# QUANTUM MEASURE THEORY

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**Abstract**

We first present some basic properties of a quantum measure space. Compatibility of sets with respect to a quantum measure is studied and the center of a quantum measure space is characterized. We characterize quantum measures in terms of signed product measures. A generalization called a super-quantum measure space is introduced. Of a more speculative nature, we show that quantum measures may be useful for computing and predicting elementary particle masses.

## 1 Introduction

Quantum measure spaces ($q$-measure spaces, for short) were introduced by R. Sorkin in his studies of the histories approach to quantum mechanics and its applications to quantum gravity and cosmology [9]. Since then a few other papers have appeared on the subject [8, 10, 11]. These investigators have been concerned with finite $q$-measure spaces in which the number of sample points is finite and the general definition of a $q$-measure space has not been given. Our first order of business is to present such a definition. After a preliminary study of the basic properties of a $q$-measure space, there are three main results in this paper. We define compatibility of sets with respect to a $q$-measure and characterize the center of a $q$-measure space. We then characterize $q$-measures in terms of signed product measure. Finally, of a more speculative nature, we show that $q$-measures may be useful for computing and predicting elementary particle masses. We briefly consider super $q$-measure spaces which generalize $q$-measure spaces just as $q$-measure spaces generalize classical measure spaces.

# 2 Basic Properties

As usual a **measurable space** is a pair $(X, \mathcal{A})$ where $X$ is a nonempty set and $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$. If $A$ and $B$ are disjoint sets, we use the notation $A \cupdot B$ for their union. Similarly, we write $\cupdot A_i$ for the union of a sequence of mutually disjoint sets $A_i$. Denoting the set of nonnegative real numbers by $\mathbb{R}^+$, a set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is **additive** if $\mu(A \cupdot B) = \mu(A) + \mu(B)$ for all disjoint $A, B \in \mathcal{A}$ and $\mu$ is **countably additive** if $\mu(\cup A_i) = \sum \mu(A_i)$ for any sequence of mutually disjoint $A_i \in \mathcal{A}$. It is well-known that $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is countably additive if and only if $\mu$ is additive and $\lim \mu(A_i) = \mu(\cup A_i)$ for any increasing sequence $(A_i \subseteq A_{i+1})$ of sets $A_i \in \mathcal{A}$ [1, 2, 6]. If $\mu$ is countably additive, we call $\mu$ a **measure** and we call the triple $(X, \mathcal{A}, \mu)$ a **measure space**. For reasons that will become apparent later, an additive set function is called **grade**-1 **additive** and a measure space is called a **grade**-1 **measure space**. If we replace $\mathbb{R}^+$ by $\mathbb{R}$ or $\mathbb{C}$ we also have the concepts of a **signed measure** and a **complex measure**, respectively.

We now introduce a generalization of additivity. A set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is **grade**-2 **additive** if

$$\mu(A \cupdot B \cupdot C) = \mu(A \cupdot B) + \mu(A \cupdot C) + \mu(B \cupdot C) - \mu(A) - \mu(B) - \mu(C) \quad (2.1)$$

and $\mu$ is **regular** if the following two conditions hold

$$\mu(A) = 0 \Rightarrow \mu(A \cupdot B) = \mu(B)$$
$$\mu(A \cupdot B) = 0 \Rightarrow \mu(A) = \mu(B)$$

It follows from (2.1) that any grade-2 additive function $\mu$ satisfies $\mu(\emptyset) = 0$. It is easy to check that if $\mu$ is grade-1 additive, then $\mu$ is regular and grade-2 additive. We say that $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is **continuous** if $\lim \mu(A_i) = \mu(\cup A_i)$ for every increasing sequence $A_i \in \mathcal{A}$ and $\lim \mu(B_i) = \mu(\cap B_i)$ for every decreasing sequence $B_i \in \mathcal{A}$. A continuous grade-2 additive set function is a **grade**-2 **measure** and a regular grade-2 measure is a **quantum measure** ($q$-**measure**, for short). If $\mu$ is a grade-2 measure ($q$-measure), then $(X, \mathcal{A}, \mu)$ is a **grade-2 measure space** (**$q$-measure space**). Of course, a measure space is a $q$-measure space, but there are important examples which show that the converse does not hold.

In various quantum formalisms, a crucial role is played by a **decoherence functional** $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ [3, 4, 5, 7]. This functional (or at least its real part) represents the amount of interference between pairs of sets in $\mathcal{A}$ and has the following properties:

$$D(A\mathbin{\dot\cup} B,C)=D(A,C)+D(B,C) \tag{2.2}$$

$$D(A,B)=\overline{D(B,A)} \tag{2.3}$$

$$D(A,A)\geq 0 \tag{2.4}$$

$$\left|D(A,B)\right|^2\leq D(A,A)D(B,B) \tag{2.5}$$

$$A\mapsto D(A,A)\text{ is continuous} \tag{2.6}$$

As we shall see, $\mu(A)=D(A,A)$ is a $q$-measure for any decoherence functional $D$. An example of a decoherence functional is

$$D(A,B)=\operatorname{tr}\left[WE(A)E(B)\right]$$

where $W$ is a density operator and $E$ is a positive operator-valued measure on a complex Hilbert space. In this case, the $q$-measure $\mu(A)=D(A,A)$ is a measure of the interference of $A\in\mathcal{A}$ with itself for the observable $E$ and state $W$.

A simpler example of a decoherence functional is $D(A,B)=\nu(A)\overline{\nu(B)}$ where $\nu$ is a complex measure on $\mathcal{A}$. In this case $\nu$ is called an **amplitude** and we have the $q$-measure $\mu(A)=\left|\nu(A)\right|^2$. In fact quantum probabilities are frequently computed by taking the modulus squared of a complex amplitude. This example illustrates the nonadditivity of $\mu$ because

$$\begin{aligned}\mu(A\mathbin{\dot\cup} B)&=\left|\nu(A\mathbin{\dot\cup} B)\right|^2=\left|\nu(A)+\nu(B)\right|^2\\&=\mu(A)+\mu(B)+2\operatorname{Re}\left[\nu(A)\overline{\nu(B)}\right]\end{aligned}$$

Hence, $\mu(A\mathbin{\dot\cup} B)=\mu(A)+\mu(B)$ if and only if $\operatorname{Re}\left[\nu(A)\overline{\nu}(B)\right]=0$ or equivalently $\operatorname{Re}D(A,B)=0$.

**Theorem 2.1.** *If $D\colon\mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is a decoherence functional, then $\mu(A)=D(A,A)$ is a $q$-measure on $\mathcal{A}$.*

*Proof.* To prove (2.1), let $R$ be the right side of (2.1) and apply (2.2) and (2.3) to obtain

$$\begin{aligned}R &= D\left(A \cup B, A \cup B\right) + D\left(A \cup C, A \cup C)\right) + D\left(B \cup C, B \cup C\right)\\ &\quad - \mu(A) - \mu(B) - \mu(C)\\ &= 2\left[D(A,A) + D(B,B) + D(C,C) + \operatorname{Re}\left(D(A,B) + D(A,C) + D(B,C)\right)\right]\\ &\quad - \mu(A) - \mu(B) - \mu(C)\\ &= D(A,A) + D(B,B) + D(C,C) + 2\operatorname{Re}\left[D(A,B) + D(A,C) + D(B,C)\right]\\ &= D\left(A \cup B \cup C, A \cup B \cup C\right) = \mu\left(A \cup B \cup C\right)\end{aligned}$$

To prove the first regularity condition, apply (2.2) and (2.3) to obtain

$$\mu(A \cup B) = D\left(A \cup B, A \cup B\right) = \mu(A) + \mu(B) + 2\operatorname{Re} D(A,B)$$

