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LOGICS FROM QUANTUM COMPUTATION

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The theory of logical gates in quantum computation has suggested new forms of quantum logic, called *quantum computational logics*. The basic semantic idea is the following: the meaning of a sentence is identified with a *quregister* (a system of *qubits*) or, more generally, with a *mixture* of quregisters (called *qumix*). In this framework, any sentence α of the language gives rise to a *quantum tree*: a kind of *quantum circuit* that transforms the quregister (qumix) associated to the atomic subformulas of α into the quregister (qumix) associated to α. A variant of the quantum computational semantics is represented by the *quantum holistic semantics*, which permits us to represent *entangled meanings*. Physical models of quantum computational logics can be built by means of Mach-Zehnder interferometers.

Keywords: quantum computation; quantum logic.

1. Introduction

The theory of logical gates in quantum computation has suggested new forms of quantum logic that have been called *quantum computational logics*[1]. The main difference between orthodox quantum logic (first proposed by Birkhoff and von Neumann[2]) and quantum computational logics concerns a basic semantic question: how to represent the *meanings* of the sentences of a given language? The answer given by Birkhoff and von Neumann is the following: the meanings of the elementary experimental sentences of quantum theory have to be regarded as determined by convenient sets of *states* of quantum objects. Since these sets should satisfy some special closure conditions, it turns out that, in the framework of orthodox quantum logic, sentences can be adequately interpreted as *closed subspaces* of the Hilbert space associated to the physical systems under investigation[3]. The answer given in the framework of quantum computational logics is quite different. The *meaning* of a sentence is identified with a quantum information quantity: a *qubit* or a *quregister* (a system of qubits) or, more generally, a *mixture* of quregisters (briefly, a *qumix*)[7].

2. Qubits, quregisters and qumixs

We will first sum up some basic concepts of quantum computation that are used in the framework of quantum computational logics. Consider the two-dimensional Hilbert space $\mathbb{C}^2$ (where any vector $|\psi\rangle$ is represented by a pair of complex numbers). Let $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$, where $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$.

Definition 1: (Qubit).
A *qubit* is a unit vector $|\psi\rangle$ of the Hilbert space $\mathbb{C}^2$.

Recalling the Born rule, any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (with $|c_0|^2 + |c_1|^2 = 1$) can be regarded as an *uncertain piece of information*, where the answer *NO* has probability $|c_0|^2$, while the answer *YES* has probability $|c_1|^2$. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively. From a semantic point of view, they can be also regarded as the classical truth-values *Falsity* and *Truth*.

An n-qubit system (also called n*-quregister*) is represented by a unit vector in the n-fold tensor product Hilbert space $\otimes^n\mathbb{C}^2 := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$ (where $\otimes^1\mathbb{C}^2 := \mathbb{C}^2$). We will use $x, y, \ldots$ as variables ranging over the set $\{0,1\}$. At the same time, $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ of the space $\otimes^n\mathbb{C}^2$ will be called an n*-configuration* (which can be regarded as a quantum realization of a classical bit sequence of length n). Instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ we will also write $|x_1, \ldots, x_n\rangle$. Recall that the dimension of $\otimes^n\mathbb{C}^2$ is 2^n, while the set of all n-configurations $\mathcal{B}^{(n)} = \{|x_1, \ldots, x_n\rangle : x_i \in \{0,1\}\}$ is an orthonormal basis for the space $\otimes^n\mathbb{C}^2$. We will call this set a *computational basis* for the n-quregisters. Since any string $x_1, \ldots, x_n$ represents a natural number $j \in [0, 2^n - 1]$ (where $j = 2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + x_n$), any unit vector of $\otimes^n\mathbb{C}^2$ can be briefly expressed in the following form: $\sum_{j=0}^{2^n-1} c_j \, \|j\rangle\rangle$, where $c_j \in \mathbb{C}$, $\|j\rangle\rangle$ is the n-configuration corresponding to the number j and $\sum_{j=0}^{2^n-1} |c_j|^2 = 1$.

Consider now the two following sets of natural numbers:

$$C_1^{(n)} := \{i : \|i\rangle\rangle = |x_1, \ldots, x_n\rangle \textit{ and } x_n = 1\}$$

and

$$C_0^{(n)} := \{i : \|i\rangle\rangle = |x_1, \ldots, x_n\rangle \textit{ and } x_n = 0\}.$$

Let us refer to a generic unit vector of the space $\otimes^n \mathbb{C}^2$:

$$|\psi\rangle = \sum_{i=0}^{2^n-1} a_i \, \|i\rangle\rangle.$$

We obtain:

$$|\psi\rangle = \sum_{i \in C_0^{(n)}} a_i \, \|i\rangle\rangle + \sum_{j \in C_1^{(n)}} a_j \, \|j\rangle\rangle.$$

Let $P_1^{(n)}$ and $P_0^{(n)}$ be the projections onto the span of $\left\{ \|i\rangle\rangle \mid i \in C_1^{(n)} \right\}$ and $\left\{ \|i\rangle\rangle \mid i \in C_0^{(n)} \right\}$, respectively. Clearly, $P_1^{(n)} + P_0^{(n)} = I^{(n)}$, where $I^{(n)}$ is the identity operator of $\otimes^n \mathbb{C}^2$. Apparently, $P_1^{(n)}$ and $P_0^{(n)}$ are density operators iff $n = 1$. Let $k_n = \frac{1}{2^{n-1}}$ be the normalization coefficient such that $k_n P_1^{(n)}$ and $k_n P_0^{(n)}$ are density operators. From an intuitive point of view, the projection $P_1^{(n)}$ and $P_0^{(n)}$ can be regarded as the mathematical representatives of the *Truth-property* and of the *Falsity-property* in the space $\otimes^n \mathbb{C}^2$. At the same time, the density operator $k_n P_1^{(n)}$ represents a privileged information corresponding to the *Truth*, while $k_n P_0^{(n)}$ corresponds to the *Falsity*. In particular, $P_1^{(1)}$ represents the bit $|1\rangle$, while $P_0^{(1)}$ represents the bit $|0\rangle$. Let $\mathfrak{D}(\otimes^n \mathbb{C}^2)$ be the set of all density operators of $\otimes^n \mathbb{C}^2$ and let $\mathfrak{D} := \bigcup_{n=1}^{\infty} \mathfrak{D}(\otimes^n \mathbb{C}^2)$.

Definition 2: (Qumix).
A *qumix* is a density operator in $\mathfrak{D}$.

Needless to say, quregisters correspond to particular qumixs that are *pure states* (i.e. projections onto one-dimensional closed subspaces of a given $\otimes^n \mathbb{C}^2$). Recalling the Born rule, we can now define the *probability-value* of any qumix.

Definition 3: (Probability of a qumix).
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$: $\mathtt{p}(\rho) = \mathtt{tr}(P_1^{(n)} \rho)$.

From an intuitive point of view, $\mathtt{p}(\rho)$ represents the probability that the information stocked by the qumix ρ is true. In the particular case where ρ corresponds to the qubit

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle,$$

we obtain that $\mathtt{p}(\rho) = |c_1|^2$.

For any quregister $|\psi\rangle$, we will write $\mathtt{p}(|\psi\rangle)$ instead of $\mathtt{p}(P_{|\psi\rangle})$, where $P_{|\psi\rangle}$ (also indicated by $|\psi\rangle\langle\psi|$) is the density operator represented by the projection onto the one-dimensional subspace spanned by the vector $|\psi\rangle$.

3. Quantum Gates

In quantum computation, *quantum logical gates* (briefly, *gates*) are unitary operators that transform quregisters into quregisters. Being unitary, gates represent characteristic *reversible transformations*. The canonical gates (which are studied in the literature) can be naturally generalized to qumixs. Generally, gates correspond to some basic *logical operations* that admit a reversible behaviour. We will consider here the following gates: the *negation*, the *Petri-Toffoli gate*[4,5] (also called *controlled-controlled-not gate*), the *controlled-not gate*, the *square root of the negation*, the *square root of the identity*. All these gates turn out to be definable in terms of a unique gate, the *controlled-controlled-blur*.

Let us first describe our gates in the framework of quregisters.

Definition 4: (The negation).
For any $n \geq 1$, the negation on $\otimes^n \mathbb{C}^2$ is the linear operator $\mathtt{Not}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\mathtt{Not}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In other words, $\mathtt{Not}^{(n)}$ inverts the value of the last element of any basis-vector of $\otimes^n \mathbb{C}^2$.

Clearly:

$$\mathtt{Not}^{(n)} = \begin{cases} X, & \textit{if } n = 1; \\ I^{(n-1)} \otimes X, & \textit{otherwise}, \end{cases}$$

where X is the "first" Pauli matrix, i.e.,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 5: (The Petri-Toffoli gate).
For any $n \geq 1$ and any $m \geq 1$ the Petri-Toffoli gate is the linear operator $T^{(n,m,1)}$ defined on $\otimes^{n+m+1}\mathbb{C}^2$ such that for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle$ of the computational basis $\mathcal{B}^{(n+m+1)}$:

$$T^{(n,m,1)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle) = |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |x_n y_m \oplus z\rangle,$$

where $\oplus$ represents the sum modulo 2.

Clearly:

$$T^{(n,m,1)} = (I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(1)} + P_1^{(n)} \otimes P_1^{(m)} \otimes X.$$

One can easily show that both $\mathtt{Not}^{(n)}$ and $T^{(n,m,1)}$ are unitary operators.

Consider now the set $\mathfrak{R} = \bigcup_{n=1}^{\infty} \otimes^n \mathbb{C}^2$ (which contains all quregisters $|\psi\rangle$ "living" in $\otimes^n \mathbb{C}^2$, for an $n \geq 1$). The gates $\mathtt{Not}$ and T can be uniformly defined on this set in the expected way:

$$\begin{array}{ll} \mathtt{Not}(|\psi\rangle) := \mathtt{Not}^{(n)}(|\psi\rangle), & \textit{if } |\psi\rangle \in \otimes^n \mathbb{C}^2 \\ T(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle) := T^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle), & \textit{if } |\psi\rangle \in \otimes^n \mathbb{C}^2, |\varphi\rangle \in \otimes^m \mathbb{C}^2 \textit{ and } |\chi\rangle \in \mathbb{C}^2. \end{array}$$

On this basis, a conjunction `And`, a disjunction `Or`, an exclusive disjunction `Xor` can be defined for any pair of quregisters $|\psi\rangle$ and $|\varphi\rangle$:

$$\texttt{And}(|\psi\rangle, |\varphi\rangle) := T(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle);$$

$$\texttt{Or}(|\psi\rangle, |\varphi\rangle) := \texttt{Not}(\texttt{And}(\texttt{Not}(|\psi\rangle), \texttt{Not}(|\varphi\rangle)));$$

$$\texttt{Xor}(|\psi\rangle, |\varphi\rangle) := \texttt{Or}(\texttt{And}(|\psi\rangle, \texttt{Not}(|\varphi\rangle)), \texttt{And}(\texttt{Not}(|\psi\rangle), |\varphi\rangle)).$$

Clearly, $|0\rangle$ represents an "ancilla" in the definition of `And`.

One can easily verify that, when applied to classical bits, `Not`, `And`, `Or` and `Xor` behave as the standard Boolean truth-functions.

At first sight, `And`, `Or` and `Xor` may look as *irreversible transformations*. However, in this framework, $\texttt{And}(|\psi\rangle, |\varphi\rangle)$ should be regarded as a mere metalinguistic abbreviation for $T(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)$ (where T is reversible). A similar observation holds for `Or` and `Xor`.

The definition we have given for the `Xor` gate (which is also called the *controlled-not gate*) refers to a Hilbert whose dimension is at least 2^7. The following more economical definition refers to a Hilbert space whose dimension is at least 2^2.

Definition 6: (The controlled-not gate).
For any $n \geq 1$ and any $m \geq 1$ the controlled-not gate is the linear operator $\texttt{Xor}^{(n,m)}$ defined on $\otimes^{n+m}\mathbb{C}^2$ such that for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle$ of the computational basis $\mathcal{B}^{(n+m)}$:

$$\texttt{Xor}^{(n,m)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle) = |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_{m-1}\rangle \otimes |x_n \oplus y_m\rangle,$$

where $\oplus$ represents the sum modulo 2.

Clearly:

$$\texttt{Xor}^{(n,m)} = P_0^{(n)} \otimes I^{(m)} + P_1^{(n)} \otimes \texttt{Not}^{(m)}.$$

The gate `Xor` can be uniformly defined in the expected way:

$$\texttt{Xor}(|\psi\rangle \otimes |\varphi\rangle) := \texttt{Xor}^{(n,m)}(|\psi\rangle \otimes |\varphi\rangle) \quad \textit{if } |\psi\rangle \in \otimes^n\mathbb{C}^2 \textit{ and } |\varphi\rangle \in \otimes^m\mathbb{C}^2.$$

The quantum logical gates we have considered so far are, in a sense, "semi-classical". A quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, such operators turn out to behave as classical (reversible) truth-functions. We will now consider two important *genuine quantum gates* that transform classical registers (elements of $\mathcal{B}^{(n)}$) into quregisters that are superpositions: the *square root of the negation* and the *square root of the identity*.

Definition 7: (The square root of the negation).
For any $n \geq 1$, the square root of the negation on $\otimes^n\mathbb{C}^2$ is the linear operator $\sqrt{\texttt{Not}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\texttt{Not}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{2}((1+i)|x_n\rangle + (1-i)|1-x_n\rangle),$$

where i is the imaginary unit.

One can easily show that $\sqrt{\mathtt{Not}}^{(n)}$ is a unitary operator. The basic property of $\sqrt{\mathtt{Not}}^{(n)}$ is the following:

$$\textit{for any } |\psi\rangle \in \otimes^n \mathbb{C}^2,\ \sqrt{\mathtt{Not}}^{(n)}(\sqrt{\mathtt{Not}}^{(n)}(|\psi\rangle)) = \mathtt{Not}^{(n)}(|\psi\rangle).$$

In other words, applying twice the square root of the negation means negating.

Clearly:

$$\sqrt{\mathtt{Not}}^{(n)} = \begin{cases} M, & \textit{if } n = 1; \\ I^{(n-1)} \otimes M, & \textit{otherwise}, \end{cases}$$

where

$$M := \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Interestingly enough, the gate $\sqrt{\mathtt{Not}}$ has some natural physical models and implementations. As an example, consider an idealized atom with a single electron and two energy levels: a *ground state* (identified with $|0\rangle$) and an *excited state* (identified with $|1\rangle$). By shining a pulse of light of appropriate intensity, duration and wavelength, it is possible to force the electron to change energy level. As a consequence, the state (bit) $|0\rangle$ is transformed into the state (bit) $|1\rangle$, and vice versa:

$$|0\rangle \mapsto |1\rangle;\ |1\rangle \mapsto |0\rangle.$$

We have obtained a typical physical model for the gate $\mathtt{Not}$. Now, by using a light pulse of half the duration as the one needed to perform the $\mathtt{Not}$ operation, we effect a half-flip between the two logical states. The state of the atom after the half pulse is neither $|0\rangle$ nor $|1\rangle$, but rather a superposition of both states.

In Sec. 9 we will see another physical model for the gate $\sqrt{\mathtt{Not}}$ as a particular 50:50 beam splitter in a Mach-Zehnder interferometer.

As observed by Deutsch, Ekert, Lupacchini[6]:

> Logicians are now entitled to propose a new logical operation $\sqrt{\mathtt{Not}}$. Why? Because a faithful physical model for it exists in nature.

Interestingly enough, the gate $\sqrt{\mathtt{Not}}$ seems to have also some linguistic "models". For instance, consider the French language. Put:

$$\sqrt{\mathtt{Not}} = \textit{"ne"} = \textit{"pas"}.$$

We obtain:

$$\sqrt{\mathtt{Not}}\sqrt{\mathtt{Not}} = \textit{"ne...pas"} = \mathtt{Not}.$$

Needless to observe, our linguistic example is only a partial model of the gate $\sqrt{\mathtt{Not}}$. In French, neither the expression "il ne pleut" nor the expression "il pleut pas" are grammatically correct sentences. And in the spoken language "il pleut pas" is simply used as an abbreviation for the correct "il ne pleut pas". In quantum computation, instead, for any quregister $|\psi\rangle$, the vector $\sqrt{\mathtt{Not}}(|\psi\rangle)$ is a quregister that is essentially different from the quregister $\mathtt{Not}(|\psi\rangle)$.

From a logical point of view, $\sqrt{\mathtt{Not}}^{(n)}$ can be regarded as a "tentative partial negation" (a kind of "half negation") that transforms *precise pieces of information* into *maximally uncertain* ones. For, we have:

$$\mathtt{p}(\sqrt{\mathtt{Not}}^{(1)}(|1\rangle)) = \frac{1}{2} = \mathtt{p}(\sqrt{\mathtt{Not}}^{(1)}(|0\rangle)).$$

As expected, the square root of the negation has no Boolean counterpart.

Lemma 8: *There is no function* $f : \{0,1\} \to \{0,1\}$ *such that for any* $x \in \{0,1\}$: $f(f(x)) = 1 - x$.

Proof: Suppose, by contradiction, that such a function f exists. Two cases are possible: (i) $f(0) = 0$; (ii) $f(0) = 1$.
(i) By hypothesis, $f(0) = 0$. Thus, $1 = f(f(0)) = f(0) = 0$, contradiction.
(ii) By hypothesis, $f(0) = 1$. Thus, $1 = f(f(0)) = f(1)$. Hence, $f(0) = f(1)$. Therefore, $1 = f(f(0)) = f(f(1)) = 0$, contradiction. □

Interestingly enough, $\sqrt{\mathtt{Not}}$ also does not have a continuous fuzzy counterpart.

Lemma 9: *There is no continuous function* $f : [0,1] \to [0,1]$ *such that for any* $x \in [0,1]$: $f(f(x)) = 1 - x$.

