any subtree $Y \subseteq X$ such that for every node of Y all but finitely many X-children lie in Y. The coalgebra structure ψ is tree-tupling: to every element $[X]$ of $T/{\equiv}$, the congruence class of the countably branching tree X, it assigns the right-hand summand, 1, of $H(T/{\equiv})$, if X is root only. Otherwise X has children X_n, $n \in \mathbb{N}$, and

$$\psi\big([X]\big) = \big([X_0], [X_1], [X_2], \dots\big).$$

There are elements of the final coalgebra that do not correspond to any well-pointed coalgebra:

For example, consider the tree Y



whose n-th son Y_n has the depth n for $n = 0, 1, 2, \dots$ Then $[Y]$ lies in each of the subcoalgebras

$$\big\{[Y]\big\} \cup \big\{[Y_n], [Y_{n+1}], [Y_{n+2}], \dots\big\}.$$

Since none of them is the least one, this element $[Y]$ of the final coalgebra does not correspond to a well-pointed coalgebra.

3D Initial algebras

Notation 3.21. The collection of all well-founded, well-pointed coalgebras (up to isomorphism) is denoted by

$$\mu H.$$

For every well-founded coalgebra $a\colon A \longrightarrow HA$ we have a function

$$a^+\colon A \longrightarrow \mu H$$

assigning to every element $x\colon 1 \longrightarrow A$ the well-founded, well-pointed coalgebra of Notation 3.11:

$$a^+(x) = (\bar{A}_0, \bar{a}_0, x_0).$$

Indeed, $(\bar{A}_0, \bar{a}_0)$ is well-founded due to Lemma 2.21 and Corollary 3.4.

Theorem 3.22. *H has an initial algebra iff it has only a set of well-founded, well-pointed coalgebras up to isomorphism. And, if it is the case, μH is an initial algebra.*

Remark. Whenever μH is a set, it carries a canonical coalgebra structure

$$\bar{\psi}\colon \mu H \longrightarrow H(\mu H).$$

It assigns to every member (A, a, x) of μH the following element of $H(\mu H)$:

$$1 \xrightarrow{x} A \xrightarrow{a} HA \xrightarrow{Ha^+} H(\mu H).$$

We prove below that this is a final well-founded coalgebra. Thus, by Theorem 2.38, μH is an initial algebra with the structure given by the inverse of $\bar{\psi}$.

Proof. (1) If H has an initial algebra I, then by Theorem 2.38 this is a final well-founded coalgebra. Every well-founded, well-pointed coalgebra is simple, whence a subcoalgebra of I since the unique homomorphism into I is monomorphic by Remark 3.7. Consequently, μH is a set.

(2) Let H have a set μH of representatives of well-founded, well-pointed coalgebras. The proof that for every well-founded coalgebra (A, a) the map $a^+ : A \longrightarrow \mu H$ is a unique coalgebra homomorphism into $\bar{\psi}\colon \mu H \longrightarrow H(\mu H)$ is completely analogous to the proof of finality of $\psi\colon \nu H \longrightarrow H(\nu H)$ in Theorem 3.15. Just recall that subcoalgebras and quotients of a well-founded coalgebra are all well-founded (by Lemma 2.21 and Corollary 3.4).

It remains to prove that $(\mu H, \bar{\psi})$ is a well-founded coalgebra. To this end notice that for every well-pointed, well-founded coalgebra (A, a, x) in μH we have

$$a^+(x) = (A, a, x).$$

Now take the coproduct (in $\mathbf{Coalg}\,H$) of all (A, a) for which there is an $x \in A$ such that (A, a, x) lies in μH. This coproduct is a well-founded coalgebra by Lemma 2.21, and, as we have just seen, unique induced homomorphism from the coproduct into $(\mu H, \bar{\psi})$ is epimorphic, whence μH is a quotient coalgebra of the coproduct. Thus, another application of Lemma 2.21 shows that it is a well-founded coalgebra as desired. $\square$

Remark 3.23. Analogously to Corollary 3.18, every set functor H has a, possibly large, initial algebra. That is, the extension $\bigcirc H$ of H to classes always has an initial algebra: denote, again,

$$\mu H = \text{all well-founded, well-pointed algebras}$$

(up to isomorphism). Then this is a subcoalgebra of νH of Remark 3.17. And as an algebra for $\bigcirc H$ it is initial:

Corollary 3.24. *For every (intersection preserving) set functor H the large coalgebra μH is the final well-founded coalgebra for $\bigcirc H$. Thus, the large initial algebra is μH w.r.t. the inverse of $\bar{\psi}$.*

The first statement follows from the Small Subcoalgebra Lemma of [2] and the fact that subcoalgebras of well-founded coalgebras are well-founded (Corollary 3.4). The second statement is proved precisely as Theorem 2.38.

Example 3.25. The initial algebra for $\mathscr{P}$ consists of all well-pointed graphs.

3E Initial iterative algebras

Remark 3.26. We know, from Theorems 2.38 and 3.22, that μH has a double role: an initial algebra and a final well-founded coalgebra. Also νH has a double role. Recall from [17] that an algebra $a\colon HA \longrightarrow A$ is *completely iterative* if for every (equation) morphism $e\colon X \longrightarrow HX + A$ there exists a unique solution, i.e., a unique morphism $e^\dagger\colon X \longrightarrow A$ such that the square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\
HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[a,A]}"']
\end{tikzcd}$$

commutes.

Theorem 3.27 (see [17]). *For every endofunctor*

$$\textit{final coalgebra} = \textit{initial completely iterative algebra.}$$

Remark 3.28. (a) Let H be a *finitary* set functor, i.e., every element $x \in HA$ lies, for some finite subset $m\colon A' \longrightarrow A$ in the image of Hm. Then an algebra $a\colon HA \longrightarrow A$ is called *iterative* provided that for every equation morphism $e\colon X \longrightarrow HX + A$ with X finite, there exists a unique solution $e^\dagger\colon X \longrightarrow A$.

This concept was studied for classical Σ-algebras by Nelson [19] and Tiurin [28], and for H-algebras in general in [6].

(b) Form the colimit C, in **Set**, of the diagram of all finite coalgebras $a\colon A \longrightarrow HA$ with the colimit cocone $a^+\colon A \longrightarrow C$. Then there exists a unique morphism $c\colon C \longrightarrow HC$ with $c{\cdot}a^+ = Ha^+{\cdot}a$. It was proved in [6] that c is invertible and the resulting algebra is the initial iterative algebra for H.

Example 3.29 (see [6]). (a) The initial iterative algebra of $HX = X^I \times \{0,1\}$ consists of all finite minimal automata. This is isomorphic to its description as all regular languages.

(b) The initial iterative algebra of the finite power-set functor consists of all finite well-pointed graphs. See Section 4 for a description using rational trees.