By (2.5) if $\mu(A) = 0$, then $D(A,B) = 0$ so that $\mu(A \cup B) = \mu(B)$. To prove the second regularity condition, applying (2.2)–(2.5) we have

$$\begin{aligned}\mu(A \cup B) &= \mu(A) + \mu(B) + 2\operatorname{Re} D(A,B) \geq \mu(A) + \mu(B) - 2\left|D(A,B)\right|\\ &\geq \mu(A) + \mu(B) - 2\mu(A)^{1/2}\mu(B)^{1/2} = \left[\mu(A)^{1/2} - \mu(B)^{1/2}\right]^2\end{aligned}$$

Hence, $\mu(A \cup B) = 0$ implies that $\mu(A) = \mu(B)$. Finally, continuity of $\mu$ follows from (2.6). $\square$

Part (a) of the next theorem gives a characterization of grade-2 additivity and (b) shows that grade-2 additivity can be extended to more than three mutually disjoint sets [8, 9, 10]. We denote the complement of a set $A$ by $A'$ and the symmetric difference of $A$ and $B$ by

$$A \Delta B = (A \cap B') \cup (A' \cap B)$$

**Theorem 2.2.** (a) *A map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is grade-2 additive if and only if $\mu$ satisfies*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) + \mu(A \Delta B) - \mu(A \cap B') - \mu(A' \cap B) \tag{2.7}$$

(b) *If $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is grade-2 additive, then for any $n \geq 3$ we have*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i<j=1}^{n} \mu(A_i \cup A_j) - (n-2)\sum_{i=1}^{n} \mu(A_i) \tag{2.8}$$

*Proof.* (a) If $\mu$ is grade-2 additive , we have

$$\begin{aligned}\mu(A \cup B) &= \mu\left[(A \cap B') \cup (A' \cap B) \cup (A \cap B)\right] \\ &= \mu(A \Delta B) + \mu(A) + \mu(B) - \mu(A \cap B') - \mu(A' \cap B) - \mu(A \cap B)\end{aligned}$$

which is (2.7). Conversely, if (2.7) holds, then letting $A_1 = A \dot{\cup} C$, $B_1 = B \dot{\cup} C$ we have

$$\begin{aligned}\mu(A \dot{\cup} B \dot{\cup} C) &= \mu(A_1 \cup B_1) = \mu(A_1) + \mu(B_1) - \mu(A_1 \cap B_1) + \mu(A_1 \Delta B_1) \\ &\quad - \mu(A_1 \cap B_1') - \mu(A_1' \cap B_1) \\ &= \mu(A \dot{\cup} C) + \mu(B \dot{\cup} C) - \mu(C) + \mu(A \dot{\cup} B) - \mu(A) - \mu(B)\end{aligned}$$

which is grade-2 additivity.
(b) We prove the result by induction on $n$. The result holds for $n = 3$. Assuming the result holds for $n \geq 3$ we have

$$\begin{aligned}\mu\left(\dot{\bigcup_{i=1}^{n}} A_i\right) &= \mu\left[A_1 \dot{\cup} \cdots \dot{\cup} (A_{n-1} \dot{\cup} A_n)\right] \\ &= \sum_{i<j=1}^{n-2} \mu\left(A_i \dot{\cup} A_j\right) + \sum_{i=1}^{n-2} \mu\left[A_i \dot{\cup} (A_{n-1} \dot{\cup} A_n)\right] \\ &\quad - (n-3)\left[\sum_{i=1}^{n-2} \mu(A_i) + \mu\left(A_{n-1} \dot{\cup} A_n\right)\right] \\ &= \sum_{i<j=1}^{n-2} \mu(A_i \dot{\cup} A_j) + \sum_{i=1}^{n-2} \mu(A_i \dot{\cup} A_{n-1}) + \sum_{i=1}^{n-2} \mu(A_i \dot{\cup} A_n) \\ &\quad + (n-2)\mu(A_{n-1} \dot{\cup} A_n) - \sum_{i=1}^{n-2} \mu(A_i) - (n-2)\mu(A_{n-1}) \\ &\quad - (n-2)\mu(A_n) - (n-3)\left[\sum_{i=1}^{n-2} \mu(A_i) + \mu(A_{n-1} \dot{\cup} A_n)\right] \\ &= \sum_{i<j=1}^{n} \mu(A_i \dot{\cup} A_j) - (n-2)\sum_{i=1}^{n} \mu(A_i)\end{aligned}$$

The result follows by induction. □

We now give an example of a $q$-measure space. We call this the **particle-antiparticle** example. Let $X = [0,1]$ and let $\nu$ be Lebesgue measure restricted to $[0,1]$. For $A \in B(X)$ define

$$\mu(A) = \nu(A) - 2\nu\left(\left\{x \in A \colon x + \tfrac{3}{4} \in A\right\}\right) = \nu(A) - 2\nu\left[A \cap \left(A - \tfrac{3}{4}\right)\right]$$

For example $\mu(X) = 1/2$ and $\mu([0,3/4]) = 3/4$. We think of pairs $(x, x+3/4)$ for $0 \le x \le 1/4$ as being destructive (or particle-antiparticle)pairs. Thus the $\mu$ measure of $A$ is the Lebesgue measure of $A$ after the destructive pairs of $A$ annihilate each other. We now show that $(X, \mathcal{B}(X), \mu)$ is a $q$-measure space.

**Theorem 2.3.** *In the particle-antiparticle example, $\mu$ is a $q$-measure.*

*Proof.* If $\mu(A) = 0$, then $A = \emptyset$ or $A$ has the form $A = C \cup (C + 3/4)$ for some $C \in \mathcal{B}(X)$ with $C \subseteq [0, 1/4]$. If $B \in \mathcal{B}(X)$ with $A \cap B = \emptyset$, then

$$\mu(A \mathbin{\dot\cup} B) = \nu(A) + \nu(B) - 2\nu\left[A \cap \left(A - \tfrac{3}{4}\right)\right] - 2\nu\left[B \cap \left(B - \tfrac{3}{4}\right)\right] = \mu(B)$$

Next suppose $\mu(A \mathbin{\dot\cup} B) = 0$. Then $\nu\left[(A \mathbin{\dot\cup} B) \cap (1/4, 3/4)\right] = 0$ and we have

$$\begin{aligned} \mu(A) &= \nu\left(\{x \in A \colon x + 3/4 \in B\}\right) + \nu\left(\{x + 3/4 \in A \colon x \in B\}\right) \\ &= \nu\left(\{x + 3/4 \in B \colon x \in A\}\right) + \nu\left(\{x \in B \colon x + 3/4 \in A\}\right) = \mu(B) \end{aligned}$$

We conclude that $\mu$ is regular. To prove grade-2 additivity let $A_1, A_2, A_3 \in \mathcal{B}(X)$ be mutually disjoint. If $x \in A_r$ and $x + 3/4 \in A_s$, $r, s = 1, 2, 3$ we call $(x, x + 3/4)$ an $rs$-**pair**. We then have

$$\begin{aligned} &\mu(A_1 \mathbin{\dot\cup} A_2) + \mu(A_1 \mathbin{\dot\cup} A_3) + \mu(A_2 \mathbin{\dot\cup} A_3) - \mu(A_1) - \mu(A_2) - \mu(A_3) \\ &\quad = \nu(A_1) + \nu(A_2) - 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is an } rs\text{-pair, } r, s = 1, 2\right\}\right) \\ &\qquad + \nu(A_1) + \nu(A_3) - 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is an } rs\text{-pair, } r, s = 1, 3\right\}\right) \\ &\qquad + \nu(A_2) + \nu(A_3) - 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is an } rs\text{-pair, } r, s = 2, 3\right\}\right) \\ &\qquad - \nu(A_1) + 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is a 11-pair}\right\}\right) \\ &\qquad - \nu(A_2) + 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is a 22-pair}\right\}\right) \\ &\qquad - \nu(A_3) + 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is a 33-pair}\right\}\right) \\ &\quad = \nu(A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3) - 2\nu\left(\left\{x \colon \left(x, x + \tfrac{3}{4}\right) \text{ is an } rs\text{-pair, } r, s = 1, 2, 3\right\}\right) \\ &\quad = \mu(A_1, \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3) \end{aligned}$$