Proof: Suppose, by contradiction, that such a function f exists. First, we prove that $f(\frac{1}{2}) = \frac{1}{2}$. By hypothesis, $f(f(\frac{1}{2})) = 1 - \frac{1}{2} = \frac{1}{2}$. Hence, $f(f(f(\frac{1}{2}))) = f(\frac{1}{2})$. Thus, $1 - f(\frac{1}{2}) = f(\frac{1}{2})$. Therefore, $f(\frac{1}{2}) = \frac{1}{2}$. Consider now $f(0)$. One can easily show: $f(0) \neq 0$ and $f(0) \neq 1$. Clearly, $f(0) \neq \frac{1}{2}$ since otherwise we would obtain $1 = f(f(0)) = f(\frac{1}{2}) = \frac{1}{2}$. Thus, only two cases are possible: (i) $0 < f(0) < \frac{1}{2}$; (ii) $\frac{1}{2} < f(0) < 1$.
(i) By hypothesis, $0 < f(0) < \frac{1}{2} < 1 = f(f(0))$. Consequently, by continuity, $\exists x \in (0, f(0))$ such that $\frac{1}{2} = f(x)$. Accordingly, $\frac{1}{2} = f(\frac{1}{2}) = f(f(x)) = 1 - x$. Hence, $x = \frac{1}{2}$, which contradicts $x < f(0) < \frac{1}{2}$.
(ii) By hypothesis, $f(\frac{1}{2}) = \frac{1}{2} < f(0) < 1 = f(f(0))$. By continuity, $\exists x \in (\frac{1}{2}, f(0))$ such that $f(x) = f(0)$. Thus, $1 - x = f(f(x)) = f(f(0)) = 1$. Hence, $x = 0$, which contradicts $x > \frac{1}{2}$. □

Definition 10: (The square root of the identity).
For any $n \geq 1$, the square root of the identity on $\otimes^n \mathbb{C}^2$ is the linear operator $\sqrt{\mathtt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

The basic property of $\sqrt{\mathtt{I}}^{(n)}$ is the following:

$$\textit{for any } |\psi\rangle \in \otimes^n\mathbb{C}^2,\ \sqrt{\mathtt{I}}^{(n)}(\sqrt{\mathtt{I}}^{(n)}(|\psi\rangle)) = |\psi\rangle.$$

Clearly:

$$\sqrt{\mathtt{I}}^{(n)} = \begin{cases} H, & \textit{if } n = 1; \\ I^{(n-1)} \otimes H, & \textit{otherwise}, \end{cases}$$

where H is the Hadamard matrix:

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

As happens in the case of $\sqrt{\mathtt{Not}}^{(n)}$, also $\sqrt{\mathtt{I}}^{(n)}$ can be regarded as a "tentative partial assertion" (a kind of "half assertion") that transforms *precise pieces of information* into *maximally uncertain* ones. Apparently, one application of $\sqrt{\mathtt{I}}^{(n)}$ to a precise information produces a *maximal disorder*, while two applications of $\sqrt{\mathtt{I}}^{(n)}$ lead back to the initial information.

As expected, also the gates $\sqrt{\mathtt{Not}}$ and $\sqrt{\mathtt{I}}$ can be uniformly defined on the set $\mathfrak{R}$ of all quregisters.

An interesting gate is represented by the *controlled-controlled-blur*. One is dealing with a quite strong operator that permits us to define all the gates we have considered so far.

Definition 11: (The controlled-controlled-blur gate).
For any $n \geq 1$, $m \geq 1$ and $t \geq 1$ the controlled-controlled-blur gate is the linear operator $\mathtt{CCBlur}^{(n,m,t)}$ defined on $\otimes^{n+m+t}\mathbb{C}^2$ such that for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_t\rangle$ of the computational basis $\mathcal{B}^{(n+m+t)}$:

$$\begin{aligned}
&\mathtt{CCBlur}^{(n,m,t)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_t\rangle) \\
&= |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_{t-1}\rangle \\
&\otimes \Big(\Big((1 - x_n y_m)\, \tfrac{(-1)^{z_t}}{\sqrt{2}} + x_n y_m \tfrac{1+i}{2}\Big) |z_t\rangle + \Big((1 - x_n y_m)\, \tfrac{1}{\sqrt{2}} + x_n y_m \tfrac{1-i}{2}\Big) |1 - z_t\rangle \Big).
\end{aligned}$$

Apparently, $\mathtt{CCBlur}$ is a genuine quantum logical gate, which behaves as a kind of *fuzzyfier* operator that *blurs* any bit-information according to some parameters.

In the case of $n = m = t = 1$, we obtain:

$$\begin{aligned}
\mathtt{CCBlur}^{(1,1,1)}(|000\rangle) &= |0\rangle \otimes |0\rangle \otimes \sqrt{\mathtt{I}}(|0\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|001\rangle) &= |0\rangle \otimes |0\rangle \otimes \sqrt{\mathtt{I}}(|1\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|010\rangle) &= |0\rangle \otimes |1\rangle \otimes \sqrt{\mathtt{I}}(|0\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|011\rangle) &= |0\rangle \otimes |1\rangle \otimes \sqrt{\mathtt{I}}(|1\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|100\rangle) &= |1\rangle \otimes |0\rangle \otimes \sqrt{\mathtt{I}}(|0\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|101\rangle) &= |1\rangle \otimes |0\rangle \otimes \sqrt{\mathtt{I}}(|1\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|110\rangle) &= |1\rangle \otimes |1\rangle \otimes \sqrt{\mathtt{Not}}(|0\rangle) \\
\mathtt{CCBlur}^{(1,1,1)}(|111\rangle) &= |1\rangle \otimes |1\rangle \otimes \sqrt{\mathtt{Not}}(|1\rangle).
\end{aligned}$$

Clearly:

$$\mathtt{CCBlur}^{(n,m,t)} = (I^{(n)} \otimes I^{(m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes \sqrt{\mathtt{I}}^{(t)} + P_1^{(n)} \otimes P_1^{(m)} \otimes \sqrt{\mathtt{Not}}^{(t)}.$$

The definitions of other gates in terms of the `CCBlur` can be now given as follows (for any $|\psi\rangle$ quregister of $\otimes^n \mathbb{C}^2$ and any $|\varphi\rangle$ quregister of $\otimes^m \mathbb{C}^2$):

$$\begin{aligned}
&\mathtt{Not}(|\psi\rangle) := (\mathtt{CCBlur}^{(1,1,n)})^2(|1\rangle \otimes |1\rangle \otimes |\psi\rangle)\\
&\sqrt{\mathtt{Not}}(|\psi\rangle) := \mathtt{CCBlur}^{(1,1,n)}(|1\rangle \otimes |1\rangle \otimes |\psi\rangle)\\
&\sqrt{\mathtt{I}}(|\psi\rangle) := \mathtt{CCBlur}^{(1,1,n)}(|0\rangle \otimes |0\rangle \otimes |\psi\rangle)\\
&\mathtt{And}(|\psi\rangle, |\varphi\rangle) = T^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle) := (\mathtt{CCBlur}^{(n,m,1)})^2(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)\\
&\sqrt{\mathtt{And}}(|\psi\rangle, |\varphi\rangle) = \sqrt{T}^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle) := \mathtt{CCBlur}^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)\\
&\mathtt{Xor}(|\psi\rangle, |\varphi\rangle) := (\mathtt{CCBlur}^{(1,n,m)})^2(|1\rangle \otimes |\psi\rangle \otimes |\varphi\rangle)\\
&\sqrt{\mathtt{Xor}}(|\psi\rangle, |\varphi\rangle) := \mathtt{CCBlur}^{(1,n,m)}(|1\rangle \otimes |\psi\rangle \otimes |\varphi\rangle).
\end{aligned}$$

The gates considered so far can be naturally generalized to qumixs. When our gates will be applied to density operators, we will write: `NOT`, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$, `AND` (instead of `Not`, $\sqrt{\mathtt{Not}}$, $\sqrt{\mathtt{I}}$, `And`).

Definition 12: (The negation).
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\mathtt{NOT}^{(n)}(\rho) = \mathtt{Not}^{(n)} \rho\, \mathtt{Not}^{(n)}.$$

Definition 13: (The square root of the negation).
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\sqrt{\mathtt{NOT}}^{(n)}(\rho) = \sqrt{\mathtt{Not}}^{(n)} \rho \sqrt{\mathtt{Not}}^{(n)*},$$

where $\sqrt{\mathtt{Not}}^{(n)*}$ is the adjoint of $\sqrt{\mathtt{Not}}^{(n)}$.

Definition 14: (The square root of the identity).
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\sqrt{\mathbb{I}}^{(n)}(\rho) = \sqrt{\mathtt{I}}^{(n)} \rho \sqrt{\mathtt{I}}^{(n)}.$$

It is easy to see that for any $n \in \mathbb{N}^+$, $\mathtt{NOT}^{(n)}(\rho)$, $\sqrt{\mathtt{NOT}}^{(n)}(\rho)$ and $\sqrt{\mathbb{I}}^{(n)}(\rho)$ are qumixs of $\mathfrak{D}(\otimes^n \mathbb{C}^2)$. Furthermore: $\mathtt{NOT}^{(n)}\mathtt{NOT}^{(n)} = I^{(n)}$.

Definition 15: (The conjunction).
Let $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$.

$$\mathtt{AND}^{(n,m,1)}(\rho, \sigma) = \mathbb{T}^{(n,m,1)}(\rho, \sigma, P_0^{(1)}) := T^{(n,m,1)}(\rho \otimes \sigma \otimes P_0^{(1)}) T^{(n,m,1)}.$$

Like in the quregister-case, the gates `NOT`, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$, $\mathbb{T}$, `AND` can be uniformly defined on the set $\mathfrak{D}$ of all qumixs.

The following theorems describe some basic properties of our gates.

Theorem 16: [7]

(i) $\mathtt{NOT}(k_n\, P_0^{(n)}) = k_n\, P_1^{(n)}$;
(ii) $\mathtt{NOT}(k_n\, P_1^{(n)}) = k_n\, P_0^{(n)}$;
(iii) $\mathtt{p}(\mathtt{NOT}(\rho)) = 1 - \mathtt{p}(\rho)$.

Consider now the "second" Pauli's matrix:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This matrix can be naturally generalized to an operator $R^{(n)}$ defined on $\otimes^n \mathbb{C}^2$ (for any $n \in \mathbb{N}^+$):

$$R^{(n)} := \begin{cases} Y, & \textit{if } n = 1; \\ I^{(n-1)} \otimes Y, & \textit{otherwise.} \end{cases}$$

Lemma 17: [7] *For any $n \in \mathbb{N}^+$, the following properties hold:*

(i) $\mathtt{tr}(R^{(n)}) = 0$;
(ii) $\mathtt{tr}(R^{(n)} P_1^{(n)}) = 0$;
(iii) $\mathtt{tr}(R^{(n)} P_0^{(n)}) = 0$.

Theorem 18: [7]

(i) $\sqrt{\mathtt{NOT}}(\sqrt{\mathtt{NOT}}(\rho)) = \mathtt{NOT}(\rho)$;
(ii) $\sqrt{\mathtt{NOT}}(\mathtt{NOT}(\rho)) = \mathtt{NOT}(\sqrt{\mathtt{NOT}}(\rho))$;
(iii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1}{2} - \frac{1}{2}\mathtt{tr}(R^{(n)}\rho)$;
(iv) $\forall n \in \mathbb{N}^+$: $\mathtt{p}(\sqrt{\mathtt{NOT}}(k_n P_1^{(n)})) = \mathtt{p}(\sqrt{\mathtt{NOT}}(k_n P_0^{(n)})) = \frac{1}{2}$.

Theorem 19: [7,8]

(i) $\mathtt{p}(\mathtt{AND}(\rho, \sigma)) = \mathtt{p}(\rho)\mathtt{p}(\sigma)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho, \sigma))) = \frac{1}{2}$.

Theorem 20:

(i) $\sqrt{\mathbb{I}}(\sqrt{\mathbb{I}}(\rho)) = \rho$;

(ii) $\forall n \in \mathbb{N}^+$: $\mathtt{p}(\sqrt{\mathbb{I}}(k_n P_1^{(n)})) = \mathtt{p}(\sqrt{\mathbb{I}}(k_n P_0^{(n)})) = \frac{1}{2}$;

(iii) $\forall n \in \mathbb{N}^+$: $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(k_n P_1^{(n)}))) = \mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(k_n P_0^{(n)}))) = \frac{1}{2}$;

(iv) $\forall n \in \mathbb{N}^+$: $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(k_n P_1^{(n)}))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(k_n P_0^{(n)}))) = \frac{1}{2}$;

(v) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\rho))) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho))$;

(vi) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\rho))) = 1 - \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$;

(vii) $\mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho,\sigma))) = \frac{1}{2}$;

(viii) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho,\sigma)))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho,\sigma)))) = \frac{1}{2}$.

Proof: (i)-(vi) Easy.
(vii) $\mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho,\sigma)))$
$= \mathtt{tr}(P_1^{(n+m+1)}\sqrt{\mathtt{I}}^{(n+m+1)}T^{(n,m,1)}(\rho\otimes\sigma\otimes P_0^{(1)})T^{(n,m,1)}\sqrt{\mathtt{I}}^{(n+m+1)})$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma)(I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})\otimes P_1^{(1)}HP_0^{(1)}H$
$+ P_1^{(n)}\rho P_1^{(n)}\otimes P_1^{(m)}\sigma P_1^{(m)}\otimes P_1^{(1)}HP_1^{(1)}H)$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma))\mathtt{tr}(P_1^{(1)}HP_0^{(1)}H)$
$+ \mathtt{tr}(P_1^{(n)}\rho)\mathtt{tr}(P_1^{(m)}\sigma)\mathtt{tr}(P_1^{(1)}HP_1^{(1)}H)$
$= (1 - \mathtt{p}(\rho)\mathtt{p}(\sigma))\mathtt{p}(\sqrt{\mathbb{I}}(P_0^{(1)})) + \mathtt{p}(\rho)\mathtt{p}(\sigma)\mathtt{p}(\sqrt{\mathbb{I}}(P_1^{(1)})) = \frac{1}{2}$.
(viii) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho,\sigma))))$
$= \mathtt{tr}(P_1^{(n+m+1)}\sqrt{\mathtt{I}}^{(n+m+1)}\sqrt{\mathtt{Not}}^{(n+m+1)}T^{(n,m,1)}(\rho\otimes\sigma\otimes P_0^{(1)})$
$T^{(n,m,1)}\sqrt{\mathtt{Not}}^{(n+m+1)*}\sqrt{\mathtt{I}}^{(n+m+1)})$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma)(I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})\otimes P_1^{(1)}HMP_0^{(1)}M^*H$
$+ P_1^{(n)}\rho P_1^{(n)}\otimes P_1^{(m)}\sigma P_1^{(m)}\otimes P_1^{(1)}HMP_1^{(1)}M^*H)$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma))\mathtt{tr}(P_1^{(1)}HMP_0^{(1)}M^*H)$
$+ \mathtt{tr}(P_1^{(n)}\rho)\mathtt{tr}(P_1^{(m)}\sigma)\mathtt{tr}(P_1^{(1)}HMP_1^{(1)}M^*H)$
$= (1 - \mathtt{p}(\rho)\mathtt{p}(\sigma))\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(P_0^{(1)}))) + \mathtt{p}(\rho)\mathtt{p}(\sigma)\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(P_1^{(1)}))) = \frac{1}{2}$.
$\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho,\sigma))))$
$= \mathtt{tr}(P_1^{(n+m+1)}\sqrt{\mathtt{Not}}^{(n+m+1)}\sqrt{\mathtt{I}}^{(n+m+1)}T^{(n,m,1)}(\rho\otimes\sigma\otimes P_0^{(1)})$
$T^{(n,m,1)}\sqrt{\mathtt{I}}^{(n+m+1)}\sqrt{\mathtt{Not}}^{(n+m+1)*})$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma)(I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})\otimes P_1^{(1)}MHP_0^{(1)}HM^*$
$+ P_1^{(n)}\rho P_1^{(n)}\otimes P_1^{(m)}\sigma P_1^{(m)}\otimes P_1^{(1)}MHP_1^{(1)}HM^*)$
$= \mathtt{tr}((I^{(n+m)} - P_1^{(n)}\otimes P_1^{(m)})(\rho\otimes\sigma))\mathtt{tr}(P_1^{(1)}MHP_0^{(1)}HM^*)$
$+ \mathtt{tr}(P_1^{(n)}\rho)\mathtt{tr}(P_1^{(m)}\sigma)\mathtt{tr}(P_1^{(1)}MHP_1^{(1)}HM^*)$
$= (1 - \mathtt{p}(\rho)\mathtt{p}(\sigma))\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(P_0^{(1)}))) + \mathtt{p}(\rho)\mathtt{p}(\sigma)\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(P_1^{(1)}))) = \frac{1}{2}$. □

4. Reversible and irreversible quantum computational structures

An interesting relation connects quregisters and qumixs with the real numbers of the interval $[0,1]$. Any real number $\lambda \in [0,1]$ uniquely determines an n-quregister $|\psi\rangle_\lambda$ and a qumix $\rho_\lambda^{(n)}$ (for any $n \in \mathbb{N}^+$):

- $|\psi\rangle_\lambda := \begin{cases} \sqrt{1-\lambda}|0\rangle + \sqrt{\lambda}|1\rangle, & \textit{if } n = 1; \\ \sqrt{(1-\lambda)k_n}\sum_{j=0}^{2^{n-1}-1}\|j\rangle\rangle|0\rangle + \sqrt{\lambda k_n}\sum_{j=0}^{2^{n-1}-1}\|j\rangle\rangle|1\rangle, & \textit{if } n > 1; \end{cases}$
- $\rho_\lambda^{(n)} := (1-\lambda)k_nP_0^{(n)} + \lambda k_nP_1^{(n)}$.

Clearly, $|\psi\rangle_\lambda \in \mathfrak{R}(\otimes^n\mathbb{C}^2)$ and $\rho_\lambda^{(n)} \in \mathfrak{D}(\otimes^n\mathbb{C}^2)$. From an intuitive point of view, $|\psi\rangle_\lambda$ represents a *maximal information* that might correspond to the *Truth* with probability λ, while $\rho_\lambda^{(n)}$ represents a *mixture of pieces of information* that might correspond to the *Truth* with probability λ.

From a physical point of view, the pure state $|\psi\rangle_\lambda$ describes a particular preparation of the quantum system such that our system might satisfy the properties of a pure state ending with the bit $|0\rangle$ with probability $1-\lambda$ and might satisfy the properties of a pure state ending with the bit $|1\rangle$ with probability λ. A similar interpretation holds for the mixed state $\rho_\lambda^{(n)}$, *mutatis mutandis*. It is worthwhile recalling that the random polarized states of the photon are represented by the density operator $\rho_{1/2}^{(1)} = \frac{1}{2} I^{(1)}$.

The following lemmas describe some important properties of the quregister $|\psi\rangle_\lambda$ and the qumix $\rho_\lambda^{(n)}$.

Lemma 21:

(i) $\forall n \in \mathbb{N}^+ \, \forall \lambda \in [0,1]$: $\mathtt{p}(|\psi\rangle_\lambda) = \lambda$;
(ii) $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle_\lambda)) = \frac{1}{2}$;
(iii) $\mathtt{p}(\sqrt{\mathtt{I}}(|\psi\rangle_\lambda)) = \frac{1}{2} - \sqrt{(1-\lambda)\lambda}$.

Proof: Easy. □

Lemma 22:

(i) $\forall n \in \mathbb{N}^+ \, \forall \lambda \in [0,1]$: $\mathtt{p}(\rho_\lambda^{(n)}) = \lambda$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho_\lambda^{(n)})) = \frac{1}{2}$;
(iii) $\mathtt{p}(\sqrt{\mathbb{I}}(\rho_\lambda^{(n)})) = \frac{1}{2}$.