Definition 3.30 (see [18]). *A coalgebra is called **locally finite** if every element lies in a finite subcoalgebra.*

Theorem 3.31 (see [18]). *Let H be a finitary set functor. Then*

$$\textit{initial iterative algebra} = \textit{final locally finite coalgebra.}$$

Moreover, the final locally finite coalgebra is the colimit of all finite coalgebras in **Coalg** H*.*

Remark 3.32. We prove below that given a finitary set functor, the set of all finite well-pointed coalgebras forms the initial iterative algebra. For this result we do not need to assume (as in the rest of this section) that the functor preserves intersections. This can be deduced from the following

Lemma 3.33. *For every finitary set functor H there exists a functor $\bar{H}$ preserving (wide) intersections that agrees with H on all nonempty sets and functions.*

Proof. The functor $\bar{H}$ of Proposition 2.36 is obviously also finitary. It preserves finite intersections, and we deduce that it preserves all intersections. Given subobjects $m_i\colon A_i \longrightarrow B$ $(i \in I)$ with an intersection $m\colon A \longrightarrow B$, let $x \in \bar{H}B$ lie in the image of each Hm_i; we are to prove that x lies in the image of $\bar{H}m$. Choose a subset $n\colon C \longrightarrow B$ of the smallest (finite) cardinality with x lying in the image of $\bar{H}n$. Since $\bar{H}$ preserves the intersection of n and m_i, the simplicity of C guarantees that $n \subseteq m_i$ (for every $i \in I$). Thus, $n \subseteq m$, proving that x lies in the image of $\bar{H}m$. ◻

Notation 3.34. For every finitary set functor denote by

$$\varrho H$$

the set of all finite well-pointed coalgebras up to isomorphism.

Given a finite coalgebra $a\colon A \longrightarrow HA$ we define a function

$$a^+\colon A \longrightarrow \varrho H$$

by assigning to every element $x\colon 1 \longrightarrow A$ the well-pointed coalgebra of Notation 3.11:

$$a^+(x) = (\bar{A}_0, \bar{a}_0, x_0).$$

This is well-defined due to Lemma 2.21 and Corollary 3.4 since H and $\bar{H}$ have the same pointed coalgebras.

Theorem 3.35. *Every finitary set functor H has the initial iterative algebra ϱH.*

Remark. ϱH has the canonical coalgebra structure

$$\bigcirc\psi\colon \varrho H \longrightarrow H(\varrho H).$$

It assigns to every element (A, a, x) of ϱH the following element of $H(\varrho H)$:

$$1 \xrightarrow{x} A \xrightarrow{a} HA \xrightarrow{Ha^+} H(\varrho H).$$

We prove below that this is the final locally finite coalgebra. Thus, ϱH is the initial iterative algebra w.r.t. the inverse of $\bigcirc\psi$, by 3.31.

Proof. Analogously to the proof of Theorem 3.15 one verifies that the morphisms

$$a^+\colon (A, a) \longrightarrow (\varrho H, \bigcirc\psi) \qquad (A \text{ finite})$$

are coalgebra homomorphisms forming a cocone. By Remark 3.28(b) it remains to prove that this is a colimit in $\mathbf{Coalg}\, H$, we only need verifying that all a^+'s form a colimit cocone in $\mathbf{Set}$. That is:

(i) Every element of ϱH has the form $a^+(x)$ for some finite coalgebra (A, a) and some $x \in A$. Indeed, for every element (A, a, x) of ϱH we have $a^+(x) = (A, a, x)$.

(ii) Whenever

$$a^+(x) = b^+(y)$$

holds for two finite coalgebras (A, a) and (B, b) and for elements $x \in A$, $y \in B$ (turning them into pointed coalgebras), then there exists a zig-zag of homomorphism of finite pointed coalgebras connecting (A, a, x) with (B, b, y). For that recall $a^+(x) = (\bar{A}_0, \bar{a}_0, x_0)$ in the notation 3.11. Here is the desired zig-zag based on $a^+(x) = b^{\#}(y)$:

$$\begin{tikzcd}
(A,a,x) \arrow[dr, "e_{(A,a)}"'] & & (\bar{A}_0, \bar{a}_0, x_0) \arrow[dl, "m_{(A,a)}"] \arrow[dr, "e_{(B,b)}"'] & & (B,b,y) \arrow[dl, "m_{(B,b)}"] \\
& (\bar{A}, \bar{a}, e_{(A,a)}\cdot x) & & (\bar{B}, \bar{b}, e_{(B,b)}\cdot y) &
\end{tikzcd}$$

□

Remark 3.36. For non-finitary set functors H the set ϱH also carries the above structure of a coalgebra. But this is in general not a fixed point of H. For example the functor $HX = X^{\mathbb{N}} + 1$ has the final coalgebra consisting of all countably branching trees (see Corollary 4.27). And ϱH is the set of all rational trees, i.e., those having only finitely many subtrees (up to isomorphism), see Example 4.29. This is a subcoalgebra of the final coalgebra, but not a fixed point of H.

4 Examples of well-pointed coalgebras

For a number of important set functors H we are going to apply the results of Section 3 and compare them to the well-known description of the three fixed points of interest: the final coalgebra, the initial algebra, and the initial iterative algebra (= final locally finite coalgebra). Throughout this section pointed coalgebras are considered up to (point-preserving) isomorphism. Recall that

$$\nu H = \text{all well-pointed coalgebras}$$
$$\mu H = \text{all well-founded well-pointed coalgebras}$$

and in case H is a finitary functor

$$\varrho H = \text{all finite well-pointed coalgebras.}$$

We are using various types of labeled trees throughout this section. Trees, too, are considered up to (label-preserving) isomorphism. Unless explicitly stated, trees are ordered, i.e., a linear ordering on the children of every node is always given.

In all our examples the endofunctors H used preserve intersections and weak pullbacks. Recall from Rutten [21] that this implies that

(a) congruences on a coalgebra A are precisely the kernel equivalences of homomorphisms $f\colon A \longrightarrow A$

and

(b) for every coalgebra the largest congruence is precisely the bisimilarity equivalence.
Also recall that, for these functors, every pointed coalgebra yields a well-pointed one by first forming "reachable part" and then simplifying (since the preservation of strong monomorphisms and weak pullbacks implies the preservation of inverse images, see Remark 3.13).

In pictures of pointed coalgebras the choice of the point q_0 is depicted by

$$\hookrightarrow q_0$$

4A Moore automata

Given a set I of inputs and a set J of outputs, a *Moore automaton* on a set Q (of states) is given by a next-state function $\delta\colon Q \times I \longrightarrow Q$ curried as

$$\operatorname{curry}\delta\colon Q \longrightarrow Q^I$$

an output function

$$\mathrm{out}\colon Q \longrightarrow J$$

and an initial state $q_0 \in Q$. The first two items form a coalgebra of

$$HX = X^I \times J,$$

thus we work with pointed coalgebras of this functor, with q_0 as the chosen point. The *behavior* of an autmaton is the function

$$\beta\colon I^* \longrightarrow J$$

which to every input word $w \in I^*$ assigns the output of the state reached from q_0 by applying the inputs in w. A function $\beta\colon I^* \longrightarrow J$ is called *regular* if the set of all functions $\beta(w-)\colon I^* \longrightarrow J$ for $w \in I^*$ is finite.