Finally, to show that $\mu$ is continuous, let $A_i$ be an increasing sequence in $\mathcal{B}(X)$. Then $B_i = A_i \cap (A_i - 3/4)$ is an increasing sequence in $\mathcal{B}(X)$. Hence,

$$\begin{aligned}\lim_{i\to\infty} \mu(A_i) = \lim_{i\to\infty} \nu(B_i) - 2\lim_{i\to\infty} \nu(B_i) = \nu(\cup A_i) - 2\nu(\cup B_i)\\ \nu(\cup A_i) - 2\nu\left[(\cup A_i) \cap (\cup A_i - 3/4)\right]\\ = \mu(\cup A_i)\end{aligned}$$

A similar result holds for a decreasing sequence of sets in $\mathcal{B}(X)$. □

We shall study this example further in the next section.

# 3 Compatibility and the Center

Let $(X, \mathcal{A}, \mu)$ be a $q$-measure space. We say that $A, B \in \mathcal{A}$ are $\mu$-**compatible** and we write $A\mu B$ if

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

If $A\mu B$, then $\mu$ acts like a measure on $A \cup B$ so in some weak sense, $A$ and $B$ do not interfere with each other. Notice that $A\mu A$ for every $A \in \mathcal{A}$. It follows from (2.7) that $A\mu B$ if and only if

$$\mu(A\Delta B) = \mu(A \cap B') + \mu(A' \cap B) \tag{3.1}$$

The $\mu$-**center** of $\mathcal{A}$ is

$$Z_\mu = \{A \in \mathcal{A} \colon A\mu B \text{ for all } B \in \mathcal{A}\}$$

The elements of $Z_\mu$ are called **macroscopic** [10].

**Lemma 3.1.** (a) *If $A \subseteq B$, then $A\mu B$.* (b) *If $A\mu B$, then $A'\mu B'$.* (c) $\phi, X \in Z_\mu$. (d) *If $A \in Z_\mu$ then $A' \in Z_\mu$.*

*Proof.* (a) If $A \subseteq B$, then

$$\mu(A \cup B) = \mu(B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Hence, $A\mu B$. (b) If $A\mu B$, then by (3.1) we have

$$\mu(A'\Delta B') = \mu(A\Delta B) = \mu(A \cap B') + \mu(A' \cap B)$$

Hence, by (3.1), $A'\mu B'$. (c) follows from (a) and (d) follows from (b). □

A set $A \in \mathcal{A}$ is $\mu$-**splitting** if $\mu(B) = \mu(B \cap A) + \mu(B \cap A')$ for every $B \in \mathcal{A}$.

**Lemma 3.2.** *$A$ is $\mu$-splitting if and only if $A \in Z_\mu$.*

*Proof.* Suppose $A$ is $\mu$-splitting. Then for any $B \in \mathcal{A}$ we have

$$\begin{aligned} \mu(A \cup B) &= \mu\left[(A \cup B) \cap A\right] + \mu\left[(A \cup B) \cap A'\right] \\ &= \mu(A) + \mu(B \cap A') = \mu(A) + \mu(B) - \mu(A \cap B) \end{aligned}$$

Hence, $A \in Z_\mu$. Conversely, suppose $A \in Z_\mu$. Then for any $B \in \mathcal{A}$ we have

$$\mu(A \cup B) = \mu\left[A \,\dot\cup\, (B \cap A')\right] = \mu(A) + \mu(B \cap A')$$

Thus,

$$\mu(B) = \mu(A \cup B) - \mu(A) + \mu(A \cap B) = \mu(B \cap A) + \mu(B \cap A')$$

so $A$ is $\mu$-splitting. □

**Theorem 3.3.** *$Z_\mu$ is a sub $\sigma$-algebra of $\mathcal{A}$ and the restriction $\mu \mid Z_\mu$ of $\mu$ to $Z_\mu$ is a measure. Moreover, if $A_i \in Z_\mu$, $i = 1, 2, \ldots$, are mutually disjoint, then for every $B \in \mathcal{A}$ we have*

$$\mu\left[\dot\cup (B \cap A_i)\right] = \sum \mu(B \cap A_i)$$

*Proof.* By Lemma 3.1, $X \in Z_\mu$ and $A' \in Z_\mu$ whenever $A \in Z_\mu$. Now suppose $A, B \in Z_\mu$ and $C \in \mathcal{A}$. Since $A$ is $\mu$-splitting, we have

$$\begin{aligned} \mu\left[C \cap (A \cup B)\right] &= \mu\left[(C \cap A) \cap (A \cup B)\right] + \mu\left[(C \cap A') \cap (A \cup B)\right] \\ &= \mu(C \cap A) + \mu(C \cap A' \cap B) \end{aligned}$$

Since $B$ is $\mu$-splitting, we conclude that

$$\begin{aligned} \mu(C) &= \mu(C \cap A) + \mu(C \cap A') \\ &= \mu(C \cap A) + \mu(C \cap A' \cap B) + \mu(C \cap A' \cap B') \\ &= \mu\left[C \cap (A \cup B)\right] + \mu\left[C \cap (A \cup B)'\right] \end{aligned}$$

It follows that $A \cup B$ is $\mu$-splitting so by Lemma 3.2, $A \cup B \in Z_\mu$. Hence, $Z_\mu$ is a subalgebra of $\mathcal{A}$. Moreover, $\mu \mid Z_\mu$ is additive because if $A, B \in Z_\mu$ with $A \cap B = \emptyset$, since $A \mu B$ we have that $\mu(A \,\dot\cup\, B) = \mu(A) + \mu(B)$. To show that

$Z_\mu$ is a $\sigma$-algebra, let $A_i \in Z_\mu$, $i = 1, 2, \ldots$, let $S_n = \cup_{i=1}^n A_i$ and $S = \cup_{i=1}^\infty A_i$. Since $S_n \in Z_\mu$ we have for every $B \in \mathcal{A}$ that $\mu(B) = \mu(B \cap S_n) + \mu(B \cap S_n')$. Since $S_n$ is increasing with $\cup S_n = S$ and $S_n'$ is decreasing with $\cap S_n' = S'$ we have by continuity that

$$\mu(B) = \lim_{n\to\infty} \mu(B \cap S_n) + \lim_{n\to\infty} \mu(B \cap S_n') = \mu(B \cap S) + \mu)B \cap S')$$

Hence, $S \in Z_\mu$ so $Z_\mu$ is a $\sigma$-algebra. To show that $\mu \mid Z_\mu$ is a measure, let $A_i \in Z_\mu$ be mutually disjoint. We then have

$$\mu\left(\dot{\bigcup_{i=1}^{\infty}} A_i\right) = \lim_{n\to\infty} \mu\left(\dot{\bigcup_{i=1}^{n}} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i)$$