Proof: Easy. □

We will now introduce three interesting relations that can be defined on the set of all qumixs. All of them turn out to be a preorder-relation. We will speak of *weak*, of *strong* and of *super-strong preorder*, respectively.

Definition 23: (Weak preorder).
$\rho \preccurlyeq_w \sigma$ *iff* $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$.

Definition 24: (Strong preorder).
$\rho \preccurlyeq_s \sigma$ *iff the following conditions hold*:

(i) $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sigma)) \leq \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$.

Definition 25: (Super-strong preorder).
$\rho \preccurlyeq_{ss} \sigma$ *iff the following conditions hold*:

(i) $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sigma)) \leq \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$;

(iii) $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) \leq \mathtt{p}(\sqrt{\mathbb{I}}(\sigma))$.

Clearly:

$$\rho \preccurlyeq_{ss} \sigma \Longrightarrow \rho \preccurlyeq_{s} \sigma \Longrightarrow \rho \preccurlyeq_{w} \sigma,$$

but not the other way around. One immediately shows that the three relations are reflexive and transitive, but not antisymmetric.

Consider now the following three structures:

- $\left(\mathfrak{D}\,, \preccurlyeq_w\,, \mathtt{AND}\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0^{(1)}, P_1^{(1)}\,, \rho_{1/2}^{(1)}\right)$
- $\left(\mathfrak{D}\,, \preccurlyeq_s\,, \mathtt{AND}\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0^{(1)}, P_1^{(1)}\,, \rho_{1/2}^{(1)}\right)$
- $\left(\mathfrak{D}\,, \preccurlyeq_{ss}\,, \mathtt{AND}\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0^{(1)}, P_1^{(1)}\,, \rho_{1/2}^{(1)}\right)$.

We will call these structures *the standard reversible weak quantum computational structure* (briefly, *the WQC-structure*), *the standard reversible strong quantum computational structure* (briefly, *the SQC-structure*), *the standard reversible super-strong quantum computational structure* (briefly, *the SSQC-structure*), respectively.

In the following we will generally write I, P_0, P_1 and $\rho_{1/2}$ instead of $I^{(1)}$, $P_0^{(1)}$,$P_1^{(1)}$, $\rho_{1/2}^{(1)}$. From an intuitive point of view, P_0, P_1 and $\rho_{1/2}$ represent privileged pieces of information that are *false*, *true*, *indeterminate*, respectively. Generally, our qumixs fail to satisfy *Duns Scotus law.* Only in the case of the WQC-structure we have: $\forall \rho \in \mathfrak{D} : P_0^{(1)} \preccurlyeq_w \rho \preccurlyeq_w P_1^{(1)}$. In this situation, it is interesting to isolate the elements that have a *Scotian* behaviour in the strong and in the super-strong structure. Let us first refer to the SQC-structure.

Definition 26: (Down and up scotian qumixs).
Let ρ be a qumix of $\mathfrak{D}$.

(i) ρ is *down Scotian* iff $P_0 \preccurlyeq_s \rho$;
(ii) ρ is *up Scotian* iff $\rho \preccurlyeq_s P_1$;
(iii) ρ is *Scotian* iff ρ is both down and up Scotian.

Lemma 27: [8]

(i) $\rho \preccurlyeq_s \sqrt{\mathtt{NOT}}(P_1)$ *iff* $\mathtt{p}(\rho) \leq \frac{1}{2}$*;*
(ii) $\sqrt{\mathtt{NOT}}(P_0) \preccurlyeq_s \rho$ *iff* $\mathtt{p}(\rho) \geq \frac{1}{2}$.

Theorem 28: [8]

(i) ρ is down Scotian iff $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) \leq \frac{1}{2}$ iff $\sqrt{\mathtt{NOT}}(\rho) \preccurlyeq_s \sqrt{\mathtt{NOT}}(P_1)$;
(ii) ρ is up Scotian iff $\frac{1}{2} \leq \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$ iff $\sqrt{\mathtt{NOT}}(P_0) \preccurlyeq_s \sqrt{\mathtt{NOT}}(\rho)$;
(iii) ρ is Scotian iff $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1}{2}$;
(iv) $\forall n \in \mathbb{N}^+$: $k_n P_0^{(n)}, k_n P_1^{(n)}, \rho_{1/2}^{(n)}$ are Scotian;

(v) For any $\in \mathbb{N}^+$, the set $\mathfrak{D}(\otimes^n\mathbb{C}^2)$ contains uncountably many Scotian density operators.

In a similar way, we can define the scotian elements of the SSQC-structure.

Definition 29: (Super-down and super-up scotian qumixs).
Let ρ be a qumix of $\mathfrak{D}$.

(i) ρ is *super-down Scotian* iff $P_0 \preccurlyeq_{ss} \rho$;
(ii) ρ is *super-up Scotian* iff $\rho \preccurlyeq_{ss} P_1$;
(iii) ρ is *super-Scotian* iff ρ is both super-down and super-up Scotian.

Theorem 30:

(i) ρ is super-down Scotian iff $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) \leq \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) \geq \frac{1}{2}$;
(ii) ρ is super-up Scotian iff $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) \geq \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) \leq \frac{1}{2}$;
(iii) ρ is super-Scotian iff $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \frac{1}{2}$;
(iv) $\forall n \in \mathbb{N}^+$: $k_n P_0^{(n)}, k_n P_1^{(n)}, \rho_{1/2}^{(n)}$ are super-Scotian;
(v) For any $\in \mathbb{N}^+$, the set $\mathfrak{D}(\otimes^n\mathbb{C}^2)$ contains uncountably many super-Scotian density operators.

Proof: (i)–(iv) Easy.
(v) It is sufficient to show that $\mathfrak{D}(\mathbb{C}^2)$ contains uncountably many super-Scotian elements. Let $\lambda \in [-1, 1] \subset \mathbb{R}$. Consider the operator

$$\rho(\lambda) := \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$$

Clearly, $\rho(\lambda) \in \mathfrak{D}(\mathbb{C}^2)$. An easy computation shows that $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho(\lambda))) = \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho(\lambda))) = \frac{1}{2}$. Thus, by (iii) we can conclude that $\rho(\lambda)$ is super-Scotian. $\square$

The gates we have considered so far represent typical *reversible* logical operations. From a logical point of view, it might be interesting to consider also some *irreversible* operations. An important example is represented by a Łukasiewicz-like disjunction.

Definition 31: (The Łukasiewicz disjunction).
Let $\tau \in \mathfrak{D}(\otimes^n\mathbb{C}^2)$ and $\sigma \in \mathfrak{D}(\otimes^m\mathbb{C}^2)$.

$$\tau \oplus \sigma := \rho^{(1)}_{\mathtt{p}(\tau)\oplus\mathtt{p}(\sigma)},$$

where $\oplus$ in $\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)$ is the Łukasiewicz "truncated sum" defined on the real interval $[0, 1]$ (i.e. $\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma) = \min\{1, \mathtt{p}(\tau) + \mathtt{p}(\sigma)\}$)[9].

The following lemmas sum up some basic properties of the Łukasiewicz disjunction:

Lemma 32:

(i)

$$\tau \oplus \sigma = \begin{cases} \rho^{(1)}_{\mathtt{p}(\tau)+\mathtt{p}(\sigma)}, & \textit{if } \mathtt{p}(\tau)+\mathtt{p}(\sigma) \le 1; \\ P^{(1)}_1, & \textit{otherwise}; \end{cases}$$

(ii) $\mathtt{p}(\tau \oplus \sigma) = \mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)$;
(iii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\tau \oplus \sigma)) = \frac{1}{2}$;
(iv) $\mathtt{p}(\sqrt{\mathbb{I}}(\tau \oplus \sigma)) = \frac{1}{2}$;
(v) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\tau \oplus \sigma))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\tau \oplus \sigma))) = \frac{1}{2}$.

Proof: (i) Straightforward.
(ii) The proof follows from Lemma 22(i).
(iii) The proof follows from Lemma 22(ii).
(iv) The proof follows from Lemma 22(iii).
(v) If $\mathtt{p}(\tau) + \mathtt{p}(\sigma) > 1$,then the proof follows from Theorem 20(iii)-(iv). Otherwise,
$\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\tau \oplus \sigma)))$
$= \mathtt{tr}(P_1\sqrt{\mathtt{I}}\sqrt{\mathtt{Not}}((1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))P_0 + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))P_1)\sqrt{\mathtt{Not}}^*\sqrt{\mathtt{I}})$
$= (1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))\mathtt{tr}(P_1HMP_0M^*H) + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))\mathtt{tr}(P_1HMP_1M^*H)$
$= (1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))\frac{1}{2} + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))\frac{1}{2} = \frac{1}{2}$;
$\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\tau \oplus \sigma)))$
$= \mathtt{tr}(P_1\sqrt{\mathtt{Not}}\sqrt{\mathtt{I}}((1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))P_0 + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))P_1)\sqrt{\mathtt{I}}\sqrt{\mathtt{Not}}^*)$
$= (1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))\mathtt{tr}(P_1MHP_0HM^*) + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))\mathtt{tr}(P_1MHP_1HM^*)$
$= (1-\mathtt{p}(\tau)-\mathtt{p}(\sigma))\frac{1}{2} + (\mathtt{p}(\tau)+\mathtt{p}(\sigma))\frac{1}{2} = \frac{1}{2}$. □

Lemma 33: *Let* $\rho \in \mathfrak{D}(\otimes^n\mathbb{C}^2)$.

(i) $\forall n \in \mathbb{N}^+$: $\rho \oplus k_nP^{(n)}_1 = P^{(1)}_1$;
(ii) $\forall n \in \mathbb{N}^+$: $\rho \oplus k_nP^{(n)}_0 = \rho^{(1)}_{\mathtt{p}(\rho)}$;
(iii) $\rho \oplus \mathtt{NOT}(\rho) = P^{(1)}_1$.

Proof: Straightforward. □

From Lemma 33 it follows that $\mathtt{p}(\rho \oplus k_nP^{(n)}_1) = 1$, $\mathtt{p}(\rho \oplus k_nP^{(n)}_0) = \mathtt{p}(\rho)$ and $\mathtt{p}(\rho \oplus \mathtt{NOT}(\rho)) = 1$.

The preorder $\preccurlyeq$ (where $\preccurlyeq$ represents either $\preccurlyeq_w$ or $\preccurlyeq_s$ or $\preccurlyeq_{ss}$) permits us to define on the set of all qumixs an equivalence relation $\equiv$ (where $\equiv$ represents either $\equiv_w$ or $\equiv_s$ or $\equiv_{ss}$, respectively) in the expected way.

Definition 34:

$$\rho \equiv \sigma \textit{ iff } \rho \preccurlyeq \sigma \textit{ and } \sigma \preccurlyeq \rho.$$

Clearly, $\equiv$ is an equivalence relation. Let

$$[\mathfrak{D}]_\equiv := \{[\rho]_\equiv \,:\, \rho \in \mathfrak{D}\}\,.$$

Unlike the qumixs (which are only preordered by $\preccurlyeq$), the equivalence-classes of $[\mathfrak{D}]_\equiv$ can be partially ordered in a natural way.

Definition 35:

$$[\rho]_\equiv \preccurlyeq [\sigma]_\equiv \textit{ iff } \rho \preccurlyeq \sigma.$$

The relation $\preccurlyeq$ (which is well defined) is a partial order.

Lemma 36:

(i) $\forall n \in \mathbb{N}^+$: $[P_1]_\equiv = \left[k_n P_1^{(n)}\right]_\equiv$;
(ii) $\forall n \in \mathbb{N}^+$: $[P_0]_\equiv = \left[k_n P_0^{(n)}\right]_\equiv$;
(iii) $\forall n \in \mathbb{N}^+ \, \forall \lambda \in [0,1]$: $\left[\rho_\lambda^{(1)}\right]_\equiv = \left[\rho_\lambda^{(n)}\right]_\equiv$.

Proof: (i)-(ii) The proof follows from Theorem 18 (iv), Theorem 20 (ii) and from the fact that $\forall n \in \mathbb{N}^+$: $\mathtt{p}(P_1^{(1)}) = 1 = \mathtt{p}(k_n P_1^{(n)})$ and $\mathtt{p}(P_0^{(1)}) = 0 = \mathtt{p}(k_n P_0^{(n)})$.
(iii) The proof follows from Lemma 22. $\square$

We will now consider three quotient-structures based on the three quotient-sets $[\mathfrak{D}]_{\equiv_{ss}}$, $[\mathfrak{D}]_{\equiv_s}$ and $[\mathfrak{D}]_{\equiv_w}$, respectively.

Theorem 37:

(i) $\equiv_{ss}$ is a congruence with respect to the operations $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$;
(ii) $\equiv_s$ is a congruence with respect to $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$ and is not a congruence with respect to $\sqrt{\mathbb{I}}$;
(iii) $\equiv_w$ is a congruence with respect to $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$ and is not a congruence with respect to $\sqrt{\mathtt{NOT}}$ and $\sqrt{\mathbb{I}}$.

Proof: The proof that $\equiv$ is a congruence with respect to $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$ is straightforward. The relation $\equiv_s$ is not a congruence with respect to $\sqrt{\mathbb{I}}$, because the following situation is possible: $[\rho]_{\equiv_s} = [\sigma]_{\equiv_s}$ and $[\sqrt{\mathbb{I}}(\rho)]_{\equiv_s} \neq [\sqrt{\mathbb{I}}(\sigma)]_{\equiv_s}$. Consider for example the following qubit $|\psi\rangle_{1/2} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and qumix $\rho_{1/2} = \frac{1}{2}P_0 + \frac{1}{2}P_1$. It turns out that $\mathtt{p}(|\psi\rangle_{1/2}) = \mathtt{p}(\rho_{1/2}) = \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle_{1/2})) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho_{1/2})) = \frac{1}{2}$. Accordingly, $[P_{|\psi\rangle_{1/2}}]_{\equiv_s} = [\rho_{1/2}]_{\equiv_s}$. However, $\mathtt{p}(\sqrt{\mathtt{I}}(|\psi\rangle_{1/2})) = 0$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho_{1/2})) = \frac{1}{2}$. Consequently, $[P_{|\psi\rangle_{1/2}}]_{\equiv_{ss}} \neq [\rho_{1/2}]_{\equiv_{ss}}$. The relation $\equiv_w$ is not a congruence with respect to $\sqrt{\mathtt{NOT}}$, because the following situation is possible: $[\rho]_{\equiv_w} = [\sigma]_{\equiv_w}$ and $[\sqrt{\mathtt{NOT}}(\rho)]_{\equiv_w} \neq [\sqrt{\mathtt{NOT}}(\sigma)]_{\equiv_w}$. Consider for example the following unit vectors of $\mathbb{C}^2$: $|\psi\rangle := \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$ and $|\varphi\rangle := \frac{\sqrt{2}}{2}|0\rangle + \frac{1+i}{2}|1\rangle$. It turns out that $\mathtt{p}(|\psi\rangle) = \mathtt{p}(|\varphi\rangle) = \frac{1}{2}$. Accordingly, $[P_{|\psi\rangle}]_{\equiv_w} = [P_{|\varphi\rangle}]_{\equiv_w}$. However, $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle)) = \frac{1}{2}$ and $\mathtt{p}(\sqrt{\mathtt{Not}}(|\varphi\rangle)) = \frac{1}{2} - \frac{\sqrt{2}}{4} \approx 0.146447$. Consequently, $[P_{|\psi\rangle}]_{\equiv_s} \neq [P_{|\varphi\rangle}]_{\equiv_s}$. $\square$

In this framework, we can define, in the expected way, the operations:

- $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$ on $[\mathfrak{D}]_{\equiv}$;
- $\sqrt{\mathtt{NOT}}$ on $[\mathfrak{D}]_{\equiv_s}$;
- $\sqrt{\mathtt{NOT}}$ and $\sqrt{\mathbb{I}}$ on $[\mathfrak{D}]_{\equiv_{ss}}$.

Let $\equiv_{s*}$ represent either $\equiv_s$ or $\equiv_{ss}$.

Definition 38:
Let $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$.

(i) $[\rho]_{\equiv} \mathtt{AND} [\sigma]_{\equiv} = [\mathtt{AND}(\rho, \sigma)]_{\equiv}$;
(ii) $[\rho]_{\equiv} \oplus [\sigma]_{\equiv} = [\rho \oplus \sigma]_{\equiv}$;
(iii) $\mathtt{NOT}([\rho]_{\equiv}) = [\mathtt{NOT}(\rho)]_{\equiv}$;
(iv) $\sqrt{\mathtt{NOT}}([\rho]_{\equiv_{s*}}) = [\sqrt{\mathtt{NOT}}(\rho)]_{\equiv_{s*}}$;
(v) $\sqrt{\mathbb{I}}([\rho]_{\equiv_{ss}}) = [\sqrt{\mathbb{I}}(\rho)]_{\equiv_{ss}}$.

Lemma 39:

(i) The operation $\mathtt{AND}$ *is associative and commutative;*
(ii) The operation $\oplus$ *is associative and commutative;*
(iii) $\mathtt{NOT}(\mathtt{NOT}([\rho]_{\equiv})) = [\rho]_{\equiv}$;
(iv) $\sqrt{\mathtt{NOT}}(\sqrt{\mathtt{NOT}}([\rho]_{\equiv_{s*}})) = \mathtt{NOT}([\rho]_{\equiv_{s*}})$;
(v) $\sqrt{\mathbb{I}}(\sqrt{\mathbb{I}}([\rho]_{\equiv_{ss}})) = [\rho]_{\equiv_{ss}}$.

Proof: Straightforward. □

On this basis, we can define the following three quotient-structures:

- $\left([\mathfrak{D}]_{\equiv_w}, \mathtt{AND}, \oplus, \mathtt{NOT}, [P_0]_{\equiv_w}, [P_1]_{\equiv_w}, [\rho_{1/2}]_{\equiv_w}\right)$
- $\left([\mathfrak{D}]_{\equiv_s}, \mathtt{AND}, \oplus, \mathtt{NOT}, \sqrt{\mathtt{NOT}}, [P_0]_{\equiv_s}, [P_1]_{\equiv_s}, [\rho_{1/2}]_{\equiv_s}\right)$
- $\left([\mathfrak{D}]_{\equiv_{ss}}, \mathtt{AND}, \oplus, \mathtt{NOT}, \sqrt{\mathtt{NOT}}, \sqrt{\mathbb{I}}, [P_0]_{\equiv_{ss}}, [P_1]_{\equiv_{ss}}, [\rho_{1/2}]_{\equiv_{ss}}\right)$.

We will call such structures *the standard irreversible weak quantum computational algebra* (briefly, *the IWQC-algebra*), *the standard irreversible strong quantum computational algebra* (briefly, *the ISQC-algebra*), *the standard irreversible super-strong quantum computational algebra* (briefly, *the ISSQC-algebra*), respectively.