Lemma 4.1. *The largest congruence on a Moore automaton merges states q and q' iff by applying an arbitrary finite sequence of inputs to each of them, we obtain states with the same output.*

This is well-known and easy to prove. Automata satisfying this condition are called *simple*. Another well-known fact is the following

Theorem 4.2. *For every function $\beta\colon I^* \longrightarrow J$ there exists a reachable and simple Moore automaton with the behavior β. This automaton is unique up to isomorphism. It is finite iff β is regular.*

Corollary 4.3. *For Moore automata, $HX = X^I \times J$, we have*

$$\begin{aligned}
&\nu H \cong J^{I^*}, \quad \textit{all functions } \beta\colon I^* \longrightarrow J;\\
&\varrho H \cong \textit{all regular functions } \beta\colon I^* \longrightarrow J;\\
&\mu H = \emptyset.
\end{aligned}$$

The coalgebra structure of νH (and ϱH) assigns to every $\beta\colon I^* \longrightarrow J$ the pair in $(\nu H)^I \times J$ consisting of the function $i \longmapsto \beta(i-)$ for $i \in I$ and the element $\beta(\varepsilon)$ of J.

Indeed, the isomorphism between νH, the set of all reachable and simple automata, and J^{I^*} is given by the above theorem. And the structure map ψ of Theorem 3.15 is easily seen to correspond to the above map taking β to $(i \longmapsto \beta(i-), \beta(\varepsilon))$. Also the isomorphism of ϱH and regular functions follows from the above theorem, and from Theorem 3.35 we know that ϱH is a subcoalgebra of νH.

Finally, $\mu H = \emptyset$ since no well-pointed coalgebra (A, a) is well-founded due to the cartesian subcoalgebra

$$\begin{tikzcd}
\emptyset \arrow[r, "\mathrm{id}"] \arrow[d, "m"'] & H\emptyset = \emptyset \arrow[d, "Hm"] \\
A \arrow[r, "a"'] & A^I \times J
\end{tikzcd}$$

Example 4.4. In case $J = \{0, 1\}$ we get $\nu H = \mathscr{P} I^*$ and $\varrho H =$ regular languages, see Examples 3.16(a) and 3.29.

4B Mealy automata

For *Mealy automata* the next-state function has the form $\delta\colon Q \times I \longrightarrow Q \times J$ and in curried form this is a coalgebra

$$HX = (X \times J)^I.$$

Given a state q of a Mealy automaton Q, its *response function* f_q is the function $f_q\colon I^\omega \longrightarrow J^\omega$ assigning to an infinite word of input symbols the infinite word of output symbols (delayed by one time unit) of the inputs given by the transitions as the computations of the inputs are performed, starting in q. Observe that f_q is a *causal function*: for every infinite word w the n-th component of $f_q(w)$ depends only on the first n components of w.

Remark 4.5. Given a causal function $f\colon I^\omega \longrightarrow J^\omega$ the above property with $n = 0$ tells us that the component 0 of $f(w)$ only depends on w_0. We thus obtain a derived function

$$f^0\colon I \longrightarrow J$$

with $f(iw) = f^0(i)w'$ (for convenient w') for all $w \in I^\omega$.

Lemma 4.6. *For every Mealy automaton the largest congruence merges precisely the pairs of states with the same response function.*

Proof. Let Q be a Mealy automaton, then the above equivalence $q \sim q'$ iff $f_q = f_{q'}$ is obviously a congruence. We have a structure of a Mealy automaton $\bar{\delta}$ on $Q/{\sim}$ derived from that of Q: Given a state $[q] \in Q/{\sim}$ and an input $i \in I$, the pair $\delta(q, i) = (q', j)$ yields $\bar{\delta}([q], i) = ([q'], j)$. It is easy to verify that the canonical map $c\colon Q \longrightarrow Q/{\sim}$ is a coalgebra homomorphism $c\colon (Q, \delta, q_0) \longrightarrow (Q/{\sim}, \bar{\delta}, [q_0])$. Conversely, every congruence is contained in $\sim$ because given a coalgebra homomorphism $h\colon Q \longrightarrow \bar{Q}$ then for every state $q \in Q$ we have $f_q = f_{h(q)}$. Thus, the kernel congruence of h is contained in $\sim$. $\square$

Corollary 4.7. *The well-pointed Mealy automata are precisely those with an initial state q_0 such that the automaton is*

(a) *reachable: every state can be reached from q_0*

and

(b) *simple: different states have different response functions.*

The automata satisfying (a) and (b) together are called "minimal". The following theorem can be found in Eilenberg's Volume A, see XII.4.1 in [11]:

Theorem 4.8. *For every causal function f there exists a unique well-pointed coalgebra whose initial state has the response function f.*

Remark 4.9. Eilemberg also proves that a minimal Mealy automaton is finite iff f has the property that the set of all functions $f(w-)$ where $w \in I^*$ is finite. Let us call such causal functions *regular*.

Corollary 4.10. *For Mealy automata, $HX = (X \times J)^I$, we have*

$$\begin{aligned} &\nu H \cong \textit{all causal functions from } I^\omega \textit{ to } J^\omega \\ &\varrho H \cong \textit{all regular causal functions} \\ &\mu H = \emptyset. \end{aligned}$$

The coalgebra structure ψ of νH (and the one $\bigcirc\psi$ of ϱH) assigns to every causal function $f\colon I^\omega \longrightarrow J^\omega$ the map

$$I \longrightarrow \nu H \times J, \qquad i \longmapsto \big(f(i-), f^0(i)\big)$$

for $f_0\colon I \longrightarrow J$ in Remark 4.5.

Indeed, the first two statements below follow from the above theorem and the last one follows again from $H\emptyset = \emptyset$. The above description of the final coalgebra is due to Rutten [22]. Samuel Eilenberg works with functions $f\colon I^* \longrightarrow J^*$ preserving length and prefixes, but it is immediate that these are just another way of coding all causal functions between infinite streams.

Remark 4.11. An alternative description of the final coalgebra for $HX = (X \times J)^I$ is:

$$\nu H \cong J^{I^+}, \qquad \text{all functions } \beta\colon I^+ \longrightarrow J.$$

Here and below, l^+ is the set of finite non-empty words on the set l. The coalgebra structure assigns to every β the mapping from I to $\nu H \times J$ given by

$$i \longmapsto \big(\beta(i-), \beta(i)\big) \qquad \text{for } i \in I.$$

Indeed, this coalgebra is isomorphic to that of all causal functions $f\colon I^\omega \longrightarrow J^\omega$: to every function $\beta\colon I^+ \longrightarrow J$ assign the causal function $f(i_0 i_1 i_2 \dots) = (\beta(i_0), \beta(i_0 i_1), \beta(i_0 i_1 i_2), \dots)$.