To prove the last statement of the theorem, let $A_i \in Z_\mu$, $i = 1, \ldots, n$, be mutually disjoint and let $S_r = \dot{\cup}_{i=1}^r A_i$, $r \le n$. We prove by induction on $r$ that for $B \in \mathcal{A}$ we have $\mu(B \cap S_r) = \sum_{i=1}^r \mu(B \cap A_i)$. The case $r = 1$ is obvious. Suppose the result holds for $r < n$. Since $S_r \in Z_\mu$ we have

$$\begin{aligned}
\mu(B \cap S_{r+1}) &= \mu(B \cap S_{r+1} \cap S_r) + \mu(B \cap S_{r+1} \cap S_r') \\
&= \mu(B \cap S_r) + \mu(B \cap A_{r+1}) \\
&= \sum_{i=1}^r \mu(B \cap A_i) + \mu(B \cap A_{r+1}) = \sum_{i=1}^{r+1} \mu(B \cap A_i)
\end{aligned}$$

By induction, the result holds for $r = n$ so that

$$\mu\left[\dot{\bigcup_{i=1}^{n}} (B \cap A_i)\right] = \mu(B \cap S_n) = \sum_{i=1}^n \mu(B \cap A_i)$$

The last statement follows by continuity. $\square$

We now illustrate these ideas for the particle-antiparticle example of Section 2. All the results in the rest of this section apply to the $q$-measure space $(X, \mathcal{B}(X), \mu)$ of that example. Using the notation $\overline{A} = A \cap (A - 3/4)$ we have that $\mu(A) = \nu(A) - 2\nu(\overline{A})$ for all $A \in \mathcal{B}(X)$.

**Theorem 3.4.** *For $A, B \in \mathcal{B}(X)$, $A \mu B$ if and only if*

$$\nu\left[(\overline{A \cap B'} + 3/4) \cap A' \cap B\right] = \nu\left[(\overline{A' \cap B} + 3/4) \cap A \cap B'\right] = 0 \qquad (3.2)$$

*Proof.* We have that $A\mu B$ if and only if (3.1) holds. But (3.1) is equivalent to

$$\nu(\overline{A\Delta B}) = \nu(\overline{A\cap B'}) + \nu(\overline{A'\cap B}) \tag{3.3}$$

and (3.3) is equivalent to

$$\begin{aligned}\nu\left[(A\Delta B)\cap(A\Delta B-3/4)\right] &= \nu\left[(A\cap B')\cap(A\cap B'-3/4)\right]\\ &\quad+\nu\left[(A'\cap B)\cap(A'\cap B-3/4)\right]\end{aligned} \tag{3.4}$$

The left side of (3.4) becomes

$$\begin{aligned}&\nu\left[((A\cap B')\cup(A'\cap B))\cap((A\cap B')\cup(A'\cap B)-3/4)\right]\\ &\quad=\nu\left\{[(A\cap B')\cap((A\cap B')\cup(A'\cap B)-3/4)]\right.\\ &\qquad\left.\cup[(A'\cap B)\cap((A\cap B')\cup(A'\cap B)-3/4)]\right\}\\ &\quad=\nu\left[(A\cap B')\cup((A\cap B')\cup(A'\cap B)-3/4)\right]\\ &\qquad+\nu\left[(A'\cap B)\cap((A\cap B')\cup(A'\cap B)-3/4)\right]\end{aligned} \tag{3.5}$$

But the last expression in (3.5) coincides with the right side of (3.4) if and only if (3.2) holds. □

**Corollary 3.5.** *For $A\in\mathcal{B}(X)$, $A\in Z_\mu$ if and only if $\nu(\overline{A})+\nu(\overline{A'})=1/4$.*

*Proof.* If $A\in Z_\mu$, then $A\mu A'$. By Theorem 3.4 we have

$$\nu\left[(\overline{A}+3/4)\cap A'\right]=\nu\left[(\overline{A'}+3/4)\cap A\right]=0$$

Hence,

$$\nu(\overline{A})=\nu(\overline{A}+3/4)=\nu\left[(\overline{A}+3/4)\cap A\right]=\nu\left(A\cap\left[0,\tfrac{1}{4}\right]\right)$$

Similarly, $\nu(\overline{A'})=\nu\left(A'\cap[0,1/4]\right)$ so that

$$\nu(\overline{A})+\nu(\overline{A'})=\nu\left(\left[0,\tfrac{1}{4}\right]\right)=\tfrac{1}{4}$$

Conversely, suppose that $\nu(\overline{A})+\nu(\overline{A'})=1/4$. Then for any $B\in\mathcal{B}(X)$ we have

$$\begin{aligned}\mu(B\cap A)+\mu(B\cap A')&=\nu(B\cap A)-2\nu(\overline{B\cap A})+\nu(B\cap A')-2\nu(\overline{B\cap A'})\\ &=\nu(B)-2\nu(\overline{B})=\mu(B)\end{aligned}$$

It follows that $A\in Z_\mu$. □

**Corollary 3.6.** *The following statements are equivalent:* (a) $A\mu A'$, (b) $A \in Z_\mu$, (c) $\mu(A) + \mu(A') = 1/2$.

*Proof.* That (a) implies (b) follows from Theorem 3.4 and Corollary 3.5. That (b) implies (c) is trivial and that (c) implies (a) follows from Corollary 3.5. □

We have seen in Theorem 3.3 that $\mu \mid Z_\mu$ is a measure. In fact, in this example for every $B \in Z_\mu$ we have

$$\mu(B) = \nu\left(B \cap \left[\tfrac{1}{4}, \tfrac{3}{4}\right]\right)$$

which is clearly a measure.

# 4 Characterization of Quantum Measures

If $(X, \mathcal{A})$ is a measurable space, we can form the Cartesian product measurable space $(X \times X, \mathcal{A} \times \mathcal{A})$ in the usual way [1, 6]. In this case, $\mathcal{A} \times \mathcal{A}$ is the $\sigma$-algebra generated by the product sets $A \times B$. We say that a signed measure $\lambda$ on $\mathcal{A} \times \mathcal{A}$ is **symmetric** if $\lambda(A \times B) = \lambda(B \times A)$ for all $A, B \in \mathcal{A}$. The next lemma shows that a symmetric signed measure $\lambda$ on $\mathcal{A} \times \mathcal{A}$ is determined by its values $\lambda(A \times A)$ for $A \in \mathcal{A}$.

**Lemma 4.1.** *If $\lambda$ is a symmetric signed measure on $\mathcal{A} \times \mathcal{A}$, then for every $A, B \in \mathcal{A}$ we have*

$$\begin{aligned}\lambda(A \times B) = \tfrac{1}{2}\{&\lambda\left[(A \cup B) \times (A \cup B)\right] + \lambda\left[(A \cap B) \times (A \cap B)\right] \\ &-\lambda\left[(A \cap B') \times (A \cap B')\right] - \lambda\left[(A' \cap B) \times (A' \cap B)\right]\}\end{aligned}$$

*Proof.* For $A, B \in \mathcal{A}$ we have

$$\begin{aligned}
&\lambda\left[(A\cup B)\times(A\cup B)\right]\\
&\quad=\lambda(A\times A)+2\lambda(A\times B)+\lambda(B\times B)-2\lambda\left[A\times(A\cap B)\right]\\
&\qquad-2\lambda\left[(A\cap B)\times(A\cap B)\right]-2\lambda\left[(A\cap B)\times B\right]\\
&\qquad+4\lambda\left[(A\cap B)\times(A\cap B)\right]-\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&\quad=\lambda(A\times A)+2\lambda(A\times B)+\lambda(B\times B)-2\lambda\left[A\times(A\cap B)\right]\\
&\qquad-2\lambda\left[B\times(A\cap B)\right]+\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&\quad=\lambda(A\times A)+\lambda(B\times B)+2\lambda(A\times B)\\
&\qquad-2\lambda\left[(A\cup B)\times(A\cap B)\right]\\
&\qquad-2\lambda\left[(A\cap B)\times(A\cap B)\right]+\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&\quad=\lambda(A\times A)+\lambda(B\times B)+2\lambda(A\times B)-2\lambda\left[(A\Delta B)\times(A\cap B)\right]\\
&\qquad-3\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&\quad=\lambda(A\times A)+\lambda(B\times B)+2\lambda(A\times B)-2\lambda\left[(A\cap B')\times(A\cap B)\right]\\
&\qquad-2\lambda\left[(A'\cap B)\times(A\cap B)\right]-3\lambda\left[(A\cap B)\times(A\cap B)\right]
\end{aligned}$$