An interesting relation between the weak, the strong and the super-strong preorder is described by the following theorem.

Theorem 40: For any $\rho, \sigma \in \mathfrak{D}$:
$[\rho]_{\equiv_w} \preccurlyeq_w [\sigma]_{\equiv_w}$ *iff* $[\rho]_{\equiv_s} \mathtt{AND} [P_1]_{\equiv_s} \preccurlyeq_s [\sigma]_{\equiv_s} \mathtt{AND} [P_1]_{\equiv_s}$
iff $[\rho]_{\equiv_{ss}} \mathtt{AND} [P_1]_{\equiv_{ss}} \preccurlyeq_{ss} [\sigma]_{\equiv_{ss}} \mathtt{AND} [P_1]_{\equiv_{ss}}$.

Proof: Suppose $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$. By Theorem 19(i), we obtain

$$\mathtt{p}(\mathtt{AND}(\rho, P_1)) = \mathtt{p}(\rho) \leq \mathtt{p}(\sigma) = \mathtt{p}(\mathtt{AND}(\sigma, P_1)).$$

By Theorem 19(ii) and Theorem 20 (vii),
$\mathtt{p}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho, P_1))) = \frac{1}{2} = \mathtt{p}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\sigma, P_1)))$

$\mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho, P_1))) = \frac{1}{2} = \mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{AND}(\sigma, P_1)))$.
Thus, $[\rho]_{\equiv_{ss}}\ \mathtt{AND}\ [P_1]_{\equiv_{ss}} \preccurlyeq_{ss} [\sigma]_{\equiv_{ss}}\ \mathtt{AND}\ [P_1]_{\equiv_{ss}}$
$([\rho]_{\equiv_s}\ \mathtt{AND}\ [P_1]_{\equiv_s} \preccurlyeq_s [\sigma]_{\equiv_s}\ \mathtt{AND}\ [P_1]_{\equiv_s})$.
Vice versa, suppose $[\rho]_{\equiv_{ss}}\ \mathtt{AND}\ [P_1]_{\equiv_{ss}} \preccurlyeq_{ss} [\sigma]_{\equiv_{ss}}\ \mathtt{AND}\ [P_1]_{\equiv_{ss}}$
$([\rho]_{\equiv_s}\ \mathtt{AND}\ [P_1]_{\equiv_s} \preccurlyeq_s [\sigma]_{\equiv_s}\ \mathtt{AND}\ [P_1]_{\equiv_s})$.
Then,

$$\mathtt{p}(\rho) = \mathtt{p}(\rho)\mathtt{p}(P_1) = \mathtt{p}(\mathtt{AND}(\rho, P_1)) \leq \mathtt{p}(\mathtt{AND}(\sigma, P_1)) = \mathtt{p}(\sigma). \qquad \square$$

5. The Poincaré quantum computational structures

We will now restrict our analysis to the qumixs living in the two-dimensional space $\mathbb{C}^2$. As is well known, every density operator of $\mathfrak{D}(\mathbb{C}^2)$ has the following matrix representation:

$$\frac{1}{2}\left(I + r_1 X + r_2 Y + r_3 Z\right),$$

where r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$ and X, Y, Z are the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It turns out that a density operator $\frac{1}{2}\left(I + r_1 X + r_2 Y + r_3 Z\right)$ is pure iff $r_1^2 + r_2^2 + r_3^2 = 1$. Consequently,

- Pure density operators are in 1 : 1 correspondence with the points of the surface of the Poincaré sphere;
- Proper mixtures are in 1 : 1 correspondence with the inner points of the Poincaré sphere.

Let ρ be a density operator of $\mathfrak{D}(\mathbb{C}^2)$. We will denote by $\bar{\rho}$ the point of the Poincaré sphere that is univocally associated to ρ.

Let (r_1, r_2, r_3) be a point of the Poincaré sphere. We will denote by $\widehat{(r_1, r_2, r_3)}$ the density operator univocally associated to (r_1, r_2, r_3).

Lemma 41: *Let $\rho \in \mathfrak{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$. The following conditions hold:*

(i) $\mathtt{p}(\rho) = \frac{1-r_3}{2}$, $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1-r_2}{2}$, $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \frac{1-r_1}{2}$;
(ii) $0 < \mathtt{p}(\rho) < 1$, $0 < \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) < 1$ *and* $0 < \mathtt{p}(\sqrt{\mathbb{I}}(\rho)) < 1$, *whenever ρ is a proper mixture.*

Proof: (i) Easy computation;
(ii) Since proper mixtures are in 1:1 correspondence with inner points of the Poincaré sphere, we have: $r_1^2 + r_2^2 + r_3^2 < 1$. Hence: $r_1^2, r_2^2, r_3^2 < 1$ and $-1 < r_1, r_2, r_3 < 1$. Consequently: $0 < \mathtt{p}(\rho) = \frac{1-r_3}{2} < 1$, $0 < \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1-r_2}{2} < 1$ and $0 < \mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \frac{1-r_1}{2} < 1$. $\square$

An irreversible conjunction can be now naturally defined on the set of all qumixs of $\mathfrak{D}(\mathbb{C}^2)$.

Definition 42: (The irreversible conjunction).
Let $\tau, \sigma \in \mathfrak{D}(\mathbb{C}^2)$.

$$\mathtt{IAND}(\tau, \sigma) := \rho^{(1)}_{\mathtt{p}(\tau)\mathtt{p}(\sigma)}$$

Interestingly enough, the density operator $\mathtt{IAND}(\tau, \sigma)$ can be described as a *reduced state* of $\mathtt{AND}(\tau, \sigma)$. Suppose we have a compound physical system consisting of r (possibly compound) subsystems, and let

$$\mathcal{H} = \mathcal{H}_1^{n_1} \otimes \ldots \otimes \mathcal{H}_r^{n_r}$$

be the Hilbert space associated to the total system (where $\mathcal{H}_j^{n_j} = \otimes^{n_j}\mathbb{C}^2$).

Let $\rho \in \mathfrak{D}(\mathcal{H})$ and $1 \leq j \leq r$. The *reduced state* of ρ with respect to the j-th subsystem is the unique density operator $red^j(\rho)$ that satisfies the following condition, for any self-adjoint operator A^j of $\mathcal{H}_j^{n_j}$:

$$\mathtt{tr}(A^j\, red^j(\rho)) = \mathtt{tr}((I^{(n_1)} \otimes \ldots \otimes I^{(n_{j-1})} \otimes A^j \otimes I^{(n_{j+1})} \otimes \ldots \otimes I^{(n_r)})\rho),$$

(where $I^{(n_h)}$ is the identity operator of $\mathcal{H}_h^{n_h}$).

Clearly, ρ and $red^j(\rho)$ turn out to be statistically equivalent with respect to the j-th subsystem of the total system.

One can prove that:

$$\mathtt{IAND}(\tau, \sigma) = red^3(\mathtt{AND}(\tau, \sigma)) = red^3(\mathbb{T}(\tau, \sigma, P_0)).$$

In other words, $\mathtt{IAND}(\tau, \sigma)$ represents the reduced state of $\mathtt{AND}(\tau, \sigma)$ on the third subsystem.

An interesting situation arises when both τ and σ are pure states. For instance, suppose that:

$$\tau = P_{|\psi\rangle} \; and \, \sigma = P_{|\varphi\rangle},$$

where $|\psi\rangle$ and $|\varphi\rangle$ are proper qubits. Then,

$$\mathtt{AND}(\tau, \sigma) = P_{T^{(1,1,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)},$$

which is a pure state. At the same time, we have:

$$\mathtt{IAND}(\tau, \sigma) = red^3(P_{T^{(1,1,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)}),$$

which is a proper mixture. Apparently, when considering only the properties of the third subsystem, we loose some information. As a consequence, we obtain a final state that does not represent a maximal knowledge. As is well known, situations where the state of a compound system represents a maximal knowledge, while the states of the subsystems are proper mixtures, play an important role in the framework of entanglement-phenomena.

Lemma 43:

(i) $\mathtt{IAND}$ *is associative and commutative;*
(ii) $\mathtt{IAND}(\rho, P_0) = P_0$;
(iii) $\mathtt{IAND}(\rho, P_1) = \rho_{\mathtt{p}(\rho)}$;
(iv) $\mathtt{p}(\mathtt{IAND}(\rho, \sigma)) = \mathtt{p}(\rho)\mathtt{p}(\sigma)$;
(v) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\mathtt{IAND}(\rho, \sigma))) = \frac{1}{2}$;
(vi) $\mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{IAND}(\rho, \sigma))) = \frac{1}{2}$.

Proof: Easy. □

Consider now the structure

$$\left(\mathfrak{D}(\mathbb{C}^2)\,, \mathtt{IAND}\,, \oplus\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0, P_1\,, \rho_{1/2}\right).$$

We will call such a structure *the Poincaré irreversible quantum computational algebra* (briefly, *the Poincaré IQC-algebra*).

We can refer to the relation $\upharpoonright\equiv$, representing the restriction of $\equiv$ to $\mathfrak{D}(\mathbb{C}^2)$. For any $\rho \in \mathfrak{D}(\mathbb{C}^2)$, let

$$[\rho]_{\upharpoonright\equiv} := \left\{\sigma \in \mathfrak{D}(\mathbb{C}^2) \,:\, \rho \equiv \sigma\right\}.$$

Furthermore, define

$$[\mathfrak{D}(\mathbb{C}^2)]_{\upharpoonright\equiv} := \left\{[\rho]_{\upharpoonright\equiv} \,:\, \rho \in \mathfrak{D}(\mathbb{C}^2)\right\}.$$

The operations $\mathtt{IAND}\,, \oplus\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}$ and the relation $\preccurlyeq$ can be defined on $[D(\mathbb{C}^2)]_{\upharpoonright\equiv}$ in the expected way.

Consider now the three quotient-structures

- $\left([\mathfrak{D}(\mathbb{C}^2)]_{\upharpoonright\equiv_w}\,, \mathtt{IAND}\,, \oplus\,, \mathtt{NOT}\,, [P_0]_{\upharpoonright\equiv_w}\,, [P_1]_{\upharpoonright\equiv_w}\,, [\rho_{1/2}]_{\upharpoonright\equiv_w}\right)$
- $\left([\mathfrak{D}(\mathbb{C}^2)]_{\upharpoonright\equiv_s}\,, \mathtt{IAND}\,, \oplus\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, [P_0]_{\upharpoonright\equiv_s}\,, [P_1]_{\upharpoonright\equiv_s}\,, [\rho_{1/2}]_{\upharpoonright\equiv_s}\right)$
- $\left([\mathfrak{D}(\mathbb{C}^2)]_{\upharpoonright\equiv_{ss}}\,, \mathtt{IAND}\,, \oplus\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, [P_0]_{\upharpoonright\equiv_{ss}}\,, [P_1]_{\upharpoonright\equiv_{ss}}\,, [\rho_{1/2}]_{\upharpoonright\equiv_{ss}}\right)$.

We will call these structures *the contracted Poincaré irreversible weak quantum computational algebra* (briefly, *the contracted Poincaré IWQC-algebra*), *the contracted Poincaré irreversible strong quantum computational algebra* (briefly, *the contracted Poincaré ISQC-algebra*), *the contracted Poincaré irreversible super-strong quantum computational algebra* (briefly, *the contracted Poincaré ISSQC-algebra*), respectively. By *contracted Poincaré algebra* we will mean anyone of this three structures.

Theorem 44: The contracted Poincaré algebra is isomorphic to the corresponding standard irreversible quantum computational algebra, via the map
$g\,:\, [\mathfrak{D}(\mathbb{C}^2)]_{\upharpoonright\equiv} \to [\mathfrak{D}]_{\equiv}$ such that $\forall\rho \in \mathfrak{D}(\mathbb{C}^2)$:

$$g([\rho]_{\upharpoonright\equiv}) = [\rho]_{\equiv}.$$

Moreover, for any $\rho\,, \sigma \in \mathfrak{D}(\mathbb{C}^2)$: $[\rho]_{\upharpoonright\equiv} \preccurlyeq [\sigma]_{\upharpoonright\equiv}$ iff $g(\,[\rho]_{\upharpoonright\equiv}) \preccurlyeq g([\sigma]_{\upharpoonright\equiv})$.

Proof: Let us consider the contracted Poincaré ISSQC-algebra. One can readily see that g preserves the operation $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$ and $\oplus$. By Theorem 19, Theorem 20(vii) and Lemma 43(iv-vi), g preserves also the operation $\mathtt{IAND}$. Clearly, the map g is injective. Let us prove that g is also surjective. To this aim, it is sufficient to show that for any $n \in \mathbb{N}^+$ and for any $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$, there exists a density operator $\rho' \in \mathfrak{D}(\mathbb{C}^2)$ such that:

(i) $\mathtt{p}(\rho) = \mathtt{p}(\rho')$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho'))$;
(iii) $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho'))$.

Let $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ and let ρ' be the reduced state of ρ with respect to the n-th subsystem. Accordingly, for any self-adjoint operator A of $\mathbb{C}^2$, we have:

$$\mathtt{tr}((I^{(n-1)} \otimes A)\rho) = \mathtt{tr}(A\,\rho'). \tag{1}$$

Thus, $\mathtt{p}(\rho) = \mathtt{tr}(P_1^{(n)}\rho) = \mathtt{tr}((I^{(n-1)} \otimes P_1^{(1)})\rho) = \mathtt{tr}(P_1^{(1)}\,\rho') = \mathtt{p}(\rho')$.
We now prove (ii).
$\mathtt{p}(\sqrt{\mathtt{NOT}}\rho) = \mathtt{tr}(P_1^{(n)}(I^{(n-1)} \otimes M)\rho(I^{(n-1)} \otimes M^*))$
$= \mathtt{tr}((I^{(n-1)} \otimes M^* P_1^{(1)} M)\rho)$
$= \mathtt{tr}(M^* P_1^{(1)} M \rho')$ (1)
$= \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho'))$. Finally, we prove (iii).
$\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \mathtt{tr}(P_1^{(n)}(I^{(n-1)} \otimes H)\rho(I^{(n-1)} \otimes H))$
$= \mathtt{tr}((I^{(n-1)} \otimes H P_1^{(1)} H)\rho)$
$= \mathtt{tr}(H P_1^{(1)} H \rho')$ (1)
$= \mathtt{p}(\sqrt{\mathbb{I}}(\rho'))$. In a similar way, one can prove the theorem for the contracted Poincaré ISQC-algebra and for the contracted Poincaré IWQC-algebra. □

Interestingly enough, any density operator ρ of $\mathbb{C}^2$ is associated to a qubit $|\psi_\rho\rangle$ that is "statistically equivalent" to ρ. In a sense, $|\psi_\rho\rangle$ represents a "purification" of ρ.

Lemma 45: *For any $\rho \in \mathfrak{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$, there exists a qubit $|\psi_\rho\rangle$ that satisfies the following conditions:*

(i) $\mathtt{p}(\rho) = \mathtt{p}(|\psi_\rho\rangle)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \mathtt{p}(\sqrt{\mathtt{Not}}(|\psi_\rho\rangle))$.

Proof: Let $\rho \in \mathfrak{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$. Consider the vector

$$|\psi_\rho\rangle = \frac{\sqrt{1 - r_2^2 - r_3^2} - i r_2}{\sqrt{2(1 - r_3)}}|0\rangle + \sqrt{\frac{1 - r_3}{2}}|1\rangle,$$

which turns out to be a qubit. An easy computation shows that

$$\mathtt{p}(|\psi_\rho\rangle) = \frac{1 - r_3}{2} \quad \textit{and} \quad \mathtt{p}(\sqrt{\mathtt{Not}}|\psi_\rho\rangle) = \frac{1 - r_2}{2}.$$

Thus by Lemma 41(i), we can conclude that

$$\mathtt{p}(|\psi_\rho\rangle) = \mathtt{p}(\rho) \quad \textit{and} \quad \mathtt{p}(\sqrt{\mathtt{Not}}(|\psi_\rho\rangle)) = \mathtt{p}(\sqrt{\mathtt{NOT}}\rho). \qquad \Box$$

As an interesting application of Lemma 45 consider a density operator whose form is: $\rho_\lambda = (1-\lambda)P_0 + \lambda P_1$. Then, by Lemma 45, there exists a qubit $|\psi_{\rho_\lambda}\rangle$ such that $\mathtt{p}(|\psi_{\rho_\lambda}\rangle) = \lambda$. It turns out that

$$|\psi_{\rho_\lambda}\rangle = \sqrt{1-\lambda}|0\rangle + \sqrt{\lambda}|1\rangle.$$

One can easily prove that for any choice of a proper mixture $\rho \in \mathfrak{D}(\mathbb{C}^2)$ there exists no qubit $|\psi\rangle$ such that $\mathtt{p}(|\psi\rangle) = \mathtt{p}(\rho)$, $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle)) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$ and $\mathtt{p}(\sqrt{\mathtt{I}}(|\psi\rangle)) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho))$. As an example, consider $\rho_{1/2}$ that is a fixed point of $\sqrt{\mathtt{NOT}}$ and $\sqrt{\mathbb{I}}$. Let ψ be any qubit such that $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle)) = \mathtt{p}(|\psi\rangle) = \frac{1}{2}$. Hence, $|\psi\rangle = \frac{e^{i\vartheta}}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, but $\mathtt{p}\left(\sqrt{\mathtt{I}}\left(\frac{e^{i\vartheta}}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\right) = 0$ and $\mathtt{p}\left(\sqrt{\mathtt{I}}\left(\frac{e^{i\vartheta}}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\right) = 1$.

Theorem 46: Let $f : \mathfrak{D}^n \to \mathfrak{D}(\mathbb{C}^2)$. Consider the set $\mathfrak{Q}$ of all qubits. Then, there exists a map

$$f_{\mathfrak{Q}} : \mathfrak{Q}^n \to \mathfrak{Q}$$

such that for any qubits $|\psi_1\rangle, \ldots, |\psi_n\rangle$ the following conditions hold:

(i) $\mathtt{p}(f_{\mathfrak{Q}}(|\psi_1\rangle, \ldots, |\psi_n\rangle)) = \mathtt{p}(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}))$;
(ii) $\mathtt{p}(\sqrt{\mathtt{Not}}(f_{\mathfrak{Q}}(|\psi_1\rangle, \ldots, |\psi_n\rangle))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})))$.