4C Streams

Consider the coalgebras for

$$HX = X \times I + 1.$$

Jan Rutten [21] interprets them as dynamical systems with outputs in I and with final states (where no next state is given). Every state q yields a stream, finite or infinite, over I by starting in q and traversing the dynamic system as long as possible. We call it the *response* of q. It is an element of $I^\omega + I^*$.

Lemma 4.12. *For a dynamic system the largest congruence merges two states iff they yield the same response.*

Proof. Let $\sim$ be the equivalence from the statement of the lemma. Then we have an obvious dynamic system on $Q/\sim$, thus, $\sim$ is a congruence. Every coalgebra homomorphism $h\colon Q \longrightarrow \bar{Q}$ fulfils: the response of q and $h(q)$ is always the same. Therefore, $\sim$ is the largest congruence. $\square$

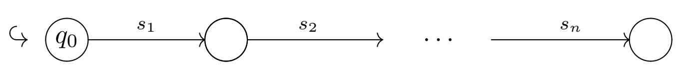
Corollary 4.13. *A well-pointed coalgebra is a dynamic system with an initial state q_0 such that the system is*

(a) *reachable: every state can be reached from q_0*

and

(b) *simple: different states yield different responses.*

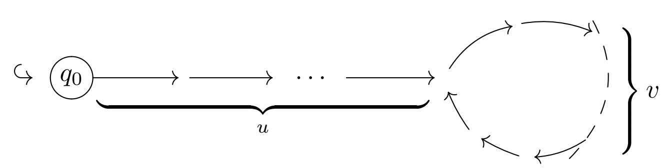
Example 4.14. (a) For every word $s_1 \dots s_n$ in I^* we have a well-founded dynamic system



(b) For every *eventually periodic* stream in I^ω,

$$w = uv^\omega \qquad \text{for } u, v \in I^*,$$

we have a pointed dynamic system



If we choose, for the given stream w, the words u and w of minimum length, then this system is well-pointed.

The following was already proved by Arbib and Manes, [16], 10.2.5.

Corollary 4.15. *For $HX = X \times I + 1$ we have*

$$\nu H \cong I^* + I^\omega, \quad \textit{all finite and infinite streams,}$$
$$\varrho H \cong \textit{all finite and eventually periodic streams,}$$
$$\mu H \cong I^*, \quad \textit{all finite streams.}$$

The coalgebra structure assigns to every nonempty stream w the pair

$$(\operatorname{tail} w, \operatorname{head} w) \qquad \textit{in} \quad \nu H \times I$$

and to the empty stream the right-hand summand of $H(\nu H) = \nu H \times I + 1$.

Indeed, the first statement follows from Corollary 4.13 since by forming the response of q_0 we get a bijection between well-pointed coalgebras and streams in $I^* + I^\omega$. For the second statement observe that a well-pointed system yields a finite or eventually periodic response iff it has finitely many states. The last statement follows from the observation that a dynamic system is well-founded iff every run of a state is finite. Indeed, given a coalgebra $a \colon A \longrightarrow A \times I + 1$, form the subset $m \colon A' \longrightarrow A$ of all states with finite runs. We obtain a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, hook, "m"'] & A' \times I + 1 \arrow[d, hook, "m \times I + 1"] \\ A \arrow[r, "a"'] & A \times I + 1 \end{tikzcd}$$

Indeed, whenever a state $q \in A$ has the property that q is final or the next state lies in A', then q lies in A'. Thus, well-founded, well-pointed coalgebras are precisely those of Example 4.14(a).

4D Binary trees

Coalgebras for the functor

$$HX = X \times X + 1$$

are given, as observed by Jan Rutten [21], by a set Q of states which are either final or have precisely two next states according to a binary input, say $\{l, r\}$. Every state $q \in Q$ yields an ordered binary tree T_q (i.e, nodes that are not leaves have a left-hand child and a right-hand one) by *tree expansion*: the root is q and a node is either a leaf, if it is a final state, or has the two next states as children (left-hand for input l, right-hand for input r). Binary trees are considered up to isomorphism.

Lemma 4.16. *For every system the largest congruence merges precisely the pairs of states having the same tree expansion.*

Proof. Let $\sim$ be the equivalence with $q \sim q'$ iff $T_q = T_{q'}$. There is an obvious structure of a coalgebra on $Q/\!\sim$ shoving that $\sim$ is a congruence. For every coalgebra homomorphism $h\colon Q \longrightarrow \bar{Q}$ the tree expansion of $q \in Q$ is always the same as the tree expansion of $h(q)$ in $\bar{Q}$. Thus $\sim$ is the largest congruence. $\square$

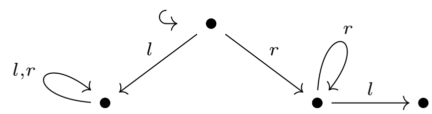
Corollary 4.17. *A well-pointed system is a system with an initial state q_0 which is*

(a) *reachable: every state can be reached from q_0*

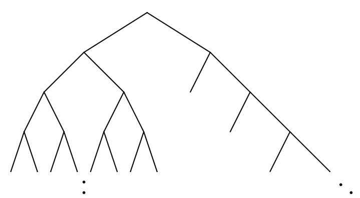
and

(b) *simple: different states have different tree expansions.*

Moreover, tree expansion is a bijection between well-pointed coalgebras and binary trees (see Proposition 4.25 below). For instance, the dynamic system



defines the tree



Observe that this tree has only 4 subtrees (up to isomorphism): this follows from the fact that the dynamic systems has 4 states. In general, the finite dynamic systems correspond to the *rational trees*, i.e., trees having (up to isomorphism) only finitely many subtrees. This concept is due to Ginali [12].

Corollary 4.18. *For the functor $HX = X \times X + 1$ we have*

$$\begin{aligned} \nu H &\cong \textit{all binary trees}, \\ \varrho H &\cong \textit{all rational binary trees}, \\ \mu H &\cong \textit{all finite binary trees}. \end{aligned}$$

The coalgebra structure is, in each case, the inverse of tree tupling: it assigns to the root-only tree the right-hand summand of $\nu H \times \nu H + 1$ and to any other tree the pair of its maximum subtrees.

Indeed, we only need to explain the last item. Given a coalgebra $a\colon A \longrightarrow A \times A + 1$, let $m\colon A' \hookrightarrow A$ be the set of all states defining a finite subtree. This is a cartesian subcoalgebra

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, hook, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \times A' + 1 \arrow[d, hook, "m \times m + \mathrm{id}"] \\ A \arrow[r, "a"'] & A \times A + 1 \end{tikzcd}$$

For this square is a pullback: whenever a state $q \in A$ has both next states in A' or whenever q is final, then $q \in A'$. Thus, if A is well-founded, then $A = A'$. The converse implication is easy: recall the subsets A_i^* of Notation 2.17. Here A_i^* is the set of all states whose binary tree has depth at most i. Thus, if $A = A_i$ for some i, the initial state defines a tree of depth at most i.