Now if $C \cap D = \emptyset$ we have

$$\lambda\left[(C\cup D)\times(C\cup D)\right]=\lambda(C\times C)+2\lambda(C\times D)+\lambda(D\times D)$$

Hence,

$$2\lambda(C\times D)=\lambda\left[(C\cup D)\times(C\cup D)\right]-\lambda(C\times C)-\lambda(D\times D)$$

We conclude that

$$\begin{aligned}
\lambda\left[(A\cup B)\times(A\cup B)\right]&=\lambda(A\times A)+\lambda(B\times B)+2\lambda(A\times B)-\lambda(A\times A)\\
&\quad+\lambda\left[(A\cap B')\times(A\cap B')\right]+2\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&\quad-\lambda(B\times B)+\lambda\left[(A'\cap B)\times(A'\cap B)\right]\\
&\quad-3\lambda\left[(A\cap B)\times(A\cap B)\right]\\
&=2\lambda(A\times B)+\lambda\left[(A\cap B')\times(A\cap B')\right]\\
&\quad+\lambda\left[(A'\cap B)\times(A'\cap B)\right]-\lambda\left[(A\cap B)\times(A\cap B)\right]
\end{aligned}$$

This gives the result. $\square$

We now characterize $q$-measures in terms of signed product measures. We say that a signed measure $\lambda$ on $\mathcal{A}\times\mathcal{A}$ is **diagonally positive** if $\lambda(A\times A)\geq 0$ for all $A\in\mathcal{A}$.

**Theorem 4.2.** *A set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is a grade-2 measure if and only if there exists a diagonally positive symmetric signed measure $\lambda$ on $\mathcal{A} \times \mathcal{A}$ such that $\mu(A) = \lambda(A \times A)$ for all $A \in \mathcal{A}$. Moreover, $\lambda$ is unique.*

*Proof.* Uniqueness follows from Lemma 4.1. Let $\lambda$ be a diagonally positive symmetric signed measure on $\mathcal{A} \times \mathcal{A}$ and define $\mu\colon \mathcal{A} \to \mathbb{R}^+$ by $\mu(A) = \lambda(A \times A)$. To show that $\mu$ is grade-2 additive, letting $A, B, C \in \mathcal{A}$ be mutually disjoint we have

$$\begin{aligned}\mu(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) &= \lambda\left[(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) \times (A \mathbin{\dot\cup} B \mathbin{\dot\cup} C)\right]\\ &= \lambda(A \times A \mathbin{\dot\cup} A \times B \mathbin{\dot\cup} A \times C \mathbin{\dot\cup} B \times A \mathbin{\dot\cup} B \times B \mathbin{\dot\cup} B \times C\\ &\quad \mathbin{\dot\cup} C \times A \mathbin{\dot\cup} C \times B \mathbin{\dot\cup} C \times C\\ &= \lambda(A \times A) + \lambda(B \times B) + \lambda(C \times C)\\ &\quad + 2\left[\lambda(A \times B) + \lambda(A \times C) + \lambda(B \times C)\right]\\ &= \lambda(A \mathbin{\dot\cup} B \times A \mathbin{\dot\cup} B) + \lambda(A \mathbin{\dot\cup} C \times A \mathbin{\dot\cup} C)\\ &\quad + \lambda(B \mathbin{\dot\cup} C \times B \mathbin{\dot\cup} C) - \lambda(A \times A) - \lambda(B \times B) - \lambda(C \times C)\\ &= \mu(A \mathbin{\dot\cup} B) + \mu(A \mathbin{\dot\cup} C) + \mu(B \mathbin{\dot\cup} C) - \mu(A) - \mu(B) - \mu(C)\end{aligned}$$

Since the continuity of $\mu$ follows from the continuity of $\lambda$, we conclude that $\mu$ is a grade-2 measure. Conversely, let $\mu$ be a grade-2 measure on $\mathcal{A}$ and for $A, B \in \mathcal{A}$ define

$$\lambda(A \times B) = \tfrac{1}{2}\left[\mu(A \cup B) + \mu(A \cap B) - \mu(A \cap B') - \mu(A' \cap B)\right]$$

Note that $\lambda(A \times B) = \lambda(B \times A)$ and that $\lambda(A \times A) = \mu(A) \geq 0$. Let $\mathcal{A}_0$ be the algebra of finite disjoint unions of measurable rectangles in $\mathcal{A} \times \mathcal{A}$. We now show that $\lambda$ can be extended to a countably additive set function on $\mathcal{A}_0$. First suppose that $A \times B = (A \times B_1) \mathbin{\dot\cup} (A \times B_2)$ for disjoint $B_1, B_2 \in \mathcal{A}$. We then have

$$\begin{aligned}\lambda(A \times B) &= \lambda\left[A \times (B_1 \mathbin{\dot\cup} B_2)\right]\\ &= \tfrac{1}{2}\left\{\mu\left[(A \cup B_1) \cup (A \cup B_2)\right] + \mu\left[(A \cap B_1) \mathbin{\dot\cup} (A \cap B_2)\right]\right.\\ &\quad \left. -\mu(A \cap B_1' \cap B_2') - \mu\left[(A' \cap B_1) \mathbin{\dot\cup} (A' \cap B_2)\right]\right\} \qquad (4.1)\end{aligned}$$

By (2.7) we have

$$\begin{aligned}
&\mu\left[(A\cup B_1)\cup(A\cup B_2)\right]\\
&\qquad=\mu\left[(A\cup B_1)\Delta(A\cup B_2)\right]-\mu\left[A\cup B_1)\cap(A\cup B_2)'\right]\\
&\qquad\quad-\mu\left[(A\cup B_1)'\cap(A\cup B_2)\right]+\mu(A\cup B_1)+\mu(A\cup B_2)\\
&\qquad\quad-\mu\left[(A\cup B_1)\cap(A\cup B_2)\right]\\
&\qquad=\mu\left[(A'\cap B_1)\mathbin{\dot\cup}(A'\cap B_2)\right]-\mu(A'\cap B_1)-\mu(A'\cap B_2)\\
&\qquad\quad+\mu(A\cup B_1)+\mu(A\cup B_2)-\mu(A)
\end{aligned}\tag{4.2}$$

Since $\mu$ is grade-2 additive, we have

$$\begin{aligned}
\mu(A)&=\mu\left[(A\cap B_1)\mathbin{\dot\cup}(A\cap B_2)\mathbin{\dot\cup}(A\cap B_1'\cap B_2')\right]\\
&=\mu(\left[(A\cap B_1)\mathbin{\dot\cup}(A\cap B_2)\right]+\mu\left[(A\cap B_1)\mathbin{\dot\cup}(A\cap B_1'\cap B_2')\right]\\
&\quad+\mu\left[(A\cap B_2)\mathbin{\dot\cup}(A\cap B_1'\cap B_2')\right]-\mu(A\cap B_1)-\mu(A\cap B_2)\\
&\quad-\mu(A\cap B_1'\cap B_2')\\
&=\mu\left[(A\cap B_1)\mathbin{\dot\cup}(A\cap B_2)\right]+\mu(A\cap B_2')+\mu(A\cap B_1')-\mu(A\cap B_1)\\
&\quad-\mu(A\cap B_2)-\mu(A\cap B_1'\cap B_2')
\end{aligned}$$