Proof: Let $|\psi_1\rangle, \ldots, |\psi_n\rangle \in \mathfrak{Q}$. Then $P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle} \in \mathfrak{D}$ and $f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}) \in \mathfrak{D}(\mathbb{C}^2)$. By lemma 45, there exists a qubit $|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle$ such that $\mathtt{p}(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})) = \mathtt{p}(|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle)$ and $\mathtt{p}(\sqrt{\mathtt{NOT}}(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}))) = \mathtt{p}(\sqrt{\mathtt{Not}}(|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle))$.
Thus, we can put $f_{\mathfrak{Q}}(|\psi_1\rangle, \ldots, |\psi_n\rangle) := |\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle$. $\Box$

As a significant application of Theorem 46, we obtain that a Łukasiewicz disjunction $\oplus_{\mathfrak{Q}}$ and an irreversible conjunction $\mathtt{IAnd}_{\mathfrak{Q}}$ can be naturally defined for any qubits $|\varphi\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\chi\rangle = b_0|0\rangle + b_1|1\rangle$:

$$|\varphi\rangle \oplus_{\mathfrak{Q}} |\chi\rangle := \begin{cases} \sqrt{1 - |a_1|^2 - |b_1|^2}|0\rangle + \sqrt{|a_1|^2 + |b_1|^2}|1\rangle, & \textit{if } |a_1|^2 + |b_1|^2 \le 1; \\ |1\rangle, & \textit{otherwise}; \end{cases}$$

$$\mathtt{IAnd}_{\mathfrak{Q}}(|\varphi\rangle, |\chi\rangle) := \sqrt{1 - |a_1 b_1|^2}|0\rangle + |a_1 b_1||1\rangle.$$

From an intuitive point of view, it is interesting to compare $\mathtt{IAnd}_{\mathfrak{Q}}(|\varphi\rangle, |\chi\rangle)$ with $\mathtt{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$ and with $\mathtt{And}(|\varphi\rangle, |\chi\rangle)$. As we already know, $\mathtt{And}(|\varphi\rangle, |\chi\rangle)$ represents a pure state of a compound physical system (living in the space $\otimes^3\mathbb{C}^2$). Hence, one is dealing with a *maximal knowledge*, that also includes a maximal knowledge about the component systems (described by the pure states $|\varphi\rangle$ and $|\chi\rangle$, respectively). Furthermore, the transformation $(|\varphi\rangle, |\chi\rangle) \mapsto \mathtt{And}(|\varphi\rangle, |\chi\rangle)$ is reversible. The state

$\mathtt{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$, instead, is generally a proper mixture: a non-maximal knowledge about a (non-decomposed) system, representing the output of a computation, where the original information about the component systems (the inputs) has been lost. The transformation $(P_{|\varphi\rangle}, P_{|\chi\rangle}) \mapsto \mathtt{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$ is typically irreversible.

The state $\mathtt{IAnd}_{\mathfrak{Q}}(|\varphi\rangle, |\chi\rangle)$ represents a "purification" of $\mathtt{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$: one is dealing with a maximal knowledge about the output, that does not preserve the original information about the inputs.

6. Quantum computational logics

The quantum computational structures we have investigated suggest a natural semantics, based on the following intuitive idea: any sentence α of the language is interpreted as a convenient qumix, that generally depends on the logical form of α; at the same time, the logical connectives are interpreted as operations that either are *gates* or can be conveniently defined in terms of gates. We will consider a *minimal (sentential) quantum computational language* $\mathcal{L}$ that contains a privileged atomic sentence $\mathbf{f}$ (whose intended interpretation is the truth-value *Falsity*) and the following primitive connectives: a *negation* $(\neg)$, a *square root of the negation* $(\sqrt{\neg})$, a *square root of the identity* $(\sqrt{id})$, a ternary *conjunction* $\bigwedge$ (which corresponds to the Petri-Toffoli gate). For any sentences α and β, the expression $\bigwedge(\alpha, \beta, f)$ is a sentence of $\mathcal{L}$. In this framework, the usual conjunction $\alpha \wedge \beta$ is dealt with as metalinguistic abbreviation for the ternary conjunction $\bigwedge(\alpha, \beta, \mathbf{f})$. The occurrence of $\mathbf{f}$ as the third element in the formula $\bigwedge(\alpha, \beta, \mathbf{f})$ is called a *non-genuine occurrence* of $\mathbf{f}$. All other occurrences of atomic sentences in a formula are called *genuine*.

Let $Form^{\mathcal{L}}$ be the set of all sentences of $\mathcal{L}$. We will use the following metavariables: $\mathbf{q}, \mathbf{r}, \ldots$ for atomic sentences and $\alpha, \beta, \ldots$ for sentences. The connective disjunction $(\vee)$ is supposed to be defined via de Morgan $(\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta))$, while the privileged sentence $\mathbf{t}$ representing the *Truth* is defined as the negation of $\mathbf{f}$ $(\mathbf{t} := \neg\mathbf{f})$. This minimal quantum computational language can be extended to richer languages containing other primitive connectives (for instance, a connective corresponding to the Łukasiewicz irreversible disjunction $\oplus$) that we will not consider here.

We will first introduce the notion of *reversible quantum computational model* (briefly, *RQC-model*).

Definition 47: (RQC-model).
A *RQC-model* of $\mathcal{L}$ is a function $\mathtt{Qum} : Form^{\mathcal{L}} \to \mathfrak{D}$ (which associates to any sentence α of the language a qumix):

$$\mathtt{Qum}(\alpha) := \begin{cases} \textit{a density operator of } \mathfrak{D}(\mathbb{C}^2) & \textit{if } \alpha \textit{ is an atomic sentence;} \\ P_0 & \textit{if } \alpha = \mathbf{f}; \\ \mathtt{NOT}(\mathtt{Qum}(\beta)) & \textit{if } \alpha = \neg\beta; \\ \sqrt{\mathtt{NOT}}(\mathtt{Qum}(\beta)) & \textit{if } \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathbb{I}}(\mathtt{Qum}(\beta)) & \textit{if } \alpha = \sqrt{id}\,\beta; \\ \mathbb{T}(\mathtt{Qum}(\beta), \mathtt{Qum}(\gamma), \mathtt{Qum}(\mathbf{f})) & \textit{if } \alpha = \bigwedge(\beta, \gamma, \mathbf{f}). \end{cases}$$

The concept of RQC-model seems to have a "quasi-intensional" feature: the meaning $\mathtt{Qum}(\alpha)$ of the sentence α partially reflects the logical form of α. In fact, the dimension of the Hilbert space where $\mathtt{Qum}(\alpha)$ "lives" depends on the number of occurrences of atomic sentences in α.

Definition 48: (The atomic complexity of α).

$$At(\alpha) = \begin{cases} 1 & \textit{if}\, \alpha \textit{ is an atomic sentence}; \\ At(\beta) & \textit{if}\, \alpha = \neg\beta \textit{ or } \alpha = \sqrt{\neg}\beta \textit{ or } \alpha = \sqrt{id}\,\beta; \\ At(\beta) + At(\gamma) + 1 & \textit{if}\, \alpha = \bigwedge(\beta, \gamma, \mathbf{f}). \end{cases}$$

Lemma 49: *If* $At(\alpha) = n$*, then* $\mathtt{Qum}(\alpha) \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$.

Proof: Straightforward. □

Given a reversible quantum computational model $\mathtt{Qum}$, any sentence α has a natural probability-value, which can be also regarded as its *extensional meaning* with respect to $\mathtt{Qum}$.

Definition 50: (The probability-value of α in a model $\mathtt{Qum}$).

$$\mathtt{p}_{\mathtt{Qum}}(\alpha) := \mathtt{p}(\mathtt{Qum}(\alpha)).$$

As we already know, qumixs are naturally preordered by three basic relations: the weak preorder $\preccurlyeq_w$, the strong preorder $\preccurlyeq_s$ and the super-strong preorder $\preccurlyeq_{ss}$. This suggests to introduce three different consequence relations: the *weak*, the *strong* and the *super-strong consequence*.

Definition 51: (Weak, strong and super-strong consequence in a model $\mathtt{Qum}$).

(i) A sentence β is a *weak consequence in a model* $\mathtt{Qum}$ of a sentence α ($\alpha \models^w_{\mathtt{Qum}} \beta$) iff $\mathtt{Qum}(\alpha) \preccurlyeq_w \mathtt{Qum}(\beta)$.
(ii) A sentence β is a *strong consequence in a model* $\mathtt{Qum}$ of a sentence α ($\alpha \models^s_{\mathtt{Qum}} \beta$) iff $\mathtt{Qum}(\alpha) \preccurlyeq_s \mathtt{Qum}(\beta)$.
(iii) A sentence β is a *super-strong consequence in a model* $\mathtt{Qum}$ of a sentence α ($\alpha \models^{ss}_{\mathtt{Qum}} \beta$) iff $\mathtt{Qum}(\alpha) \preccurlyeq_{ss} \mathtt{Qum}(\beta)$.

The notions of *weak*, *strong* and *super-strong truth in a model* $\mathtt{Qum}$, *weak*, *strong* and *super-strong logical consequence*, *weak*, *strong* and *super-strong logical truth* can be now defined in the expected way.

Definition 52: (Weak, strong and super-strong truth in a model $\mathtt{Qum}$).

(i) A sentence α is *weakly true in a model* $\mathtt{Qum}$ iff $\mathbf{t} \models^w_{\mathtt{Qum}} \alpha$.
(ii) A sentence α is *strongly true in a model* $\mathtt{Qum}$ iff $\mathbf{t} \models^s_{\mathtt{Qum}} \alpha$.
(iii) A sentence α is *super-strongly true in a model* $\mathtt{Qum}$ iff $\mathbf{t} \models^{ss}_{\mathtt{Qum}} \alpha$.

Definition 53: (Weak, strong and super-strong logical consequence).

(i) A sentence β is a *weak logical consequence of a sentence* α ($\alpha \models^w \beta$) iff for any model $\mathtt{Qum}, \alpha \models^w_{\mathtt{Qum}} \beta$.
(ii) A sentence β is a *strong logical consequence of a sentence* α ($\alpha \models^s \beta$) iff for any model $\mathtt{Qum}, \alpha \models^s_{\mathtt{Qum}} \beta$.
(iii) A sentence β is a *super-strong logical consequence of a sentence* α ($\alpha \models^{ss} \beta$) iff for any model $\mathtt{Qum}, \alpha \models^{ss}_{\mathtt{Qum}} \beta$.

Definition 54: (Weak, strong and super-strong logical truth).

(i) A sentence α is a *weak logical truth* iff for any model $\mathtt{Qum}$, α is weakly true in $\mathtt{Qum}$.
(ii) A sentence α is a *strong logical truth* iff for any model $\mathtt{Qum}$, α is strongly true in $\mathtt{Qum}$.
(iii) A sentence α is a *super-strong logical truth* iff for any model $\mathtt{Qum}$, α is super-strongly true in $\mathtt{Qum}$.

The weak, strong and super-strong logical consequence relations permit us to characterize semantically three different forms of *quantum computational logic*. We will indicate by $\mathbf{QCL^w}, \mathbf{QCL^s}, \mathbf{QCL^{ss}}$ the logics that are semantically characterized by the weak, strong and super-strong logical consequence relation respectively. In other words, we have:

- β is a logical consequence of α in the logic $\mathbf{QCL^w}$ ($\alpha \models_{\mathbf{QCL^w}} \beta$) iff β is a weak logical consequence of α;
- β is a logical consequence of α in the logic $\mathbf{QCL^s}$ ($\alpha \models_{\mathbf{QCL^s}} \beta$) iff β is a strong logical consequence of α;
- β is a logical consequence of α in the logic $\mathbf{QCL^{ss}}$ ($\alpha \models_{\mathbf{QCL^{ss}}} \beta$) iff β is a super-strong logical consequence of α.

Clearly, $\mathbf{QCL^{ss}}$ is a sublogic of $\mathbf{QCL^s}$ and $\mathbf{QCL^s}$ is a sublogic of $\mathbf{QCL^w}$. For:

$$\alpha \models_{\mathbf{QCL^{ss}}} \beta \ \ \textit{implies} \ \ \alpha \models_{\mathbf{QCL^s}} \beta \ \ \textit{implies} \ \ \alpha \models_{\mathbf{QCL^w}} \beta.$$

But not the other way around!

An interesting relation between the three logics $\mathbf{QCL^{ss}}$, $\mathbf{QCL^s}$ and $\mathbf{QCL^w}$ is described by the following theorem:

Theorem 55: $\alpha \models_{\mathbf{QCL^w}} \beta$ iff $\alpha \wedge \mathbf{t} \models_{\mathbf{QCL^s}} \beta \wedge \mathbf{t}$ iff $\alpha \wedge \mathbf{t} \models_{\mathbf{QCL^{ss}}} \beta \wedge \mathbf{t}$.

Proof: The theorem is a direct consequence of the definition of $\mathbf{QCL^{ss}}$, $\mathbf{QCL^s}$ and $\mathbf{QCL^w}$ and of Theorem 40. □

We will indicate by $\mathbf{QCL}$ the generic quantum computational logic (either $\mathbf{QCL^{ss}}$ or $\mathbf{QCL^s}$ or $\mathbf{QCL^w}$).

Let us now turn to the concept of *irreversible quantum computational model* (briefly, *IQC-model*), where the "quasi-intensional" character of reversible models is lost. In fact, the interpretation of a sentence in an irreversible model does not generally reflect the logical form of our sentence: the meaning of the *whole* does not include the meanings of its *parts*. In spite of this, we will prove that reversible and irreversible models turn out to characterize the same logic.

Definition 56: (IQC-model).
An *IQC-model* of $\mathcal{L}$ is a function $\mathtt{Qum}^{\mathbb{C}^2} : Form^{\mathcal{L}} \to \mathfrak{D}(\mathbb{C}^2)$ (which associates to any sentence α of the language a qumix of $\mathbb{C}^2$):

$$\mathtt{Qum}^{\mathbb{C}^2}(\alpha) := \begin{cases} P_0 & \textit{if}\, \alpha = \mathbf{f}; \\ \mathtt{NOT}(\mathtt{Qum}^{\mathbb{C}^2}(\beta)) & \textit{if}\, \alpha = \neg\beta; \\ \sqrt{\mathtt{NOT}}(\mathtt{Qum}^{\mathbb{C}^2}(\beta)) & \textit{if}\, \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathbb{I}}(\mathtt{Qum}^{\mathbb{C}^2}(\beta)) & \textit{if}\, \alpha = \sqrt{id}\,\beta; \\ \mathtt{IAND}(\mathtt{Qum}^{\mathbb{C}^2}(\beta), \mathtt{Qum}^{\mathbb{C}^2}(\gamma)) & \textit{if}\, \alpha = \bigwedge(\beta, \gamma, \mathbf{f}). \end{cases}$$

The weak, strong and super-strong notions of *consequence*, *truth*, *logical consequence*, *logical truth* are defined like in the reversible case, *mutatis mutandis*. The logics that are determined by the weak, strong and super-strong irreversible logical consequence will be denoted by $\mathbf{IQCL^w}$, $\mathbf{IQCL^s}$, $\mathbf{IQCL^{ss}}$, respectively; while **IQCL** will represent the generic irreversible quantum computational logic.

Lemma 57: *Let* $\mathtt{Qum}$ *be a RQC-model and let* $\mathtt{Qum}^{\mathbb{C}^2}$ *be an IQC-model such that for any atomic sentence* $\mathbf{q}$*:* $\mathtt{Qum}(\mathbf{q}) = \mathtt{Qum}^{\mathbb{C}^2}(\mathbf{q})$*. Then, for any sentence* $\alpha \in Form^{\mathcal{L}}$*:*

$$\mathtt{p}(\mathtt{Qum}(\alpha)) = \mathtt{p}(\mathtt{Qum}^{\mathbb{C}^2}(\alpha)).$$

Proof: The proof is by induction on the *length* (i.e. the number of connectives) of α. $\square$

Corollary 58:

(i) For any RQC-model $\mathtt{Qum}$*, there exists an IQC-model* $\mathtt{Qum}^{\mathbb{C}^2}$ *such that for any* $\alpha \in Form^{\mathcal{L}}$*:*

$$\mathtt{p}(\mathtt{Qum}(\alpha)) = \mathtt{p}(\mathtt{Qum}^{\mathbb{C}^2}(\alpha));$$

(ii) For any IQC-model $\mathtt{Qum}^{\mathbb{C}^2}$ *there exists a RQC-model* $\mathtt{Qum}$ *such that for any* $\alpha \in Form^{\mathcal{L}}$*:*

$$\mathtt{p}(\mathtt{Qum}^{\mathbb{C}^2}(\alpha)) = \mathtt{p}(\mathtt{Qum}(\alpha)).$$

Theorem 59:

(i) $\alpha \models_{\mathbf{QCL^{ss}}} \beta$ iff $\alpha \models_{\mathbf{IQCL^{ss}}} \beta$;
(ii) $\alpha \models_{\mathbf{QCL^{s}}} \beta$ iff $\alpha \models_{\mathbf{IQCL^{s}}} \beta$;
(iii) $\alpha \models_{\mathbf{QCL^{w}}} \beta$ iff $\alpha \models_{\mathbf{IQCL^{w}}} \beta$.

Proof: The theorem is a direct consequence of Corollary 58. $\square$

Hence, each **QCL** and its corresponding **IQCL** are the same logic.

So far we have considered (reversible and irreversible) models, where the meaning of any sentence is represented by a qumix. A natural question arises: do density operators have an essential role in characterizing **QCL**? This question has a negative answer in the case of $\mathbf{QCL^s}$ and $\mathbf{QCL^w}$.

Let us first introduce the notion of (reversible) *qubit-model* (which is the basic concept of the qubit-semantics described in Refs. 1 and 10).

Definition 60: (Reversible qubit-model).
A *reversible qubit-model* of $\mathcal{L}$ is a function $\mathtt{Qub} : Form^{\mathcal{L}} \to \mathfrak{R}$ (which associates to any sentence α of the language a quregister):

$$\mathtt{Qub}(\alpha) := \begin{cases} \textit{a qubit in } \mathbb{C}^2 & \textit{if}\, \alpha \;\; \textit{is an atomic sentence}; \\ |0\rangle & \textit{if}\, \alpha = \mathbf{f}; \\ \mathtt{Not}(\mathtt{Qub}(\beta)) & \textit{if}\, \alpha = \neg\beta; \\ \sqrt{\mathtt{Not}}(\mathtt{Qub}(\beta)) & \textit{if}\, \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathtt{I}}(\mathtt{Qub}(\beta)) & \textit{if}\, \alpha = \sqrt{id}\,\beta; \\ T(\mathtt{Qub}(\beta), \mathtt{Qub}(\gamma), \mathtt{Qub}(\mathbf{f})) & \textit{if}\, \alpha = \bigwedge(\beta, \gamma, \mathbf{f}). \end{cases}$$

The notions of (weak, strong and super-strong) *consequence, truth, logical consequence, logical truth* are defined like in the case of reversible qumix-models, *mutatis mutandis.*

We will write $\alpha \models^{\mathtt{Qub}}_{\mathbf{QCL^s}} \beta$, when β is a strong logical consequence of α in the qubit-semantics. In a similar way, we will write $\alpha \models^{\mathtt{Qub}}_{\mathbf{QCL^w}} \beta$ when β is a weak logical consequence in the same semantics.