4E Σ-Algebras and Σ-coalgebras

All the examples above (and a number of other interesting cases) are subsumed in the following general case. Let Σ be a signature, i.e., a set of operation symbols with given arities $\mathrm{ar}(s)$ of symbols $s \in \Sigma$; the arity is a (possibly infinite) cardinal. The classical Σ-algebras are the algebras of the corresponding *polynomial functor*

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\mathrm{ar}(\sigma)}.$$

Coalgebras for H_Σ are called *Σ-coalgebras.*

Example 4.19. Deterministic automata $HX = X^I \times \{0,1\} = X^I + X^I$ are given by two $|I|$-ary operations. Streams, $HX = X \times I + 1$, are given by $|I|$ operations of arity 1 and a constant. Binary trees $HX = X \times X + 1$ are given by one binary operation and one constant.

Definition 4.20. *A **Σ-tree** is an ordered tree with nodes labeled in Σ so that every node with n children has a label of arity n. We consider Σ-trees up to isomorphism.*

Observe that every Σ-tree T is a coalgebra: the function $a\colon T \longrightarrow H_\Sigma T$ takes every node x labelled by a symbol $\sigma \in \Sigma$ (of arity n) to the n-tuple $(x_i)_{i<n}$ of its children, an element of the σ-summand T^n of $H_\Sigma T$.

In general a Σ-coalgebra $a\colon Q \longrightarrow H_\Sigma Q$ can be viewed as a system with a state set Q labeled in Σ:

$$\bar{a}\colon Q \longrightarrow \Sigma$$

and such that every state $q \in Q$ with n-ary label has "next states" forming an n-tuple

$$\bigcirc a(q) \in Q^n.$$

Indeed, to give a function $a\colon Q \longrightarrow H_\Sigma Q$ means precisely to given a pair $(\bar{a}, \bigcirc a)$ of functions as above.

Definition 4.21. *Let $a\colon Q \longrightarrow H_\Sigma Q$ be a coalgebra.*

(a) *A **computation** of length n is a word $i_0 \cdots i_{n-1}$ in $\mathbb{N}^*$ for which there are states $q_0, \cdots, q_n$ in Q with*

$$q_{k+1} = \textit{the } i_k\textit{-component of } \bigcirc a(q_k) \qquad (k = 0, \ldots, n-1).$$

(b) *The **tree expansion** of a state q is the Σ-tree*

$$T_q$$

of all computations with initial state q. The label of a computation is $\bar{a}(q_n)$, where q_n is its last state. And the children are all one-step extensions of that computation, i.e., all words $i_0 \dots i_{n-1} j$ with $j \leq \mathrm{ar}(\bar{a}(q_n))$.

Lemma 4.22. *The greatest congruence on a Σ-coalgebra merges precisely the pairs of states with the same tree expansion.*

Proof. Let $(Q, \bar{a}, \bigcirc a)$ be a Σ-coalgebra and put $q \sim q'$ iff $T_q = T_{q'}$. Then we have a coalgebra structure on $Q/{\sim}$: the label of $[q]$ is $\bar{a}(q)$, independent of the representation. The next-state n-tuple is $([q_i])_{i<n}$ where $\bigcirc a(q) = (q_i)$. It is easy to see that this is independent of the choice of representatives. And the quotient map is a coalgebra homomorphism from Q to $Q/{\sim}$. Thus, $\sim$ is a congruence.

To prove that Σ is the largest congruence, observe that given a coalgebra homomorphism $h \colon Q \longrightarrow Q'$, then for every state $q \in Q$ we have $T_q = T_{h(q)}$. Indeed, an isomorphism $i \colon T_q \longrightarrow T_{h(q)}$ is easy to define by induction on the depth of nodes of T_q. □

Corollary 4.23. *Well-pointed Σ-coalgebras are the Σ-coalgebras with an initial state q_0 which are*

(a) *reachable: every state can be reached from q_0 by a computation*

and

(b) *simple: different states have different tree expansions.*

Example 4.24. For every Σ-tree T the equivalence on the nodes of T given by

$$x \sim y \qquad \text{iff} \qquad T_x \cong T_y \tag{4.1}$$

where T_x is the subtree rooted at node x, is a congruence. And $T/{\sim}$ carries an obvious structure of a Σ-coalgebra. Let $[r]$ be the congruence class of the root, then the pointed Σ-coalgebra $(T/{\sim}, [r])$ is well-pointed.

Indeed, this pointed coalgebra is reachable: given a node q of T let $i_0 \cdots i_{n-1}$ be the unique path from r to q, then $i_0 \cdots i_{n-1}$ is a computation in $T/{\sim}$ with initial state $[r]$ and terminal state $[q_n]$.

The simplicity of $T/{\sim}$ follows from Lemma 4.22 and the observation that the tree expansion of a state $[q]$ of $T/{\sim}$ is the subtree T_q of T.

These are all well-pointed Σ-coalgebras. Moreover:

Proposition 4.25. *Every well-pointed coalgebra is isomorphic to $(T/{\sim}, [r])$ for a unique Σ-tree T.*

Proof. Let (Q, a, q_0) be a well-pointed Σ-coalgebra.

(a) Existence. Let T denote the tree expansion of q_0. For the above equivalence (4.1) we prove that two equivalent computations always terminate in the same state. This follows from the simplicity of (Q, a): Denote by $\approx$ the equivalence with $q \approx q'$ iff there exist two equivalent (under $\sim$) computations with terminal states q and q' (respectively). This is clearly a congruence on (Q, a), so $q = q'$. We thus obtain a function

$$t\colon T/{\sim} \longrightarrow Q, \qquad [i_0 \cdots i_{n-1}] \longmapsto \text{terminal state of } i_0 \cdots i_{n-1}.$$

It is easy to see that this is a coalgebra homomorphism. Since it takes $[q_0]$ to q_0, it is surjective (use the reachability). And it is an isomorphism since two computations p and p' with the same last state q_n fulfil $T_p = T_{p'}$. (Indeed, the subtree of T at the node p is precisely the tree T_{q_n}). Hence, if t merges the equivalence classes $[p]$ and $[p']$, then $p \sim p'$.