Substituting $\mu(A)$ into (4.2) gives

$$\begin{aligned}
&\mu\left[(A\cup B_1)\cup(A\cup B_2)\right]\\
&\qquad=\mu\left[(A'\cap B_1)\mathbin{\dot\cup}(A'\cap B_2)\right]-\mu(A'\cap B_1)-\mu(A'\cap B_2)\\
&\qquad\quad+\mu(A\cup B_1)+\mu(A\cup B_2)-\mu\left[(A\cap B_1)\mathbin{\dot\cup}(A\cap B_2')\right]\\
&\qquad\quad-\mu(A\cap B_2')-\mu(A\cap B_1')+\mu(A\cap B_1)+\mu(A\cap B_2)\\
&\qquad\quad+\mu(A\cap B_1'\cap B_2')
\end{aligned}$$

Substituting this last expression into (4.1) gives

$$\begin{aligned}
\lambda(A\times B)&=\tfrac{1}{2}\left[\mu(A\cup B_1)+\mu(A\cup B_2)+\mu(A\cap B_1)+\mu(A\cap B_2)\right.\\
&\qquad\left.-\mu(A\cap B_1')-\mu(A\cap B_2')-\mu(A'\cap B_1)-\mu(A'\cap B_2)\right]\\
&=\lambda(A\times B_1)+\lambda(A\times B_2)
\end{aligned}\tag{4.3}$$

Next suppose that $A\times B=\dot\cup_{i=1}^{n}(A\times B_i)$ where $B_i\cap B_j=\emptyset$, $i\neq j$. We prove by induction on $n$ that $\lambda(A\times B)=\sum_{i=1}^{n}\lambda(A\times B_i)$. By (4.3) the result holds for $n=2$. Suppose the result holds for $n$. From the $n=2$ case we have for $A\times B=\dot\cup_{i=1}^{n+1}(A\times B_i)$ that

$$\lambda(A\times B)=\lambda\left(A\times\dot\bigcup_{i=1}^{n}B_i\right)+\lambda(A\times B_{n+1})$$

By the induction hypothesis we concluded that

$$\lambda(A\times B)=\sum_{i=1}^{n}\lambda(A\times B_i)+\lambda(A\times B_{n+1})=\sum_{i=1}^{n+1}\lambda(A\times B_i)$$

This concludes the induction proof. In a similar way we conclude that if $A\times B=\mathbin{\dot\cup}_{i=1}^{n}(A_i\times B)$ where $A_i\cap A_j=\emptyset$, $i\neq j$, then $\lambda(A\times B)=\sum_{i=1}^{n}\lambda(A_i\times B)$. By a standard measure-theoretic technique [1, 2, 6], it follows that if $A\times B=\mathbin{\dot\cup}_{i=1}^{n}(A_i\times B_i)$, then $\lambda(A\times B)=\sum\lambda(A_i\times B_i)$. For $C=\mathbin{\dot\cup}_{i=1}^{n}(A_i\times B_i)\in\mathcal{A}_0$ we define $\lambda(C)=\sum_{i=1}^{n}\lambda(A_i\times B_i)$. Our previous work shows that $\lambda$ is well-defined on $\mathcal{A}_0$. Now suppose that $A\times B=\mathbin{\dot\cup}_{i=1}^{\infty}(A\times B_i)$. By the continuity of $\mu$ we have

$$\begin{aligned}
2\sum_{i=1}^{\infty}\lambda(A\times B_i)&=2\lim_{n\to\infty}\sum_{i=1}^{n}\lambda(A\times B_i)=2\lim_{n\to\infty}\lambda\left[\dot{\bigcup_{i=1}^{n}}(A\times B_i\right]\\
&=2\lim_{n\to\infty}\lambda\left(A\times\dot{\bigcup_{i=1}^{n}}B_i\right)=\lim_{n\to\infty}\mu\left[A\cup\left(\dot{\bigcup_{i=1}^{n}}B_i\right)\right]\\
&\quad+\lim_{n\to\infty}\mu\left[A\cap\left(\bigcap_{i=1}^{n}B_i\right)\right]-\lim_{n\to\infty}\mu\left[A\cap\left(\bigcap_{i=1}^{n}B_i'\right)\right]\\
&\quad-\lim_{n\to\infty}\mu\left[\dot{\bigcup_{i=1}^{n}}(A'\cap B_i)\right]\\
&=\mu(A\cup B)+\mu(A\cap B)-\mu(A\cap B)-\mu(A'\cap B)\\
&=2\lambda(A\times B)
\end{aligned}$$

Similarly, if $A\times B=\mathbin{\dot\cup}_{i=1}^{\infty}(A_i\times B)$, then $\lambda(A\times B)=\sum_{i=1}^{\infty}\lambda(A_i\times B)$. Again, by a standard argument it follows that if $A\times B=\mathbin{\dot\cup}_{i=1}^{\infty}(A_i\times B_i)$, then $\lambda(A\times B)=\sum_{i=1}^{\infty}\lambda(A_i\times B_i)$. We conclude that $\lambda$ is countably additive on $\mathcal{A}$. By the Hahn extension theorem [1, 2, 6], $\lambda$ extends to a signed measure on $\mathcal{A}\times\mathcal{A}$. □

**Corollary 4.3.** *The set function $\mu$ in Theorem 4.2 is a q-measure if and only if $\lambda$ satisfies the following two conditions:* (a) *If $\lambda(A\times A)=0$, then $\lambda(A\times B)=0$ for all $B\in\mathcal{A}$ with $A\cap B=\emptyset$,* (b) *If $\lambda\left[(A\mathbin{\dot\cup}B)\times(A\mathbin{\dot\cup}B)\right]=0$ then*

$$\lambda(A\times B)=-\lambda(A\times A)=-\lambda(B\times B)$$

*Proof.* Suppose $\mu$ is a $q$-measure and hence, $\mu$ is regular. If $\lambda(A \times A) = 0$, then $\mu(A) = 0$. Hence, $\mu(A \cup B) = \mu(B)$ for all $B \in \mathcal{A}$ with $A \cap B = \emptyset$. We then have that

$$\begin{aligned}\lambda(A \times B) &= \tfrac{1}{2}\left[\mu(A \cup B) + \mu(A \cap B) - \mu(A \cap B') - \mu(A' \cap B)\right] \\ &= \tfrac{1}{2}\left[\mu(B) - \mu(B)\right] = 0\end{aligned}$$

Thus, (a) holds. If $\lambda\left[(A \cup B) \times (A \cup B)\right] = 0$, then $\mu(A \cup B) = 0$ so that $\mu(A) = \mu(B)$. Hence,

$$\begin{aligned}0 = \mu(A \cup B) &= \lambda\left[(A \cup B) \times (A \cup B)\right] \\ &= \lambda(A \times A) + 2\lambda(A \times B) + \lambda(B \times B) \\ &= 2\mu(A) + 2\lambda(A \times B)\end{aligned}$$

It follows that

$$\lambda(A \times B) = -\mu(A) = -\lambda(A \times A) = -\lambda(B \times B)$$

so that (b) holds. Conversely, suppose (a) and (b) hold. If $\mu(A) = 0$, then $\lambda(A \times A) = 0$ so by (a) we have $\lambda(A \times B) = 0$ whenever $B \in \mathcal{A}$ with $A \cap B = \emptyset$. Hence,

$$\mu(A \cup B) = \lambda(A \times A) + 2\lambda(A \times B) + \lambda(B \times B) = \lambda(B \times B) = \mu(B)$$