Instead of the class $\mathfrak{R}$ of all quregisters, we could equivalently refer to the class $\mathfrak{D}_{\mathfrak{R}}$ of all *pure density operators* having the form $P_{|\psi\rangle}$, where $|\psi\rangle$ is a quregister. One can easily show that $\mathfrak{D}_{\mathfrak{R}}$ is closed under the gates $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$, $\mathtt{AND}$. At the same time, $\mathfrak{D}_{\mathfrak{R}}$ is not closed under $\mathtt{IAND}$, because (as we have seen) $\mathtt{IAND}(P_{|\psi\rangle}, P_{|\varphi\rangle})$ is, generally, a proper mixture.

Lemma 61: *Consider a reversible qubit-model* $\mathtt{Qub}$ *and let* $\mathtt{Qum}$ *be a RQC-model such that for any atomic sentence* $\mathbf{q}$, $\mathtt{Qum}(\mathbf{q}) = P_{\mathtt{Qub}(\mathbf{q})}$. *Then, for any sentences* α*:*

$$\mathtt{Qum}(\alpha) \equiv P_{\mathtt{Qub}(\alpha)}.$$

Proof: Easy. □

On this basis we can prove that the qubit-semantics and the qumix-semantics characterize the same logics $\mathbf{QCL^s}$ and $\mathbf{QCL^w}$.

Theorem 62:

(i) $\alpha \models_{\mathbf{QCL^s}} \beta$ iff $\alpha \models^{\mathtt{Qub}}_{\mathbf{QCL^s}} \beta$;
(ii) $\alpha \models_{\mathbf{QCL^w}} \beta$ iff $\alpha \models^{\mathtt{Qub}}_{\mathbf{QCL^w}} \beta$.

Proof:

(i) (a) Suppose that $\alpha \models_{\mathbf{QCL^s}} \beta$. Then for any RQC-model $\mathtt{Qum}$:$\mathtt{Qum}(\alpha) \preccurlyeq_s \mathtt{Qum}(\beta)$. Hence, for any $\mathtt{Qum}$ such that $\mathtt{Qum}(\alpha)$ and $\mathtt{Qum}(\beta)$ are pure density operators: $\mathtt{Qum}(\alpha) \preccurlyeq_s \mathtt{Qum}(\beta)$.
Consequently, by Lemma 61, for any qubit-model $\mathtt{Qub}$:$\mathtt{Qub}(\alpha) \preccurlyeq_s \mathtt{Qub}(\beta)$.

(b) Suppose, by contradiction, that $\alpha \models^{\mathtt{Qub}}_{\mathbf{QCL^s}} \beta$ and $\alpha \not\models_{\mathbf{QCL^s}} \beta$. Then, by Theorem 59 there exists an irreversible model $\mathbf{Qum}^{\mathbb{C}^2}$ such that $\mathbf{Qum}^{\mathbb{C}^2}(\alpha) \not\preccurlyeq_s \mathbf{Qum}^{\mathbb{C}^2}(\beta)$. By Lemma 45, there exists a qubit-model $\mathtt{Qub}$ such that for any sentential letter $\mathbf{q}$: $\mathtt{p}(\mathtt{Qub}(\mathbf{q})) = \mathtt{p}(\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q}))$ and $\mathtt{p}(\sqrt{\mathtt{Not}}(\mathtt{Qub}(\mathbf{q}))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q})))$. One can easily prove that for any α, $\mathtt{p}(\mathtt{Qub}(\alpha)) = \mathtt{p}(\mathbf{Qum}^{\mathbb{C}^2}(\alpha))$ and $\mathtt{p}(\sqrt{\mathtt{Not}}(\mathtt{Qub}(\alpha))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\mathbf{Qum}^{\mathbb{C}^2}(\alpha)))$ (by induction on the length of α).
Consequently, $\alpha \not\models^{\mathtt{Qub}}_{\mathbf{QCL^s}} \beta$, contradiction.

(ii) Similarly. □

The proof of Theorem 62 cannot be extended to the case of $\mathbf{QCL^{ss}}$. As we have seen, for any proper mixtures $\rho \in \mathfrak{D}(\mathbb{C}^2)$ there exists no qubit $|\psi\rangle$ such that $\mathtt{p}(|\psi\rangle) = \mathtt{p}(\rho)$, $\mathtt{p}(\sqrt{\mathtt{Not}}(|\psi\rangle)) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$ and $\mathtt{p}(\sqrt{\mathtt{I}}(|\psi\rangle)) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho))$. Hence, the following situation is possible:

- $\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q})$ is a proper mixture;
- there exists no qubit-model $\mathtt{Qub}$ such that:

$$\mathtt{p}(\mathtt{Qub}(\mathbf{q})) = \mathtt{p}(\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q}));$$

$$\mathtt{p}(\sqrt{\mathtt{Not}}(\mathtt{Qub}(\mathbf{q}))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q})));$$

$$\mathtt{p}(\sqrt{\mathtt{I}}(\mathtt{Qub}(\mathbf{q}))) = \mathtt{p}(\sqrt{\mathbb{I}}(\mathbf{Qum}^{\mathbb{C}^2}(\mathbf{q}))).$$

A remarkable property of the logics **QCL** is the following: our logics do not admit any "genuine" logical truth. In other words, any sentence α, that does not contain the atomic sentence $\mathbf{f}$, cannot be a logical truth.

Let us first prove the following theorem:

Theorem 63: Let $\mathtt{Qum}$ be a RQC-model and let α be any sentence. If $\mathtt{p}(\mathtt{Qum}(\alpha)) \in \{0,1\}$, then there is an atomic subformula $\mathbf{q}$ of α such that $\mathtt{p}(\mathtt{Qum}(\mathbf{q})) \in \left\{0, \frac{1}{2}, 1\right\}$.

Proof: Suppose that $\mathtt{p}(\mathtt{Qum}(\alpha)) \in \{0,1\}$. The proof is by induction on the length of α.
(i) α is an atomic sentence. The proof is trivial.
(ii) $\alpha = \neg\beta$. By Theorem 16(iii), $\mathtt{p}(\mathtt{Qum}(\alpha)) = 1 - \mathtt{p}(\mathtt{Qum}(\beta)) \in \{0,1\}$. The conclusion follows by induction hypothesis.
(iii) $\alpha = \sqrt{\neg}\beta$. By hypothesis and by Theorem 19(ii), β cannot be a conjunction. Consequently, only the following cases are possible: (iiia) $\beta = \mathbf{q}$; (iiib) $\beta = \neg\gamma$; (iiic) $\beta = \sqrt{\neg}\gamma$; (iiid) $\beta = \sqrt{id}\,\gamma$.
(iiia) $\beta = \mathbf{q}$. By hypothesis, $\mathtt{p}(\sqrt{\neg}\beta) \in \{0,1\}$. One can easily show that $\mathtt{p}(\mathbf{q}) = \frac{1}{2}$.

(iiib) $\beta = \neg\gamma$. By Theorem 18(iii), $\mathtt{p}(\mathtt{Qum}(\sqrt{\neg}\neg\gamma)) = \mathtt{p}(\mathtt{Qum}(\neg\sqrt{\neg}\gamma)) = 1 - \mathtt{p}(\mathtt{Qub}(\sqrt{\neg}\gamma))$. The conclusion follows by induction hypothesis.
(iiic) $\beta = \sqrt{\neg}\gamma$. Then $\mathtt{p}(\mathtt{Qum}(\sqrt{\neg}\sqrt{\neg}\gamma)) = \mathtt{p}(\mathtt{Qum}(\neg\gamma)) = 1 - \mathtt{p}(\mathtt{Qum}(\gamma))$. The conclusion follows by induction hypothesis.
(iiid) $\beta = \sqrt{id}\,\gamma$. By Theorem 20(vi), $\mathtt{p}(\mathtt{Qub}(\sqrt{\neg}\sqrt{id}\,\gamma)) = 1 - \mathtt{p}(\mathtt{Qub}(\sqrt{\neg}\gamma))$. The conclusion follows by induction hypothesis.
(iv) $\alpha = \sqrt{id}\,\beta$. By hypothesis and by Theorem 20(vii), β cannot be a conjunction. Consequently, only the following cases are possible: (iva) $\beta = \mathbf{q}$; (ivb) $\beta = \neg\gamma$; (ivc) $\beta = \sqrt{\neg}\gamma$; (ivd) $\beta = \sqrt{id}\,\gamma$.
(iva) $\beta = \mathbf{q}$. By hypothesis, $\mathtt{p}(\sqrt{id}\,\beta) \in \{0, 1\}$. One can easily show that $\mathtt{p}(\mathbf{q}) = \frac{1}{2}$.
(ivb) $\beta = \neg\gamma$. By Theorem 20(v), $\mathtt{p}(\mathtt{Qum}(\sqrt{id}\,\neg\gamma)) = \mathtt{p}(\mathtt{Qum}(\sqrt{id}\,\gamma))$. The conclusion follows by induction hypothesis.
(ivc) $\beta = \sqrt{\neg}\gamma$. Then $\mathtt{p}(\mathtt{Qum}(\sqrt{id}\,\sqrt{\neg}\gamma)) = \mathtt{p}(\mathtt{Qum}(\sqrt{id}\,\gamma))$. The conclusion follows by induction hypothesis.
(ivd) $\beta = \sqrt{id}\,\gamma$. Then $\mathtt{p}(\mathtt{Qum}(\sqrt{id}\,\sqrt{id}\,\gamma)) = \mathtt{p}(\mathtt{Qum}(\gamma))$. The conclusion follows by induction hypothesis.
(v) $\alpha = \bigwedge(\beta, \gamma, \mathbf{f})$. By Theorem 19(i), $\mathtt{p}(\mathtt{Qum}(\bigwedge(\beta, \gamma, \mathbf{f})) = \mathtt{p}(\mathtt{Qum}(\beta))\mathtt{p}(\mathtt{Qum}(\gamma)) \in \{0, 1\}$. The conclusion follows by induction hypothesis. □

As a consequence, we immediately obtain the following Corollary.

Corollary 64: *If α does not contain any genuine occurrence of* $\mathbf{f}$*, then α is not a logical truth of* **QCL**.

Proof: Suppose, by contradiction, that α is a logical truth of **QCL**. Then, we obtain that: $\mathtt{p}(\alpha) = 1$. Let $\mathbf{q}_1, \ldots, \mathbf{q}_n$ be the atomic sentences genuinely occurring in α. Since α does not contain any genuine occurrence of $\mathbf{f}$, there exists a RQC-model $\mathtt{Qum}$ such that for any i ($1 \leq i \leq n$), $\mathtt{p}(\mathtt{Qum}(\mathbf{q}_i)) \notin \{0, \frac{1}{2}, 1\}$. Then, by Theorem 63, $\mathtt{p}(\mathtt{Qum}(\alpha)) \notin \{0, 1\}$, contradiction. □

We will now list some interesting logical consequences and rules that hold for the logics **QCL**. We will indicate by $\alpha \models \beta$ the logical consequence relation that refers to **QCL**. According to the usual notation we will write:

$$\frac{\alpha_1 \models \beta_1, \ldots, \alpha_n \models \beta_n}{\gamma \models \delta},$$

to be read as: if $\alpha_1 \models \beta_1, \ldots, \alpha_n \models \beta_n$, then $\gamma \models \delta$. We will also write $\alpha \equiv \beta$ as an abbreviation for: $\alpha \models \beta$ and $\beta \models \alpha$.

Since $\mathbf{QCL^{ss}}$ is a sublogic of both $\mathbf{QCL^{s}}$ and $\mathbf{QCL^{w}}$, any logical consequence that holds in $\mathbf{QCL^{ss}}$ will also hold in $\mathbf{QCL^{s}}$ and in $\mathbf{QCL^{w}}$. At the same time, some rules that hold in $\mathbf{QCL^{ss}}$ may be violated in $\mathbf{QCL^{s}}$ and in $\mathbf{QCL^{w}}$ (and, of course, vice versa). A similar relation holds for $\mathbf{QCL^{s}}$ and $\mathbf{QCL^{w}}$.

Theorem 65: (Logical consequences and rules of **QCL**).

(i) $\alpha \models \alpha$;
(identity)

(ii) $\frac{\alpha \models \beta,\, \beta \models \gamma}{\alpha \models \gamma}$;
(transitivity)

(iii) $\alpha \equiv \neg\neg\alpha$;
(double negation)

(iv) $\sqrt{\neg}\sqrt{\neg}\alpha \equiv \neg\alpha$;
(the double square root of the negation principle)

(v) $\neg\sqrt{\neg}\alpha \equiv \sqrt{\neg}\neg\alpha$;
(permutation of the negations)

(vi) $\sqrt{\neg}\mathbf{f} \models \sqrt{\neg}\mathbf{t}$;
(a "tentative negation" of the falsity implies a "tentative negation" of the truth)

(vii) $\sqrt{id}\,\sqrt{id}\,\alpha \equiv \alpha$;
(the double square root of the identity principle)

(viii) $\alpha \wedge \beta \equiv \beta \wedge \alpha, \quad \alpha \vee \beta \equiv \beta \vee \alpha$;
(commutativity)

(ix) $\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma, \quad \alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$;
(associativity)

(x) $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta, \quad \neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$;
(de Morgan)

(xi) $\alpha \wedge (\beta \vee \gamma) \models (\alpha \wedge \beta) \vee (\alpha \wedge \gamma), \quad (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \models \alpha \vee (\beta \wedge \gamma)$;
(distributivity 1)

(xii) $\mathbf{f} \wedge \mathbf{f} \equiv \mathbf{f}, \quad \mathbf{t} \wedge \mathbf{t} \equiv \mathbf{t}$;
(idempotence for the truth and the falsity)

(xiii) $\mathbf{f} \wedge \mathbf{t} \equiv \mathbf{f}, \quad \mathbf{f} \vee \mathbf{t} \equiv \mathbf{t}$;

(xiv) $\frac{\alpha \quad \equiv \quad \beta}{\neg\alpha \quad \equiv \quad \neg\beta}$;
(logical equivalence is a congruence for the negation)

(xv) $\frac{\alpha \equiv \gamma, \quad \beta \equiv \delta}{\alpha\wedge\beta \equiv \gamma\wedge\delta}$;
(logical equivalence is a congruence for the conjunction)

(xvi) $\sqrt{\neg}(\alpha \wedge \beta) \models \sqrt{\neg}\mathbf{t}$;

Proof: Easy. □

Let us now consider examples of logical consequences and rules that hold in $\mathbf{QCL^s}$ ($\mathbf{QCL^w}$) and are violated in $\mathbf{QCL^{ss}}$.

Theorem 66: (Logical rules of $\mathbf{QCL^s}$ and $\mathbf{QCL^w}$ that fail in $\mathbf{QCL^{ss}}$).

(i) $\frac{\alpha \models \beta}{\neg\beta \models \neg\alpha}$;
(contraposition for the negation)

(ii) $\frac{\sqrt{\neg}\alpha \models \sqrt{\neg}\mathbf{t}}{\alpha\wedge\beta \models \alpha}$, $\quad \frac{\sqrt{\neg}\beta \models \sqrt{\neg}\mathbf{t}}{\alpha\wedge\beta \models \beta}$;

(iii) $\frac{\sqrt{\neg}\alpha \models \sqrt{\neg}\mathbf{t}}{\mathbf{f}\models\alpha}$.
(Weak Duns Scotus)

Proof: Easy. □

Theorem 67: (Logical consequences of $\mathbf{QCL^w}$ that fail both in $\mathbf{QCL^{ss}}$ and $\mathbf{QCL^s}$).

(i) $\alpha \wedge \beta \models_{\mathbf{QCL^w}} \alpha, \quad \alpha \wedge \beta \models_{\mathbf{QCL^w}} \beta$;

(ii) $\alpha \models_{\mathbf{QCL^w}} \alpha \vee \beta, \quad \beta \models_{\mathbf{QCL^w}} \alpha \vee \beta$;

(iii) $\alpha \wedge \alpha \models_{\mathbf{QCL^w}} \alpha, \quad \alpha \models_{\mathbf{QCL^w}} \alpha \vee \alpha$;
(semiidempotence 1)

(iv) $\mathbf{f} \models_{\mathbf{QCL^w}} \alpha$.
(Duns Scotus)

Proof: Easy. □

Theorem 68: (A rule that holds both in $\mathbf{QCL^s}$ and $\mathbf{QCL^{ss}}$ and fails in $\mathbf{QCL^w}$).

$$\frac{\alpha \equiv \beta}{\sqrt{\neg}\alpha \equiv \sqrt{\neg}\beta}.$$

Proof: Easy. □

In other words, logical equivalence is a congruence for the square root of the negation.

Theorem 69: (A rule that holds in $\mathbf{QCL^{ss}}$ and fails both in $\mathbf{QCL^{s}}$ and $\mathbf{QCL^{w}}$).

$$\frac{\alpha \equiv \beta}{\sqrt{id}\,\alpha \equiv \sqrt{id}\,\beta}.$$

Proof: Easy. □

In other words, logical equivalence is a congruence for the square root of the identity.

Theorem 70: (Logical consequences that fail in **QCL**).

(i) $\alpha \not\models \alpha \wedge \alpha$;
(semiidempotence 2)
(ii) $\mathbf{t} \not\models \alpha \vee \neg\alpha$;
(excluded middle)
(iii) $\mathbf{t} \not\models \neg(\alpha \wedge \neg\alpha)$;
(non contradiction)
(iv) $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \not\models \alpha \wedge (\beta \vee \gamma)$, $\quad \alpha \vee (\beta \wedge \gamma) \not\models (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$.
(distributivity 2)

Proof: Easy. □

Apparently, the logics **QCL** turn out to be non standard forms of quantum logic. Conjunction and disjunction do not correspond to lattice operations, because they are not generally idempotent. Unlike Birkhoff and von Neumann's quantum logic, the weak distributivity principle ($(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \models \alpha \wedge (\beta \vee \gamma)$) breaks down. At the same time, the strong distributivity ($\alpha \wedge (\beta \vee \gamma) \models (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$), that is violated in orthodox quantum logic, is here valid. Both the excluded middle and the non contradiction principles are violated. As a consequence, one can say that the logics arising from quantum computation represent, in a sense, new examples of *fuzzy logics*.

The axiomatizability of **QCL** is an open problem.

7. Quantum trees

An interesting feature of the quantum computational semantics is the following: the *meaning* and the probability-value of any molecular sentence α can be naturally described (and calculated) by means of a convenient *quantum tree*, that illustrates a kind of reversible transformation of the atomic subformulas of α.

The notion of quantum tree can be dealt with either in the framework of the qubit-semantics or in the framework of the qumix-semantics. In the first case quantum trees will be called *qubit-trees*, while in the second case we will speak of *qumix-trees*. Before dealing with quantum trees, we will first introduce the notion of *syntactical tree* of a sentence α (abbreviated as $STree^\alpha$). Consider all subformulas of α.