(b) Uniqueness. This follows from the observation that two Σ-trees T and T' are isomorphic whenever the coalgebras of Example 4.24 are. Indeed, given an isomorphism $f\colon (T/{\sim}, [r]) \longrightarrow (T'/{\sim}, [r'])$, define an isomorphism $g\colon T \longrightarrow T'$ from top down. Since f preserves labels, r and r' are labeled by the same n-ary label. We put $g(r) = r'$. Let $\bigcirc a(r) = (x_i)_{i<n}$ and $\widetilde{a'}(r') = (x'_i)_{i<n}$. Since f preserves $\bigcirc a$ we have $f[x_i] = [x'_i]$ for all i. We define g on level 1 by $g(x_i) = x'_i$, $i < n$, and proceed recursively. $\square$

Proposition 4.26. *A Σ-coalgebra is well-founded iff all its tree-expansions are* ***well-founded Σ-trees****, i.e., Σ-trees with no infinite path.*

Proof. Given a Σ-coalgebra A let $m\colon A' \hookrightarrow A$ be the subset of all states $q \in A$ with T_q well-founded. This is, obviously, a subcoalgebra. And it is cartesian

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, hook, "m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & H_\Sigma A' \arrow[d, hook, "H_\Sigma m"] \\ A \arrow[r, "a"'] & H_\Sigma A \end{tikzcd}$$

Indeed, if a state q has the property that all components of $\bigcirc a$ lies in A', the q lies in A'. Thus A is well-founded iff $A = A'$. $\square$

Corollary 4.27. *For every signature Σ we have*

$$\nu H_\Sigma \cong \textit{all } \Sigma\textit{-trees}$$

and

$$\mu H_\Sigma \cong \textit{all well-founded } \Sigma\textit{-trees.}$$

The coalgebra structure is in each case inverse to tree tupling.

Indeed, the isomorphism between νH_Σ and all Σ-trees is given by Proposition 4.25. And the coalgebra structure of Theorem 3.15 corresponds to the inverse of tree-tupling, i.e., it assigns to a Σ-tree T with $\bigcirc a(r) = (x_1, \ldots, x_n)$ the n-tuple $(T_{x_1}, \ldots, T_{x_n})$ in the σ-summand of $H_\Sigma(\nu H_\Sigma)$ where σ is the label of the root.

Definition 4.28 (see [12]). *A Σ-tree is called **rational** if it has up to isomorphism only finitely many subtrees.*

Example 4.29. Given a finite Σ-coalgebra, all tree expansions of its states are rational.

Indeed, if $Q = \{q_1, \ldots, q_n\}$ is the state set, then every subtree of T_{q_i} (given by a computation with initial state q_i) has the form T_{q_j}: take q_j to be the terminal state of the computation.

Corollary 4.30. *For every finitary signature Σ we have*

$$\begin{aligned}
\nu H_\Sigma &\cong \textit{all } \Sigma\textit{-trees} \\
\varrho H_\Sigma &\cong \textit{all rational } \Sigma\textit{-trees} \\
\mu H_\Sigma &\cong \textit{all finite } \Sigma\textit{-trees.}
\end{aligned}$$

The coalgebra structure is inverse to tree-tupling.

Indeed, the isomorphism between ϱH_Σ (all finite well-pointed coalgebras) and rational Σ-trees follows from Proposition 4.25 and Example 4.29. The last item follows from König's Lemma: every well-founded finitely branching tree is finite.

Example 4.31. For the functor $HX = X^*$ we can use nonlabeled trees: we have

$$\begin{aligned}
\nu H &= \text{all finitely branching trees} \\
\varrho H &= \text{all rational finitely branching trees} \\
\mu H &= \text{all finite trees.}
\end{aligned}$$

Indeed, let Σ be the signature with one n-ary operation for every $n \in \mathbb{N}$. Then $H_\Sigma X \cong X^*$. And Σ-trees need no labeling, since operations already differ by arities.

Example 4.32. All previous examples 4A–4D are special cases of Corollary 4.30:

(a) Moore automata. The functor

$$HX = X^I \times J$$

corresponds to a signature Σ of $|J|$ operations, all of arity $|I|$. Since no nullary operation is given, every Σ-tree is a complete I-ary tree labeled by J. Now the usual representation of the complete I-ary tree is by I^*: the root is the empty word and the children of $i_1 \cdots i_n$ are all $i_1 \cdots i_n j$ for $i \in I$. Thus, a complete binary tree labeled by J is nothing else than a function from I^* to J:

$$\nu H = J^{I^*} \qquad \text{(see Corollary 4.3).}$$

The subcoalgebra ϱH is then given by the regular functions. And since we have no leaves,

$$\mu H = \emptyset$$

because a well-founded tree always has leaves.

(b) Mealy automata. The functor

$$HX = (X \times J)^I = X^I \times J^I$$

corresponds to J^I operations, all of arity I. Thus, a Σ-tree is a function from I^* to J^I. Or, by uncurrying, a function from $I^* \times I = I^+$ to J:

$$\nu H = J^{I^+} \qquad \text{(see Remark 4.11).}$$

(c) Streams. The functor

$$HX = X \times I + 1$$

corresponds to one constant and I unary operations. A Σ-tree is then a labeled tree consisting either of a path of length $n \in \mathbb{N}$, or an infinite path. The nodes are labeled by I. Thus,

$$\nu H \cong 1 + I + I^2 + \cdots + I^\omega \qquad \text{(see Corollary 4.15).}$$

(d) Binary trees. Here

$$HX = X \times X + 1$$

is given by a binary operation and a constant. Thus

$$\nu H = \text{binary trees} \qquad \text{(see Corollary 4.18).}$$

4F Graphs

Here we investigate coalgebras for the power-set functor $\mathscr{P}$. In the rest of Section 4 all trees are understood to be non-ordered. That is, a tree is a directed graph with a node (root) from which every node can be reached by a unique path.

Recall the concept of a *bisimulation* between graphs X and Y: it is a relation $R \subseteq X \times Y$ such that whenever $x \mathrel{R} y$ then every child of x is related to a child of y, and vice versa. Two nodes of a graph X are called *bisimilar* if they are related by a bisimulation $R \subseteq X \times X$.

Lemma 4.33. *The greatest congruence on a graph merges precisely the bisimilar pairs of states.*

This follows, since $\mathscr{P}$ preserves weak pullbacks, from general results of Rutten [21].

Corollary 4.34. *A pointed graph (G, q_0) is well-pointed iff it is*

(a) *reachable: every vertex can be reached from q_0 by a directed path*

and

(b) *simple: all distinct pairs of states are non-bisimilar.*

Example 4.35. Peter Aczel introduced in [1] the *canonical picture* of a (well-founded) set X. It is the graph with vertices all sets Y such that a sequence

$$Y = Y_0 \in Y_1 \in \cdots \in Y_n = X$$

of sets exists. The neighbours of a vertex Y are all of its elements. When pointed by X, this is a well-pointed graph. Indeed, reachability is clear. And suppose R is a bisimulation and $Y\ R\, Y'$, then we prove $Y = Y'$. Assuming the contrary, there exists $Z_0 \in Y$ with $Z_0 \notin Y'$, or vice versa. Since R is a bisimulation, from $Z_0 \in Y$ we deduce that $Z_0' \in Y'$ exists with $Z_0\ R\ Z_0'$. Clearly $Z_0 \neq Z_0'$. Thus, we substitute (Z_0, Z_0') by (Y, Y') and obtain $Z_1 \in Z_0$ and $Z_1' \in Z_0'$ with $Z_1\ R\ Z_1'$ but $Z_1 \neq Z_1'$ etc. This is a contradiction to the well-foundedness of X: we get an infinite sequence Z_n with

$$\cdots\ Z_2 \in Z_1 \in Z_0 \in Y.$$

Here are some concrete examples of canonical pictures and their corresponding tree expansions (cf. Remark 4.36 below):