If $\mu(A \cup B) = 0$, then by (b)

$$\mu(A) = \lambda(A \times A) = \lambda(B \times B) = \mu(B)$$

Therefore, $\mu$ is regular so $\mu$ is a $q$-measure. $\square$

# 5 Super-Quantum Measures

We say that a set function $\mu \colon \mathcal{A} \to \mathbb{R}^+$ is **grade-$n$ additive** on the $\sigma$-algebra $\mathcal{A}$ if $\mu$ satisfies

$$\begin{aligned}\mu\left(A_1 \cup \cdots \cup A_{n+1}\right) = &\sum_{i_1 < \cdots < i_n = 1}^{n+1} \mu\left(A_{i_1} \cup \cdots \cup A_{i_n}\right) \\ &- \sum_{i_1 < \cdots < i_{n-1} = 1}^{n+1} \mu\left(A_{i_1} \cup \cdots \cup A_{i_{n-1}}\right) \\ &+ \cdots (-1)^{n+1} \sum_{i=1}^{n+1} \mu(A_i) \qquad (5.1)\end{aligned}$$

A continuous grade-$n$ additive set function is a **grade-$n$ measure**. Grade-$n$ measures for $n \geq 3$ correspond to super-quantum measures and these may describe theories that are more general than quantum mechanics. It can be shown by induction that a grade-$n$ measure is a grade-$(n+1)$ measure [8, 10]. Thus, we have a hierarchy of measure grades with each grade contained in all higher grades. Instead of giving the induction proof we will just check that any grade-2 measure $\mu$ is also a grade-3 measure. Indeed, by (2.8) we have

$$\begin{aligned}
&\sum_{i<j<k=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j \mathbin{\dot\cup} A_k) - \sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{4} \mu(A_i) \\
&\quad = 2\sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) - 3\sum_{i=1}^{4} \mu(A_i) - 4\sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{4} \mu(A_i) \\
&\quad = \sum_{i<j=1} \mu(A_i \mathbin{\dot\cup} A_j) - 2\sum_{i=1}^{4} \mu(A_i) = \mu\left(A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3 \mathbin{\dot\cup} A_4\right)
\end{aligned}$$

The next result gives a general method of generating grade-$n$ measures. We denote the Cartesian product of a set $A$ with itself $n$ times by $A^n$ and we denote the $\sigma$-algebra $\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$ ($n$ factors) by $\mathcal{A}^n$. A signed measure $\lambda$ on $\mathcal{A}^n$ is **symmetric** if

$$\lambda(A_1 \times A_2 \times \cdots \times A_n) = \lambda(B_1 \times B_2 \times \cdots \times B_n)$$

where the $B_i$ form a permutation of the $A_i$. Moreover, $\lambda$ is **diagonally positive** if $\lambda(A^n) \geq 0$ for all $A \in \mathcal{A}$.

**Theorem 5.1.** *If $\lambda$ is a diagonally positive symmetric signed measure on $\mathcal{A}^n$, then $\mu(A) = \lambda(A^n)$ is a grade-$n$ measure on $\mathcal{A}$.*

*Proof.* For fixed $C \in \mathcal{A}$, notice that $\lambda_C(B) = \lambda(B \times C)$ is a symmetric signed measure on $\mathcal{A}^{n-1}$. We shall prove the theorem by induction on $n$. By Theorem 4.2 the result holds for $n = 2$. Suppose the result holds for $n - 1 \geq 1$. Let $\lambda$ be a diagonally positive symmetric signed measure on $\mathcal{A}^n$ and define $\mu(A) = \lambda(A^n)$. For $C \in \mathcal{A}$ define $\mu_C(A) = \lambda_C(A^{n-1})$. By the induction hypothesis $\mu_C$ satisfies $(n-1)$-additivity and hence, $\mu_C$ satisfies $n$-additivity. (Notice that $\lambda_C$ need not be diagonally positive and $\mu_C$ need not be nonnegative, but these are not important for this intermediate step.)

Let $A_1, \dots A_{n+1}$ be mutually disjoint elements of $\mathcal{A}$ and let $C = \dot\cup_{i=1}^{n+1} A_i$. Then

$$\mu\left(\dot{\bigcup_{i=1}^{n+1}} A_i\right) = \lambda\left[\left(\dot{\bigcup_{i=1}^{n+1}} A_i\right)^n\right] = \lambda_C\left[\left(\dot{\bigcup_{i=1}^{n+1}}\right)^{n-1}\right] = \mu_C\left(\dot{\bigcup_{i=1}^{n+1}} A_i\right)$$

$$= \sum_{i_1<\cdots<i_n=1}^{n+1} \mu_C\left(A_{i_1} \dot\cup \cdots \dot\cup A_{i_n}\right)$$

$$- \sum_{i_1<\cdots<i_{n-1}=1}^{n+1} \mu_C\left(A_{i_1} \dot\cup \cdots \dot\cup A_{i_{n-1}}\right) + \cdots (-1)^n \sum_{j=1}^{n+1} \mu_C(A_j)$$

Now we have

$$\mu_C(A_j) = \lambda_C(A_j^{n-1}) = \lambda\left(A_j^{n-1} \times \dot{\bigcup_{i=1}^{n+1}} A_i\right) = \sum_{i=1}^{n+1} \lambda(A_j^{n-1} \times A_i)$$

$$= \mu(A_j) + \sum_{\substack{i=1 \\ i\neq j}}^{n+1} \lambda(A_j^{n-1} \times A_i)$$

Also,

$$\mu_C(A_r \dot\cup A_s) = \lambda_C\left[(A_r \dot\cup A_s)^{n-1}\right] = \lambda\left[(A_r \dot\cup A_s)^{n-1} \times \dot{\bigcup_{i=1}^{n+1}} A_i\right]$$

$$= \mu(A_r \dot\cup A_s) + \sum_{\substack{i=1 \\ i\neq r,s}}^{n+1} \lambda\left[(A_r \dot\cup A_s)^{n-1} \times A_i\right]$$

Similarly,

$$\mu_C(A_r \dot\cup A_s \dot\cup A_i) = \mu(A_r \dot\cup A_s \dot\cup A_t) + \sum_{\substack{i=1 \\ i\neq r,s,t}}^{n+1} \lambda\left[(A_r \dot\cup A_s \dot\cup A_t)^{n-1} \times A_i\right]$$

Continuing this process, we obtain cancellation of the terms not involving $\mu$. Hence, $\mu$ satisfies (5.1) so $\mu$ is grade-$n$ additive. This completes the induction proof. □

We conjecture that the converse of Theorem 5.1 holds. That is, if $\mu$ is a grade-$n$ measure on $\mathcal{A}$ then there exists a diagonally positive symmetric signed measure $\lambda$ on $\mathcal{A}^n$ such that $\mu(A) = \lambda(A^n)$ for all $A \in \mathcal{A}$. This would generalize Theorem 4.2 to higher grade measures.

# 6 Particle Masses

This section is of a speculative nature. The idea is that $q$-measures can be employed to compute and predict elementary particle masses. These mass predictions are only approximate because presumably they account for the strong nuclear force and neglect weak and electromagnetic forces. Moreover, they only pertain to two-body interactions and neglect three-body and higher order interactions. Nevertheless, our preliminary computations are within about 3% of experimental values.

Following the standard model, the baryons (mesons and hadrons) are composed of constituent parts, namely quarks and gluons. A meson consists of a quark, antiquark and gluons while a hadron consists of three quarks and gluons. One of the problems is that we do not know (at least, I do not know) how many gluons are involved and we shall only make guesses about these numbers. Our base space will be a finite set $X = \{x_1 \ldots, x_n\}$ of particle constituents. Each $x_i$ will represent a quark or a gluon. For simplicity we shall not distinguish between quarks and antiquarks and will not be concerned with gluon colors. Also, we shall only consider up, down and strange quarks. In this first approximation, we shall not distinguish between an up and down quark and denote such quarks by $q$. We denote a strange quark by $q_s$ and a gluon by $g$. We also assume the existence of virtual gluons $g'$ that are massless and only interact with gluons. Thus, each of the constituents $x_i$ are either $q, q_s, g$ or $g'$.