Any subformula may be:

- an *atomic* sentence $\mathbf{q}$ (possibly $\mathbf{f}$);
- a *negated* sentence $\neg\beta$;
- a *square root negated* sentence $\sqrt{\neg}\beta$;
- a *square root* sentence $\sqrt{id}\,\beta$;
- a *conjunction* $\bigwedge(\beta, \gamma, \mathbf{f})$.

The intuitive idea of *syntactical tree* can be illustrated as follows. Every occurrence of a subformula of α gives rise to a *node* of $STree^\alpha$. The tree consists of a finite number of *levels* and each level is represented by a sequence of subformulas of α:

$$\begin{array}{c} Level_k(\alpha) \\ \vdots \\ Level_1(\alpha) \end{array}$$

The *root-level* (denoted by $Level_1(\alpha)$) consists of α. From each node of the tree at most 3 edges may branch according to the *branching-rule* (Fig. 1).

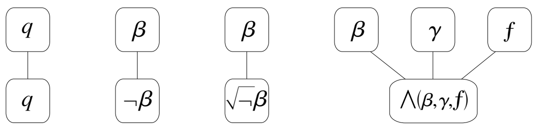

Fig. 1. Branching rules for the construction of syntactical trees.

The second level ($Level_2(\alpha)$) is the sequence of subformulas of α that is obtained by applying the branching-rule to α. The third level ($Level_3(\alpha)$) is obtained by applying the branching-rule to each element (node) of $Level_2(\alpha)$, and so on. Finally, one obtains a level represented by the sequence of all atomic occurrences of α. This represents the *last level* of $STree^\alpha$. The *height* of $Stree^\alpha$ (denoted by $Height(\alpha)$) is then defined as the number of levels of $STree^\alpha$.

A more formal definition of *syntactical tree* can be given by using some standard graph-theoretical notions.

For example, the syntactical tree of $\alpha = \neg\mathbf{q} \wedge (\mathbf{r} \wedge \sqrt{\neg}\mathbf{q})$ is the following (Fig. 2). Clearly the height of $Stree^\alpha$ is 4.

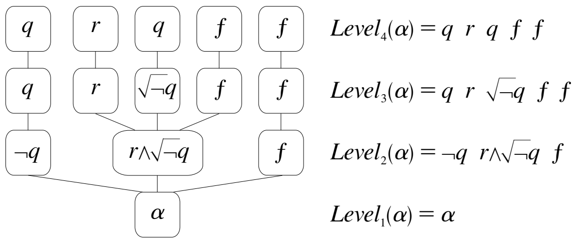

Fig. 2. The syntactical tree of $\alpha = \neg\mathbf{q} \wedge (\mathbf{r} \wedge \sqrt{\neg}\mathbf{q})$.

For any choice of a qubit-model $\texttt{Qub}$, the syntactical tree of α determines a corresponding sequence of quregisters. Consider a sentence α with n atomic occurrences $(\mathbf{q}_1, \dots, \mathbf{q}_n)$. Then $\texttt{Qub}(\alpha) \in \otimes^n \mathbb{C}^2$. We can associate a quregister $|\psi_i\rangle$ to each $Level_i(\alpha)$ of $Stree^\alpha$ in the following way. Suppose that:

$$Level_i(\alpha) = (\beta_1, \dots, \beta_r).$$

Then:

$$|\psi_i\rangle = \texttt{Qub}(\beta_1) \otimes \dots \otimes \texttt{Qub}(\beta_r).$$

Hence:

$$\begin{array}{c} |\psi_{Height(\alpha)}\rangle = \texttt{Qub}(\mathbf{q}_1) \otimes \dots \otimes \texttt{Qub}(\mathbf{q}_n) \\ \vdots \\ |\psi_1\rangle = \texttt{Qub}(\alpha) \end{array}$$

where all $|\psi_i\rangle$ belong to the same space $\otimes^n \mathbb{C}^2$.

From an intuitive point of view, $|\psi_{Height(\alpha)}\rangle$ can be regarded as a kind of *epistemic state*, corresponding to the input of a computation, while $|\psi_1\rangle$ represents the output.

We obtain the following correspondence:

$$\begin{array}{c} Level_{Height(\alpha)}(\alpha) \leftrightsquigarrow |\psi_{Height(\alpha)}\rangle : \textit{ the input} \\ \dots \leftrightsquigarrow \dots \\ Level_1(\alpha) \leftrightsquigarrow |\psi_1\rangle : \textit{ the output} \end{array}$$

The notion of *qubit-tree* of a sentence α ($QubTree^\alpha$) can be now defined as a particular sequence of unitary operators that is uniquely determined by the syntactical tree of α. As we already know, each $Level_i(\alpha)$ of $STree^\alpha$ is a sequence of subformulas of α. Let $Level_i^j(\alpha)$ represent the j-th element of $Level_i(\alpha)$. Each node $Level_i^j(\alpha)$ (where $1 \leq i < Height(\alpha)$) can be naturally associated to a unitary operator Op_i^j, according to the following *operator-rule*:

$$Op_i^j := \begin{cases} I^{(1)} & \textit{if } Level_i^j(\alpha) \textit{ is an atomic sentence;} \\ \mathtt{Not}^{(r)} & \textit{if } Level_i^j(\alpha) = \neg\beta \textit{ and } \mathtt{Qub}(\beta) \in \otimes^r\mathbb{C}^2; \\ \sqrt{\mathtt{Not}}^{(r)} & \textit{if } Level_i^j(\alpha) = \sqrt{\neg}\beta \textit{ and } \mathtt{Qub}(\beta) \in \otimes^r\mathbb{C}^2; \\ \sqrt{\mathtt{I}}^{(r)} & \textit{if } Level_i^j(\alpha) = \sqrt{id}\,\beta \textit{ and } \mathtt{Qub}(\beta) \in \otimes^r\mathbb{C}^2; \\ T^{(r,s,1)} & \textit{if } Level_i^j(\alpha) = \bigwedge(\beta,\gamma,\mathbf{f}),\ \mathtt{Qub}(\beta) \in \otimes^r\mathbb{C}^2 \textit{ and } \mathtt{Qub}(\gamma) \in \otimes^s\mathbb{C}^2. \end{cases}$$

On this basis, one can associate an operator U_i to each $Level_i(\alpha)$ (such that $1 \leq i < Height(\alpha)$):

$$U_i := \bigotimes_{j=1}^{|Level_i(\alpha)|} Op_i^j,$$

where $|Level_i(\alpha)|$ is the length of the sequence $Level_i(\alpha)$.

Being the tensor product of unitary operators, every U_i turns out to be a unitary operator. One can easily show that all U_i are defined on the same space $\otimes^n\mathbb{C}^2$, where n is the atomic complexity of α.

The notion of *qubit-tree* of a sentence can be now defined as follows.

Definition 71: (The qubit-tree of α).
The *qubit-tree* of α (denoted by $QubTree^\alpha$) is the operator-sequence

$$(U_1, \ldots, U_{Height(\alpha)-1})$$

that is uniquely determined by the syntactical tree of α.

As an example, consider the following sentence: $\alpha = \mathbf{q} \wedge \neg\mathbf{q} = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. The syntactical tree of α is the following:

$$\begin{aligned} &Level_3(\alpha) = (\mathbf{q}, \mathbf{q}, \mathbf{f}); \\ &Level_2(\alpha) = (\mathbf{q}, \neg\mathbf{q}, \mathbf{f}); \\ &Level_1(\alpha) = \textstyle\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}). \end{aligned}$$

In order to construct the qubit-tree of α, let us first determine the operators Op_i^j corresponding to each node of $Stree^\alpha$. We will obtain:

- $Op_1^1 = T^{(1,1,1)}$, because $\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ is connected with $(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ (at $Level_2(\alpha)$);
- $Op_2^1 = I^{(1)}$, because $\mathbf{q}$ is connected with $\mathbf{q}$ (at $Level_3(\alpha)$);
- $Op_2^2 = \mathtt{Not}^{(1)}$, because $\neg\mathbf{q}$ is connected with $\mathbf{q}$ (at $Level_3(\alpha)$);
- $Op_2^3 = I^{(1)}$, because $\mathbf{f}$ is connected with $\mathbf{f}$ (at $Level_3(\alpha)$).

The qubit-tree of α is represented by the operator-sequence (U_1, U_2), where:

$$\begin{aligned} &U_2 = Op_2^1 \otimes Op_2^2 \otimes Op_2^3 = I^{(1)} \otimes \mathtt{Not}^{(1)} \otimes I^{(1)}; \\ &U_1 = Op_1^1 = T^{(1,1,1)}. \end{aligned}$$

Apparently, $QubTree^\alpha$ is independent of the choice of $\mathtt{Qub}$.

Theorem 72: Let α be a sentence whose qubit-tree is the operator-sequence $(U_1, \ldots, U_{Height(\alpha)-1})$. Given a qubit-model $\mathtt{Qub}$, consider the quregister-sequence $(|\psi_1\rangle, \ldots, |\psi_{Height(\alpha)}\rangle)$ that is determined by $\mathtt{Qub}$ and by the syntactical tree of α. Then, $U_i(|\psi_{i+1}\rangle) = |\psi_i\rangle$ (for any i such that $1 \leq i < Height(\alpha)$).

Proof: Straightforward. □

The qubit-tree of α can be naturally regarded as a *quantum circuit* that computes the output $\mathtt{Qub}(\alpha)$, given the input $\mathtt{Qub}(\mathbf{q}_1), \ldots, \mathtt{Qub}(\mathbf{q}_n)$ (where $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are the atomic occurrences of α). In this framework, each U_i is the unitary operator that describes the computation performed by the i-th layer of the circuit.

Let us now turn to the notion of qumix-tree. Consider a sentence α, its syntactical tree $STree^\alpha$ and its qubit-tree $QubTree^\alpha$. Suppose that $At(\alpha) = n$. The syntactical tree of α will have the following form:

$$\begin{gathered} Level_k(\alpha) = \mathbf{q}_1, \ldots, \mathbf{q}_n \\ \vdots \\ Level_1(\alpha) = \alpha \end{gathered}$$

where k is the height of $STree^\alpha$ and $\mathbf{q}_1, \ldots \mathbf{q}_n$ are the atomic sentences occurring in α. At the same time the qubit-tree of α will have the following form:

$$U_1, \ldots, U_{k-1},$$

where each U_i $(1 \leq i \leq k-1)$ is a unitary operator of $\otimes^n \mathbb{C}^2$, which represents the "semantic space" of α.

Let $\mathtt{Qub}$ be a qubit-model of the language $\mathcal{L}$. We have:

$$\mathtt{Qub}(\alpha) \in \otimes^n \mathbb{C}^2.$$

Let $Level_i(\alpha) = \beta_1, \ldots, \beta_r$ be the i-th level of $STree^\alpha$. We will briefly write: $\mathtt{Qub}(Level_i(\alpha))$ for $\mathtt{Qub}(\beta_1) \otimes \ldots \otimes \mathtt{Qub}(\beta_r)$. Hence, we obtain:

$$\begin{gathered} \mathtt{Qub}(Level_k(\alpha)) = \mathtt{Qub}(\mathbf{q}_1) \otimes \ldots \otimes \mathtt{Qub}(\mathbf{q}_n) \\ \ldots \\ \mathtt{Qub}(Level_1(\alpha)) = \mathtt{Qub}(\alpha) \end{gathered}$$

By Theorem 72, we have:

$$U_i(\mathtt{Qub}(Level_{i+1}(\alpha))) = \mathtt{Qub}(Level_i(\alpha)).$$

We will now generalize the qubit-tree representation to the qumix-semantics. Consider a model Qum. Suppose again that $Level_i(\alpha) = \beta_1, \ldots, \beta_r$. Like in the case of qubit-models, we will briefly write $Qum(Level_i(\alpha))$ for $Qum(\beta_1) \otimes \ldots \otimes Qum(\beta_r)$. Define now, the following sequence of functions on the set $\mathfrak{D}(\otimes^n \mathbb{C}^2)$:

$$\begin{gathered} {}^{\mathcal{D}}U_{k-1}(\rho) = U_{k-1}\, \rho\, U_{k-1}^* \\ \ldots \\ {}^{\mathcal{D}}U_1(\rho) = U_1\, \rho\, U_1^*, \end{gathered}$$

where $(U_1, \ldots, U_{k-1})$ is the qubit-tree of α.

Lemma 73: *For any* $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$, ${}^{\mathcal{D}}U_i(\rho)$ *is a density operator of* $\mathfrak{D}(\otimes^n \mathbb{C}^2)$.

Lemma 74: ${}^{\mathcal{D}}U_i(Qum(Level_{i+1}(\alpha)) = Qum(Level_i(\alpha))$.

The sequence $({}^{\mathcal{D}}U_1, \ldots, {}^{\mathcal{D}}U_{k-1})$ (obtained from the qubit-tree $(U_1, \ldots, U_{k-1})$) will be also called the *qumix-tree* of α (and will be indicated by $QumTree^\alpha$).

Any U_i of $QubTree^\alpha$ is a unitary operator. Hence its inverse U_i^{-1} is a unitary operator. We have for any $\mathtt{Qub}$:

$$U_i^{-1}(\mathtt{Qub}(Level_i(\alpha))) = \mathtt{Qub}(Level_{i+1}(\alpha)).$$

One can easily show that for any i $(1 \leq i \leq k-1)$, ${}^{\mathcal{D}}U_i^{-1}$ is a function. Consider the sequence

$$({}^{\mathcal{D}}U_1^{-1}, \ldots, {}^{\mathcal{D}}U_{k-1}^{-1}).$$

Like in the pure case, one can prove:

$${}^{\mathcal{D}}U_i^{-1}(Qum(Level_i(\alpha))) = Qum(Level_{i+1}(\alpha)).$$

8. Holistic semantics and entanglement

The quantum computational semantics we have investigated so far is typically *non-holistic* (*compositional*). As happens in the case of standard classical semantics, *the meaning of a molecular sentence is determined by the meanings of its parts.* As a consequence, in this framework, the meaning of a molecular α cannot be a pure state, when some atomic parts of α are proper mixtures. An interesting question arises: is it possible to generalize the quantum computational semantics in order to represent some typical quantum holistic situations? For instance, a significant case would be the following: the meaning of a molecular α is a *maximal* information quantity that corresponds to an *entangled state*, while the meanings of the atomic parts are proper mixtures (*non-maximal* pieces of information).

Definition 75: (Holistic pseudo-model).
A *holistic pseudo-model* of the language $\mathcal{L}$ is a map $\mathtt{Hol} : Form^{\mathcal{L}} \to \mathfrak{D}$ s.t. for any sentence α whose atomic complexity is n:

$$\mathtt{Hol}(\alpha) \in \mathfrak{D}(\otimes^n \mathbb{C}^2).$$

$\mathtt{Hol}(\alpha)$ reflects the atomic complexity, but not the logical form of α!

Consider now a sentence α whose atomic complexity is n. The syntactical tree and the qumix-tree of α will have the following form (where k is the height of the tree):

- $STree^\alpha = \begin{array}{l} Level_k(\alpha) = \mathbf{q}_1, \ldots, \mathbf{q}_n \\ \ldots \\ Level_1(\alpha) = \alpha. \end{array}$

- $QumTree^{\alpha} = ({}^{\mathcal{D}}U_1, \ldots, {}^{\mathcal{D}}U_{k-1})$.

For any choice of a density operator ρ in $\mathfrak{D}(\otimes^n \mathbb{C}^2)$, $QumTree^{\alpha}$ determines the following sequence of density operators:

$$\begin{aligned}
&\rho_k = {}^{\mathfrak{D}}U_{k-1}^{-1}(\rho_{k-1}) \\
&\ldots \\
&\rho_2 = {}^{\mathfrak{D}}U_1^{-1}(\rho_1) \\
&\rho_1 = \rho
\end{aligned}$$

Suppose that $Level_i(\alpha) = (Level_i^1(\alpha), \ldots, Level_i^r(\alpha)) = (\beta_1, \ldots, \beta_r)$.

Clearly, the space $\mathcal{H}^{\alpha}$ (the semantic space of α) can be represented as the following tensor product:

$$\mathcal{H}^{\alpha} = \mathcal{H}^{Level_i^1(\alpha)} \otimes \ldots \otimes \mathcal{H}^{Level_i^r(\alpha)}, \ \textit{where}:$$

$$\mathcal{H}^{Level_i^j(\alpha)} = \mathcal{H}^{\beta_j}.$$

Of course, the space $\mathcal{H}^{\beta_j}$ (the semantic space of β_j), is a subspace of $\mathcal{H}^{\alpha}$.

Consider now $red^j(\rho_i)$, the reduced state of ρ_i with respect to the j-th subsystem. Clearly, $red^j(\rho_i) \in \mathfrak{D}(\mathcal{H}^{Level_i^j(\alpha)})$. Hence, $red^j(\rho_i)$ can be regarded as a *possible meaning* of the sentence β_j.

Suppose that the pseudo model `Hol` associates to α the qumix ρ_1, i.e.:

$$\texttt{Hol}(\alpha) = \rho_1.$$

Then, the reduced state $red^j(\rho_i)$ can be naturally regarded as the *contextual meaning* of the occurrence β_j (at the node $Level_i^j(\alpha)$) under the *global interpretation* $\texttt{Hol}(\alpha)$. We write:

$$\texttt{Hol}^{\alpha}(Level_i^j(\alpha)) = red^j(\rho_i).$$

It is worthwhile noticing that different occurrences of the same subformula may receive different *contextual meanings*!

Definition 76: (Holistic model of a sentence).
A holistic pseudo-model `Hol` of the language $\mathcal{L}$ is a *holistic model* of a sentence α with atomic occurrences $\mathbf{q}_1, \ldots, \mathbf{q}_n$ iff the following condition holds (for any $j \in \{1, \ldots, n\}$):

$$\textit{if } \mathbf{q}_j = \mathbf{f}, \ \textit{then } \texttt{Hol}^{\alpha}(\mathbf{q}_j) = P_0.$$

In other words, the contextual meaning of the false sentence $\mathbf{f}$ is the truth-value *Falsity* (P_0). This condition guarantees that conjunctions and disjunctions are well behaved.

Notice that generally:

$$\texttt{Hol}(\mathbf{q}_j) \neq \texttt{Hol}^{\alpha}(\mathbf{q}_j).$$

From an intuitive point of view, $\texttt{Hol}(\mathbf{q}_j)$ can be regarded as the *standard (non-contextual) meaning* under the *global interpretation* `Hol` (a kind of *first meaning* in a dictionary). At the same time, $\mathtt{Hol}^\alpha(\mathbf{q}_j)$ represents the *contextual meaning* of $\mathbf{q}_j$ in the *semantic environment* $\mathtt{Hol}(\alpha)$.

All this gives rise to a typically *holistic semantic situation*: **the meaning of the whole determines the contextual meanings of its parts, but not vice versa.**

The following situation is possible

- α is a sentence with atomic occurrences $\mathbf{q}_1, \ldots, \mathbf{q}_n$;
- $\mathtt{Hol}(\alpha)$ is a *pure state*;
- $\mathtt{Hol}^\alpha(\mathbf{q}_1), \ldots, \mathtt{Hol}^\alpha(\mathbf{q}_n)$ are *non pure*.