Set: Canonical picture: Tree expansion

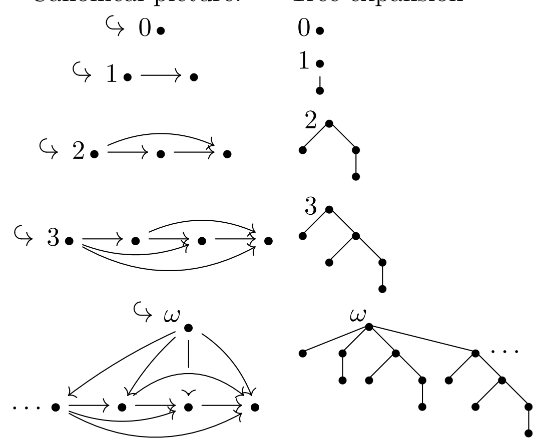
$0 = \emptyset$

$1 = \{0\}$

$2 = \{0, 1\}$

$3 = \{0, 1, 2\}$

ω



Remark 4.36. Given a vertex q of a graph, its *tree expansion* is (similarly to the ordered case, see Definition 4.21) the non-ordered tree

$$T_q$$

whose nodes are all finite directed paths from q.

The children of a node p are all one-step extensions of the path p. The root is q (considered as the path of length 0).

For every pointed graph (G, x) the *tree expansion* is the tree T_x. In the previous example we saw tree expansions of the given pointed graphs.

Definition 4.37 (James Worrell [31]). *By a* ***tree-bisimulation*** *between trees* T_1 *and* T_2 *is meant a graph bisimulation* $R \subseteq T_1 \times T_2$ *which*

(a) *relates the roots*

and

(b) $x_1 \; R \; x_2$ *implies that* x_1 *and* x_2 *are the roots or have related parents.*

A tree T is called ***strongly extensional*** *iff every tree bisimulation $R \subseteq T \times T$ is trivial: $D \subseteq \Delta_T$.*

Example 4.38. The tree expansion of a well-pointed graph (G, q_0) is strongly extensional. Indeed, given a tree bisimulation $R \subseteq T_{q_0} \times T_{q_0}$, we obtain a graph bisimulation $\bar{R} \subseteq G \times G$ consisting of all pairs (q_1, q_2) of vertices for which paths p_i from q_0 to q_i exist, $i = 1, 2$, with $p_1 \; R \; p_2$. Since G is minimal, $\bar{R} \subseteq \Delta$. Thus, for all pairs of paths:

$$\text{if } p_1 \; R \; p_2 \text{ then the last vertices of } p_2 \text{ and } p_1 \text{ are equal.}$$

We prove $p_1 \; R \; p_2$ implies $p_1 = p_2$ by induction on the maximum k of the lengths of p_1 and p_2. For $k = 0$ we have $p_1 = q_0 = p_2$. For $k+1$ we have $p_1' \; R \; p_2'$ where p_i' is the trimming of p_i by one edge (since R is a tree bisimulation). Then $p_1' = p_2'$ implies $p_1 = p_2$ because the last vertices are equal.

Furthermore, there are no other extensional trees:

Proposition 4.39. *Every strongly extensional tree is the tree expansion of a unique (up to isomorphism) well-pointed graph.*

Proof. Let T be a strongly extensional tree with root r.

(a) Existence. The coalgebra $(T/{\sim}, [r])$ where $\sim$ merges bisimilar vertices of T (looking at T as a coalgebra) is well-pointed by Lemma 4.33. Its tree expansion $T' = (T/{\sim})_{[r]}$ is (isomorphic to) the given tree T. Indeed, the relation $R \subseteq T \times T'$ of all pairs (x, p) where x is a node of T and p is the equivalence class of the unique path from r to x is clearly a tree bisimulation. Since $\mathscr{P}$ preserves weak pullbacks, it follows that the composite $R \circ R^{-1}$ of R and R^{-1} is also a tree bisimulation. But T is strongly extensional, thus $R \circ R^{-1} \subseteq \Delta$. Also T' is strongly extensional, see Example 4.38, thus $R^{-1} \circ R \subseteq \Delta$. We conclude that R is (the graph of) an isomorphism from T to T'.

(b) Uniqueness: If well-pointed graphs (G, q_0) and (G', q_0') have isomorphic tree expansions, then they are isomorphic. Arguing analogously to (a) we only need to find a graph bisimulation $R \subseteq G \times G'$ and use the simplicity of G and G'. For that, we just observe that there is a graph bisimulation between (G, q_0) and T_{q_0}: the relation $R \subseteq G \times T_{q_0}$ of all pairs (q, p) where $q \in G$ is the last vertex of the path p from q_0 to q. $\square$

Corollary 4.40. $\nu\mathscr{P}$ *= all strongly extensional trees.*

We must be careful here: $\mathscr{P}$ has no fixed points. But recall the extension of set functors to classes in Remark 3.17. For $\mathscr{P}$ this is the functor $\bigcirc\mathscr{P} = \{A; A \text{ is a set with } A \subseteq X\}$. Its (large) final coalgebra is the coalgebra of all strongly extensional trees.

Notation 4.41. Let $\mathscr{P}_\lambda$ be the subfunctor of all subsets of cardinality less than λ. (Thus $\mathscr{P}_\omega$ is the finite power-set functor.) Then by precisely the same argument as above one proves

Corollary 4.42. *For every cardinal λ*

$$\nu\mathscr{P}_\lambda = \textit{all } \lambda\textit{-branching strongly extensional trees.}$$

This was proved for $\lambda = \omega$ by Worrell [31] and for general λ by Schwencke [24]. Our proof is entirely different.

Which graphs are well-founded? This was answered by Osius [20]: precisely the graphs without an infinite directed path.

Now strong extensionality can, in the case of well-founded trees, be simplified to *extensionality* which says that for every node different children define non-isomorphic subtrees. Thus we get

Corollary 4.43. $\mu\mathscr{P}$ = *all well-founded, extensional trees. For every infinite cardinal λ*

$$\mu\mathscr{P}_\lambda = \textit{all } \lambda\textit{-branching, well-founded, extensional trees.}$$

Analogously to Example 4.29 the fixed point $\varrho\mathscr{P}_\omega$ consists of all rational strongly extensional trees, i.e., those with finitely many subtrees up to isomorphism:

Corollary 4.44. *For the finite power-set functor $\mathscr{P}_\omega$ we have*

$$\nu\mathscr{P}_\omega = \textit{all finitely branching, strongly extensional trees,}$$
$$\varrho\mathscr{P}_\omega = \textit{all finitely branching, rational, strongly extensional trees,}$$

and

$$\mu\mathscr{P}_\omega = \textit{all finite extensional trees.}$$

4G Non-well-founded sets

We revisit $\mu\mathscr{P}$ and $\nu\mathscr{P}$ here from a set-theoretic perspective. Before coming to the non-well-founded sets, let us observe that Example 4.35 has the following strengthening:

Lemma 4.45. *Well-founded, well-pointed graphs are precisely the canonical pictures of well-founded sets.*

This follows from the observation of Peter Aczel [1] that every well-pointed graph G has a unique *decoration*, i.e., coalgebra homomorphism d to the class **Set** of sets with $\in$ as the neighbourhood relation. That is, d assigns to every vertex x a set $d(x)$ so that

$$d(x) = \big\{d(y); y \in G \text{ a neighbour of } x\big\}.$$

Observe that the kernel of d is clearly a congruence on G. Thus, given a well-pointed, well-founded graph (G, q_0), we know from Remark 3.7 that d is monic. From that it follows that the canonical picture of the set $d(q_0)$ is isomorphic to (G, q_0).