Let $\mu$ be a $q$-measure on the power set $\mathcal{P}(X)$ of $X$ that measures masses of subsets of $X$. For example, $\mu(\{q, g\})$ gives the mass of the pair of constituents $(q, g)$. For simplicity, we write $\mu(x_i) = \mu(\{x_i\})$ for a singleton set $\{x_i\}$. By Theorem 2.2 (b), $\mu$ is completely determined by the values on singleton and doubleton sets. For example,

$$\mu(\{x_1, x_2, x_3\}) = \mu(\{x_1, x_2\}) + \mu(\{x_1, x_3\}) + \mu(\{x_2, x_3\}) \\ - \mu(x_1) - \mu(x_2) - \mu(x_3)$$

so once the masses on the right are known the mass on the left is determined.

Assuming that $\mu(g')=0$ and $\mu(\{g,g'\})=\mu(\{g,g\})$ we have nine unknown masses to determine. These are $\mu(q)$, $\mu(q_s)$, $\mu(g)$ and the six pairs $\mu(\{x,y\})$, $x,y=q,q_s,g$. (We assume that $\mu(\{x,g'\})=0$, $x=q,q_s$, because $g'$ does not interact with $q$ or $q_s$.) Once we have found these nine unknowns we can compute the masses of all the sets in $\mathcal{P}(X)$.

Mainly by examining the known masses of mesons the author has estimated these nine unknowns. Using these estimates and Theorem 2.2 (b) the masses of 14 baryons have been calculated. The mass estimates in $MeV$s are the following:

$$\mu(g)=97, \mu(q)=121, \mu(q_s)=646, \mu(\{q,q\})=156, \mu(\{g,g\})=187$$
$$\mu(\{q,g\})=256, \mu(\{q_s,g\})=493, \mu(\{q_s,q\})=1297, \mu(\{q_s,q_s\})=2550$$

We propose the following constituents for the mesons $\pi$, $\kappa$, $\eta'$, $\eta$, $\rho$, $f_0$ and $\kappa^*$ where $n-g$ designates $n$ gluons:

$$\pi=\{q,q,g,g'\}, \kappa=\{q,q_s,g,g'\}, \eta'=\{q_s,q_s,g,g'\}, \eta=\{q,q,3-g\}$$
$$\rho=\{q,q,5-g\}, f_0=\{q,q,7-g\}, \kappa^*=\{q,q_s,3-g\}$$

Notice that we are postulating that mesons contain an odd number of gluons. We next propose the following constituents for the hadrons $N$, $\Lambda$, $\Sigma$, $\Sigma^*$, $\Delta$, $\Xi$ and $\Omega$:

$$N=\{q,q,q,6-g\}, \Lambda=\{q,q,q_s,6-g\}, \Sigma=\{q,q,q_s,5-g\}$$
$$\Sigma^*=\{q,q,q_s,4-g\} \, \Delta=\{q,q,q,9-g\}, \Xi=\{q,q_s,q_s,6-g\}$$
$$\Omega=\{q_s,q_s,q_s,6-g\}$$

We now compute these baryon masses $M(B)$ using Theorem 2.2 (b). The first number is the calculated mass in $MeV$s and the second number in parenthesis is the experimental value.

$$
\begin{aligned}
M(\pi) &= \mu\left(\{q,q\}\right) + 2\mu\left(\{q,g\}\right) + \mu\left(\{g,g\}\right) - 4\mu(q) - 2\mu(g) = 138\,(140) \\
M(\kappa) &= \mu\left(\{q_s,q\}\right) + \mu\left(\{q_s,g\}\right) + \mu\left(\{q,g\}\right) + \mu\left(\{g,g\}\right) \\
&\quad - 2\mu(q_s) - 2\mu(q) - 2\mu(g) = 486\,(494) \\
M(\eta') &= \mu\left(\{q_s,q_s\}\right) + 2\mu\left(\{q_s,g\}\right) + \mu\left(\{g,g\}\right) - 4\mu(q_s) - 2\mu(g) \\
&= 946\,(958) \\
M(\eta) &= \mu\left(\{q,q\}\right) + 6\mu\left(\{q,g\}\right) + 3\mu\left(\{g,g\}\right) - 6\mu(q) - 9\mu(g) = 539\,(542) \\
M(\rho) &= \mu\left(\{q,q\}\right) + 10\mu\left(\{q,g\}\right) + 10\mu\left(\{g,g\}\right) - 10\mu(q) - 25\mu(g) \\
&= 764\,(770) \\
M(f_0) &= \mu\left(\{q,q\}\right) + 14\mu\left(\{q,g\}\right) + 21\mu\left(\{g,g\}\right) - 14\mu(q) - 49\mu(g) \\
&= 965\,(975) \\
M(\kappa^*) &= \mu\left(\{q_s,q\}\right) + 3\mu\left(\{q_s,g\}\right) + 3\mu\left(\{q,g\}\right) + 3\mu\left(\{g,g\}\right) \\
&\quad - 3\mu(q_s) - 3\mu(q) - 9\mu(g) = 876\,(892) \\
M(N) &= \mu\left(\{q,q\}\right) + 18\mu\left(\{q,g\}\right) + 15\mu\left(\{g,g\}\right) - 21\mu(q) - 42\mu(g) \\
&= 927\,(940) \\
M(\Lambda) &= \mu\left(\{q,q\}\right) + 2\mu\left(\{q_s,q\}\right) + 6\mu\left(\{q_s,g\}\right) + 12\mu\left(\{q,g\}\right) + 15\mu\left(\{g,g\}\right) \\
&\quad - 7\mu(q_s) - 14\mu(q) - 42\mu(g) = 1076\,(1116) \\
M(\Sigma) &= \mu\left(\{q,q\}\right) + 2\mu\left(\{q_s,q\}\right) + 5\mu\left(\{q_s,g\}\right) + 10\mu\left(\{q,g\}\right) + 10\mu\left(\{g,g\}\right) \\
&\quad - 6\mu(q_s) - 12\mu(q) - 30\mu(g) = 1222\,(1189) \\
M(\Sigma^*) &= \mu\left(\{q,q\}\right) + 2\mu\left(\{q_s,q\}\right) + 4\mu\left(\{q_s,g\}\right) + 8\mu\left(\{q,g\}\right) + 6\mu\left(\{g,g\}\right) \\
&\quad - 5\mu(q_s) - 10\mu(q) - 20\mu(g) = 1362\,(1383) \\
M(\Delta) &= 3\mu\left(\{q,q\}\right) + 27\mu\left(\{q,g\}\right) + 36\mu\left(\{g,g\}\right) - 30\mu(q) - 90\mu(g) \\
&= 1257\,(1234) \\
M(\Xi) &= \mu\left(\{q_s,q_s\}\right) + 2\mu\left(\{q_s,q\}\right) + 12\mu\left(\{q_s,g\}\right) + 6\mu\left(\{q,g\}\right) \\
&\quad + 15\mu\left(\{g,g\}\right) - 14\mu(q_s) - 7\mu(q) - 42\mu(g) = 1337\,(1321) \\
M(\Omega) &= 3\mu\left(\{q_s,q_s\}\right) + 18\mu\left(\{q_s,g\}\right) + 15\mu\left(\{g,g\}\right) - 21\mu(q_s) - 42\mu(g) \\
&= 1710\,(1672)
\end{aligned}
$$

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