As a consequence, if we invert the direction of our procedure, going from the *parts* to the *whole* (instead of from the *whole* to the *parts*), we obtain a final result that is different from the original pure state $\mathtt{Hol}(\alpha)$. In fact, the $QumTree$ determined by $STree^\alpha$ and by $\mathtt{Hol}^\alpha(\mathbf{q}_1), \ldots, \mathtt{Hol}^\alpha(\mathbf{q}_n)$ gives rise to a proper mixture that is necessarily different from the pure state $\mathtt{Hol}(\alpha)$.

Of course, compositional models turn out to be special cases of holistic models.

Definition 77: (Compositional model with respect to a sentence).
A holistic model $\mathtt{Hol}$ is called *compositional with respect to a sentence* α iff there exists a compositional model $\mathtt{Qum}$ s.t. for any node $Level_i^j(\alpha)$ in $STree^\alpha$:

$$\mathtt{Hol}(Level_i^j(\alpha)) = \mathtt{Qum}(Level_i^j(\alpha)).$$

The holistic semantics represents a natural environment that permits us to study *entangled meanings*. For instance, the meaning of a sentence α might have the typical form of a *singlet-state* (as happens in the case of EPR-like situations).

Example: (A singlet-meaning).
Consider the following sentence:

$$\alpha = \bigwedge(\mathbf{q}, \mathbf{q}, \mathbf{f}).$$

The syntactical tree of α is:

$$Level_2(\alpha) = (\mathbf{q}, \mathbf{q}, \mathbf{f})$$
$$Level_1(\alpha) = \bigwedge(\mathbf{q}, \mathbf{q}, \mathbf{f}).$$

The qubit-tree of α is:

$$(U_1),$$

where $U_1 = T$ (the Petri-Toffoli-gate).

Consider now any holistic pseudo-model $\mathtt{Hol}$ such that:

$$\mathtt{Hol}(\alpha) = P_{|\psi\rangle},$$

where

$$|\psi\rangle = \frac{1}{\sqrt{3}}|100\rangle + \sqrt{\frac{2}{3}}|010\rangle.$$

One can easily show that $\mathtt{Hol}$ is a holistic model of α. The syntactical tree of α and $\mathtt{Hol}$ determine the following sequence of quregisters:

$$|\psi_2\rangle = T^{-1}(|\psi_1\rangle) = \tfrac{1}{\sqrt{3}}|100\rangle + \sqrt{\tfrac{2}{3}}|010\rangle$$
$$|\psi_1\rangle = \tfrac{1}{\sqrt{3}}|100\rangle + \sqrt{\tfrac{2}{3}}|010\rangle$$

Hence: $|\psi_1\rangle = |\psi_2\rangle$.

At the same time, the atomic parts of α receive the following *contextual meanings*:

- $\mathtt{Hol}^\alpha(Level_2^1(\alpha)) = \mathtt{Hol}^\alpha(\mathbf{q}) = red^1(P_{|\psi_2\rangle}) = \frac{2}{3}P_0 + \frac{1}{3}P_1$
- $\mathtt{Hol}^\alpha(Level_2^2(\alpha)) = \mathtt{Hol}^\alpha(\mathbf{q}) = red^2(P_{|\psi_2\rangle}) = \frac{1}{3}P_0 + \frac{2}{3}P_1$
- $\mathtt{Hol}^\alpha(Level_2^3(\alpha)) = \mathtt{Hol}^\alpha(\mathbf{f}) = red^3(P_{|\psi_2\rangle}) = P_0$.

Consider now any other holistic pseudo model $\widehat{\mathtt{Hol}}$ such that:

$$\widehat{\mathtt{Hol}}(\alpha) = \frac{1}{3}P_1 \otimes P_0 \otimes P_0 + \frac{2}{3}P_0 \otimes P_1 \otimes P_0.$$

$\widehat{\mathtt{Hol}}$ is a holistic model of α. The atomic parts of α receive the following *contextual meanings*:

- $\widehat{\mathtt{Hol}}^\alpha(Level_2^1(\alpha)) = \widehat{\mathtt{Hol}}^\alpha(\mathbf{q}) = red^1\left(\frac{1}{3}P_1 \otimes P_0 \otimes P_0 + \frac{2}{3}P_0 \otimes P_1 \otimes P_0\right)$
$= \frac{2}{3}P_0 + \frac{1}{3}P_1$
- $\widehat{\mathtt{Hol}}^\alpha(Level_2^2(\alpha)) = \widehat{\mathtt{Hol}}^\alpha(\mathbf{q}) = red^2\left(\frac{1}{3}P_1 \otimes P_0 \otimes P_0 + \frac{2}{3}P_0 \otimes P_1 \otimes P_0\right)$
$= \frac{1}{3}P_0 + \frac{2}{3}P_1$
- $\widehat{\mathtt{Hol}}^\alpha(Level_2^3(\alpha)) = \widehat{\mathtt{Hol}}^\alpha(\mathbf{f}) = red^3\left(\frac{1}{3}P_1 \otimes P_0 \otimes P_0 + \frac{2}{3}P_0 \otimes P_1 \otimes P_0\right) = P_0$.

One can easily show that both $\mathtt{Hol}(\alpha)$ and $\widehat{\mathtt{Hol}}(\alpha)$ are not compositional with respect to α. We have: $\mathtt{Hol}(\alpha) \neq \widehat{\mathtt{Hol}}(\alpha)$. At the same time, the atomic parts of α receive the same contextual meanings.

The example of the singlet meaning (described above) represents a paradigmatic entangled semantic situation. The molecular sentence $\alpha = \bigwedge(\mathbf{q}, \mathbf{q}, \mathbf{f})$ has a global meaning, $\mathtt{Hol}(\alpha)$, that is a maximal information. At the same time, two parts of α (two different occurrences of the same atomic sentence $\mathbf{q}$) have two different (ambiguous) contextual meanings that are represented by two different mixed states ($\frac{2}{3}P_0 + \frac{1}{3}P_1$ and $\frac{1}{3}P_0 + \frac{2}{3}P_1$). These contextual meanings turn out to be also compatible with other global meanings of α (for instance, with the qumix $\widehat{\mathtt{Hol}}(\alpha)$, which is different from the pure $\mathtt{Hol}(\alpha)$). Hence, the global meaning of α determines the meanings of its parts, but not the other way around.

9. Physical models of QCL by means of Mach–Zehnder Interferometers

The conventional Mach–Zehnder (MZ) interferometer (sketched in Fig. 3) involves three essential components: symmetric *50:50 beam-splitters* (BS), relative *phase* *shifters* (PS) along the x-direction and *mirrors* (M)[11].

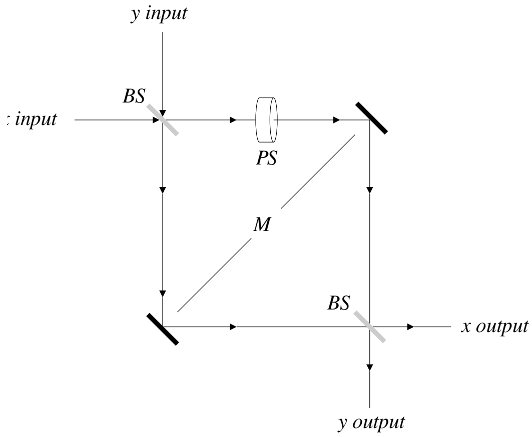

Fig. 3. Mach-Zehnder interferometer on the Hilbert space $\mathbb{C}^2$.

- A beam-splitter can be built by means of a partially silvered piece of glass, which reflects a fraction R of the incident light, and transmits $T = 1 - R$.
- A phase shifter can be built by means of a slab of transparent medium with index of refraction n different from n_0, the index of refraction of free space. Propagation in such a medium through a distance L changes a photon phase by e^{ikL}, where $k = n\omega/c_0$, and c_0 is the speed of light in vacuum.
- Highly reflective mirrors reflect photons and change their propagation direction in space. Mirrors with 0.01% loss are not unusual[12].

The standard quantum description of this scenario is based on the Hilbert space $\mathbb{C}^2$, where the basis-vectors $|0\rangle$ and $|1\rangle$ are supposed to describe photons (wave packets) that move along two given directions defined by the geometry of the interferometer. We assume that:

- $|1\rangle$ is the pure state representing the wave packet moving along the y-direction;
- $|0\rangle$ is the pure state representing the wave packet moving along the x-direction.

In this framework, 50:50 beam-splitters, relative phase shifters and mirrors are described by the following unitary operators:

$$U_{BS} = \sqrt{\texttt{Not}} \qquad U_x(\vartheta) = \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & 1 \end{pmatrix} \qquad U_M = \texttt{Not}.$$

The block diagram corresponding to the Mach–Zehnder interferometer (represented in Fig. 3) is then the following:

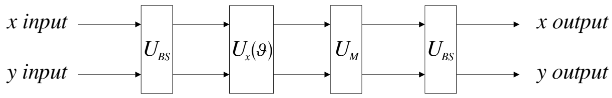

Fig. 4. Block diagram of the Mach–Zehnder interferometer.

Consequently, the global MZ interferometer is mathematically described by the following unitary operator (acting on $\mathbb{C}^2$):

$$U_{MZ}(\vartheta) = U_{BS} \circ U_M \circ U_x(\vartheta) \circ U_{BS} = \frac{1}{2}\begin{pmatrix} 1+e^{i\vartheta} & -i(1-e^{i\vartheta}) \\ i(1-e^{i\vartheta}) & 1+e^{i\vartheta} \end{pmatrix}.$$

Consider an atomic sentence $\mathbf{q}$ (of the language of **QCL**) asserting that "the wave packet moves along the y-direction". The natural semantic interpretation of $\mathbf{q}$ will be the following:

- $\mathtt{Qum}(\mathbf{q}) = P_1$, if the wave packet actually moves along the y-direction;
- $\mathtt{Qum}(\mathbf{q}) = P_0$, if the wave packet actually moves along the x-direction.

Apparently, the projection P_1 represents, at the same time, the pure state of a photon moving along the y-direction and a classical bit (which gives the answer "Yes" to the question "Does the wave packet move along the y-direction?").

Consider now the following molecular sentence: $\alpha = \sqrt{\neg}\,\neg\,\sqrt{\neg}\mathbf{q}$. Suppose the source sends a single photon along the y-direction into the MZ device with $\vartheta = 0$. Hence, according to our semantic convention, we have: $\mathtt{Qum}(\mathbf{q}) = P_1$. Consequently, the sentence $\sqrt{\neg}\mathbf{q}$ turns out to describe the internal interferometer state, corresponding to a quantum superposition of the two possible paths available to the single photon, before the mirror-action. We have: $\mathtt{Qum}(\sqrt{\neg}\mathbf{q}) = \sqrt{\mathtt{NOT}}(P_1) = P_{\frac{1-i}{2}|0\rangle+\frac{1+i}{2}|1\rangle}$. The sentence $\neg\sqrt{\neg}\mathbf{q}$, instead, describes the state of the photon after the mirror-action. We have: $\mathtt{Qum}(\neg\sqrt{\neg}\mathbf{q}) = \mathtt{NOT}(\sqrt{\mathtt{NOT}}(P_1)) = P_{\frac{1+i}{2}|0\rangle+\frac{1-i}{2}|1\rangle}$. What about the final state of the photon, after the action of the second beam splitter? According to our semantic rules, we obtain:

$$\mathtt{Qum}(\alpha) = \mathtt{Qum}(\sqrt{\neg}\,\neg\,\sqrt{\neg}\mathbf{q}) = \sqrt{\mathtt{NOT}}(\mathtt{NOT}(\sqrt{\mathtt{NOT}}(P_1))) = P_1.$$

In other words, the outgoing photon is along the y-direction, and this result agrees with the experimental evidence. Interestingly enough, the internal interferometer state could not be analyzed in terms of classical or fuzzy logics, because, as we have learnt, the square root of the negation does not have any Boolean or fuzzy counterpart.

What happens if we try to analyze the internal interferometer state by *observing* the presence of the photon in one arm? In such a case, the state of the photon before the action of the second beam splitter is represented by the density operator $\rho_{1/2}$. Accordingly, the final state of the outgoing photon will be $\rho_{1/2} \neq P_1 = \mathtt{Qum}(\alpha)$. Notice that the transformation $P_1 \mapsto \rho_{1/2}$ cannot be described by a gate (which is, by definition, a unitary operator).

In the top–down approach of the Holistic semantics we can invert the procedure that is characteristic of the compositional semantics. We can start by providing the *global meaning* of the sentence $\alpha = \sqrt{\neg}\,\neg\sqrt{\neg}\mathbf{q}$. Suppose, for example, that $\mathtt{Hol}(\alpha) = P_1$, which corresponds to sending a single photon along the y-direction backward in the MZ interferometer. The contextual meanings of the subformulas of α are determined as follows:

$$\begin{aligned} Level_2^1(\alpha) = \neg\sqrt{\neg}\mathbf{q} &\leftrightsquigarrow \mathtt{Hol}^\alpha(\neg\sqrt{\neg}\mathbf{q}) = P_{\frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle} \\ Level_3^1(\alpha) = \sqrt{\neg}\mathbf{q} &\leftrightsquigarrow \mathtt{Hol}^\alpha(\sqrt{\neg}\mathbf{q}) = P_{\frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle} \\ Level_4^1(\alpha) = \mathbf{q} &\leftrightsquigarrow \mathtt{Hol}^\alpha(\mathbf{q}) = P_1. \end{aligned}$$

So far we have considered physical models for the connectives $\neg$ and $\sqrt{\neg}$. We know that, in the simplest situation, the corresponding gates are defined on the space $\mathbb{C}^2$. How to deal with physical models of the conjunction, whose corresponding gate (the Petri-Toffoli gate) refers, in the simplest situation, to the space $\otimes^3\mathbb{C}^2$? The idea is to use the *conditional Kerr–Mach–Zehnder interferometer* (CKMZ). Such interferometer involves three components: symmetric *50:50 beam-splitters* (BS), relative *conditional phase shifters* (CPS) along the x-direction and *mirrors* (M). The main difference with respect to the standard Mach–Zehnder (outlined in Fig. 3) is the use of *Kerr's effect* to produce intensity–dependent phase shift. A substance with an intensity dependent refractive index (optical Kerr effect) is placed in both arms of the device. In such a medium the field encounters a refractive index which changes according to the field intensity; as a consequence, an intensity dependent phase shift is obtained[13].

A physical model of the Petri-Toffoli gate based on a CKMZ interferometer is a three–input/three–output device, corresponding to a unitary operator acting on the space $\otimes^3\mathbb{C}^2$. In this framework, 50:50 beam-splitters and mirrors are described by the following unitary operators (defined on $\mathbb{C}^2$):

$$U_{BS_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad U_{BS_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad U_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The conditional phase shifter is described by the unitary operator U_{CPS} that is defined for any element $|x, y, z\rangle$ of the computational basis of $\otimes^3\mathbb{C}^2$ as follows:

$$U_{CPS}(|x, y, z\rangle) = e^{i\pi xy(1-z)}|x, y, z\rangle.$$

The block diagram corresponding to the CKMZ interferometer is represented in Fig. 5.

Consequently, the global CKMZ interferometer is mathematically described by the following unitary operator acting on the space $\otimes^3\mathbb{C}^2$:

$$U_{CKMZ} = (I \otimes I \otimes U_{BS_2}) \circ (I \otimes I \otimes U_M) \circ U_{CPS} \circ (I \otimes I \otimes U_{BS_1}).$$

One can easily show that $U_{CKMZ} = T^{(1,1,1)}$. Hence the global CKMZ interferometer permits us to realize the Petri-Toffoli gate.

As an example, consider the following sentence: $\alpha = \bigwedge(\mathbf{q}, \mathbf{q}, \mathbf{f})$. The two different occurrences of the atomic sentence $\mathbf{q}$ are physically interpreted by two photons,

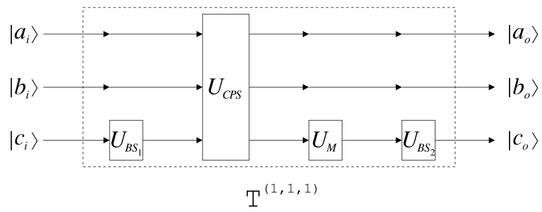

Fig. 5. Block diagram of the Mach–Zehnder interferometer with a conditional phase shifter.

prepared in the same state. Suppose that the source sends two photons randomly along the two directions x and y. Hence, we have: $\mathtt{Qum}(\mathbf{q}) = \rho_{1/2}$ (where $\rho_{1/2} \in \mathfrak{D}(\mathbb{C}^2)$). Suppose that a third incoming photon moves along the x-direction. Such a photon (which is in the pure state P_0) can be regarded as an *ancilla photon*, representing the physical interpretation of the false sentence $\mathbf{f}$ (i.e. $\mathtt{Qum}(\mathbf{q}) = P_0$). We obtain:

$$\mathtt{Qum}(\alpha) = \frac{1}{4}(P_0 \otimes P_0 \otimes P_0 + P_0 \otimes P_1 \otimes P_0 + P_1 \otimes P_0 \otimes P_0 + P_1 \otimes P_1 \otimes P_1).$$

In other words, the target photon will go out of the CKMZ interferometer along the y direction with probability $\frac{1}{4}$. Hence, the probability of the truth of the conjunction $\mathbf{q} \wedge \mathbf{q}$ is $\frac{1}{4}$; at the same time, the probability of the truth of the single sentence $\mathbf{q}$ is $\frac{1}{2}$.

In the framework of the holistic semantics we can represent a different situation. The physical interpretation of the global sentence α may be a pure state. For instance, we might have: $\mathtt{Hol}(\alpha) = P_{\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle} \otimes P_0$, which corresponds to sending a dual–rail single photon[12] along the y-direction and an ancilla photon along the x-direction backward.

The state of the dual–rail single photon is the superposition $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, while the state of the ancilla photon is $|0\rangle$. The contextual meanings of the subformulas of α are the following:

$$\begin{array}{l} Level_2^1(\alpha) = \mathbf{q} \leftrightsquigarrow \mathtt{Hol}^\alpha(\mathbf{q}) = \rho_{1/2} \\ Level_2^2(\alpha) = \mathbf{q} \leftrightsquigarrow \mathtt{Hol}^\alpha(\mathbf{q}) = \rho_{1/2} \\ Level_2^3(\alpha) = \mathbf{f} \leftrightsquigarrow \mathtt{Hol}^\alpha(\mathbf{f}) = P_0. \end{array}$$

In other words, the compositional model Qum and the holistic model Hol associate two different interpretations to the global sentence α. At the same time, the two models Qum and Hol associate the same meanings to the parts of α.

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