Corollary 4.46. *$\mu\mathscr{P}$ = all sets.*

This was proved by Rutten and Turi in [23]. The bijection between well-founded, well-pointed graphs and sets (given by the canonical picture) takes the finite graphs to the *hereditary finite sets* X, i.e., finite sets with finite elements which also have finite elements, etc. More precisely: a set is hereditary finite if all sets in the canonical picture of X are finite:

Corollary 4.47. *$\mu\mathscr{P}_\omega$ = all hereditary finite sets.*

In order to describe the final coalgebra for $\mathscr{P}$ in a similar set-theoretic manner, we must move from the classical theory to the non-well-founded set theory of Peter Aczel [1]. Recall that a **decoration** of a graph is a coalgebra homomorphism from this graph into the large coalgebra $(\mathbf{Set}, \in)$. Non-well-founded set theory is obtained by swapping the axiom of foundation, telling us that $(\mathbf{Set}, \in)$ is well-founded, with the following

Anti-foundation axiom. Every graph has a unique decoration.

Example 4.48. The decoration of a single loop is a set Ω such that $\Omega = \{\Omega\}$.

The coalgebra $(\mathbf{Set}, \in)$ where now **Set** is the class of all non-well-founded sets, is of course final: the decoration of G is the unique homomorphism $d\colon G \longrightarrow \mathbf{Set}$.

Corollary 4.49. *In the non-well-founded set theory*

$$\nu\mathscr{P} = \textit{all sets.}$$

Let us turn to the finite power-set functor $\mathscr{P}_\omega$. Its final coalgebra consists of all sets whose canonical picture is finitely branching. They are called 1-*hereditary finite*, notation $HF^1[\emptyset]$, in the monograph of Barwise and Moss [8]. The rational fixed point of $\mathscr{P}_\omega$ consists of all sets whose canonical picture is finite, these are called 1/2-*hereditary finite*, notation $HF^{1/2}[\emptyset]$. For well-founded sets (with canonical picture well-founded) both are equivalent to hereditary finite above.

Corollary 4.50. *In the non-well-founded set theory*

$$\nu\mathscr{P}_\omega = HF^1[\emptyset], \quad \textit{the 1-hereditary finite sets,}$$
$$\varrho\mathscr{P}_\omega = HF^{1/2}[\emptyset], \quad \textit{the 1/2-hereditary finite sets,}$$

and

$$\mu\mathscr{P}_\omega = \textit{the well-founded, hereditary finite sets.}$$

4H Labeled transition systems

Here we consider, for a set A of actions, coalgebras for $\mathscr{P}(-\times A)$. A *bisimulation* between two labeled transition systems (LTS) G and G' is a relation $R \subseteq G \times G'$ such that

$$\text{if} \quad x\,R\,x' \quad \text{then for every transition } x \xrightarrow{a} x' \text{ in } G \text{ there exists a transition } y \xrightarrow{a} y' \text{ with } x'\;R\;y' \text{, and vice versa.}$$

States x, y of an LTS are called *bisimilar* if $x\ R\ y$ for some bisimulation $R \subseteq G \times G$.

Lemma 4.51. *For every LTS the greatest congruence merges precisely the bisimilar pairs of states.*

This, again, follows from general results of Rutten [21] since $\mathscr{P}(-\times A)$ preserves weak pullbacks.

Corollary 4.52. *A well-pointed LTS is an LTS together with an initial state q_0 which is*

(a) *reachable: every state can be reached from q_0*

and

(b) *simple: distinct states are non-bisimilar.*

The *tree expansion* of a state q is a (non-ordered) tree with edges labeled in A, shortly, an A-labeled tree. For A-labeled trees we modify Definition 4.37 and speak about *tree bisimulation* if a bisimulation $R \subseteq T_1 \times T_2$ also fulfils (a)–(c) of Definition 4.37. An A-labeled tree T is *strongly extensional* iff every tree bisimulation $R \subseteq T \times T$ is trivial.

Proposition 4.53. *Tree expansion is a bijection between well-pointed LTS and strongly extensional A-labeled trees.*

The proof is analogous to that of Proposition 4.39. Also the rest is analogous to the case of $\mathscr{P}$ above:

Corollary 4.54. $\nu\mathscr{P}(-\times A) =$ *all strongly extensional A-labeled trees, and, for every cardinal λ the λ-branching LTS have the final coalgebra*

$$\nu\mathscr{P}_\lambda(-\times A) = \textit{all } \lambda\textit{-branching, strongly extensional } A\textit{-labeled trees.}$$

Corollary 4.55. *For the finitely branching LTS we have*

$$\begin{aligned}
\nu\mathscr{P}_\omega(-\times A) &= \textit{all finitely branching, strongly extensional } A\textit{-labeled trees}\\
\varrho\mathscr{P}_\omega(-\times A) &= \textit{all rational, finitely branching strongly extensional}\\
&\quad\ A\textit{-labeled trees}\\
\mu\mathscr{P}_\omega(-\times A) &= \textit{all finite extensional } A\textit{-labeled trees.}
\end{aligned}$$

5 Conclusion and future work

For set functors H satisfying the (mild) assumption of preservation of intersections we described (a) the final coalgebra as the set of all well-pointed coalgebras, (b) the initial algebra as the set of all well-pointed coalgebras that are well-founded, and (c) in the case where H is finitary, the initial iterative algebra as the set of all finite well-pointed coalgebras. This is based on the observation that given an element of a final coalgebra, the subcoalgebra it generates has no proper subcoalgebras nor proper quotients—shortly, this subcoalgebra is well-pointed. And different elements define nonisomorphic well-pointed subcoalgebras. We then combined this with our result that for all set functors the initial algebra is precisely the final well-founded coalgebra. This resulted in the above description of the initial algebra. Numerous examples demonstrate that this view of final coalgebras and initial algebras is useful in applications.

Whereas our result about well-founded coalgebras was proved in locally finitely presentable categories, the description of the final coalgebra was formulated for set functors only. In future research we intend to generalize this result to a wider class of base categories.

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