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Quantum picturalism

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Why did it take us 50 years since the birth of the quantum mechanical formalism to discover that unknown quantum states cannot be cloned? Yet, the proof of the 'no-cloning theorem' is easy, and its consequences and potential for applications are immense. Similarly, why did it take us 60 years to discover the conceptually intriguing and easily derivable physical phenomenon of 'quantum teleportation'? We claim that the quantum mechanical formalism doesn't support our intuition, nor does it elucidate the key concepts that govern the behaviour of the entities that are subject to the laws of quantum physics. The arrays of complex numbers are kin to the arrays of 0s and 1s of the early days of computer programming practice. Using a technical term from computer science, the quantum mechanical formalism is 'low-level'. In this review we present steps towards a diagrammatic 'high-level' alternative for the Hilbert space formalism, one which appeals to our intuition.

The diagrammatic language as it currently stands allows for intuitive reasoning about interacting quantum systems, and trivialises many otherwise involved and tedious computations. It clearly exposes limitations such as the no-cloning theorem, and phenomena such as quantum teleportation. As a logic, it supports 'automation': it enables a (classical) computer to reason about interacting quantum systems, prove theorems, and design protocols. It allows for a wider variety of underlying theories, and can be easily modified, having the potential to provide the required step-stone towards a deeper conceptual understanding of quantum theory, as well as its unification with other physical theories. Specific applications discussed here are purely diagrammatic proofs of several quantum computational schemes, as well as an analysis of the structural origin of quantum non-locality.

The underlying mathematical foundation of this high-level diagrammatic formalism relies on so-called *monoidal categories*, a product of a fairly recent development in mathematics, and its logical underpinning is *linear logic*, an even more recent product of research in logic and computer science. These monoidal categories do not only provide a natural foundation for physical theories, but also for proof theory, logic, programming languages, biology, cooking, ... So the challenge is to discover the required additional pieces of structure that allow us to predict genuine quantum phenomena. These pieces of structure, in turns, represent the capabilities nature has provided us with in order to manipulate entities subject to the laws of quantum theory.

Keywords: Diagrammatic reasoning, quantum information and computation, quantum foundations, monoidal categories and linear logic, axiomatic quantum theory

1 Historical context

With John von Neumann's "Mathematische Grundlagen der Quantenmechanik", published in 1932 [1], quantum theory reached maturity, now having been provided with a rigourous mathematical underpinning. Three year later something remarkable happened. John von Neumann wrote in a letter to the renowned American mathematician Garrett Birkhoff the following:

> *I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more* – sic [2, 3]

In other words, merely three years after completing something that is in many ways the most successful formalism physics has ever known, both in terms of experimental predictions, technological applications, and conceptual challenges, its creator denounced his own brainchild. However, today, more than 70 years later, we still teach John von Neumann's Hilbert space formalism to our students. People did try to come up with alternative formalisms, by relying on physically

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motivated mathematical structures other than Hilbert spaces. Most notably, the 'quantum logic' proposed by Birkhoff and von Neumann himself in 1936 [4]. But quantum logic's disciples failed to convince the wider physics community of this approach's virtues. There are similar alternative approaches due to Ludwig, and Foulis and Randall [5], but none proved to be compelling.

Today, more than 70 years later, we meanwhile did learn many new things. For example, we discovered new things about the quantum world and its potential for applications:

- During the previous century, a vast amount of the ongoing discourse on quantum foundations challenged in some way or another the validity of quantum theory. The source of this was the community's inability to craft a satisfactory worldview in the light of *Bell-type non-locality* for compound quantum systems, and related to this, the *measurement problem* [6]. But the position that quantum theory is in some way or another 'wrong' seems to be increasingly hard to maintain. Not only have there been impressive experiments which assert quantum theory in all of its aspects, but also, several new quantum phenomena have been observed, which radically alter the way in which we need think about nature, and which raise new kinds of conceptual challenges. Examples of experimentally established new phenomena are quantum teleportation [7] and quantum key exchange [8]. In particular, the field of quantum information has emerged from embracing 'quantum weirdness', *not as a bug, but as a feature*!
- Within this quantum informatic endeavour we are becoming increasingly conscious of how crucial the particular behaviour of *compound* systems is to quantum theory, that is, the existence of *quantum entanglement*. The first to point at the key role of quantum entanglement within quantum theory was Schrödinger in 1935 [9]. But this key role of quantum entanglement is completely ignored within all previously proposed alternatives to the Hilbert space formalism. All key concept of those approaches solely apply to individual quantum systems, and, it is a recognised soft spot of these approaches that they weren't able to reproduce entanglement in a canonical manner. In hindsight, this is not that surprising. Neither the physical evidence nor the appropriate mathematical tools were available (yet) to pursue a research program which axiomatizes quantum theory in terms of the behaviour of compound quantum systems.

But today, more than 70 years later, this situation has changed, which brings us to other important recent developments, this time not in physics itself, but in other areas of science:

- Firstly, not many might beware of the enormous effort made by the computer science community to understand the mathematical structure of general *processes*, and in particular, the way in which these *interact*, how different configurations of interacting processes might result in the same overall process, and similar fairly abstract questions. An accurate description of how concurrent processes precisely interact turns out to be far more delicate than one would imagine at first. Key to solving these problems are appropriate mathematical means for describing these processes, usually referred to as their *semantics*. This research area has produced a vast amount of new mathematical structures and is key to the design of high-level programming languages. Why do we need these high-level programming languages you may ask. Simply because otherwise there wouldn't internet, there wouldn't be operating systems for your Mac or PC, there wouldn't be mobile phone networks, and there wouldn't be secure electronic payment mechanisms, simply because these systems are so complicated that getting things right wouldn't be possible without relying on the programming paradigms present in high-level programming languages e.g. *abstraction*, *modularity*, *compositionality* ...
- These developments in computer science went hand-in-hand with developments in proof theory (that is, the study of the structure of mathematical proofs), since the study of interacting programs 'is isomorphic to' the study of interacting proofs. The subject of proof theory comprehends logic: rather than merely being interested in what is true and what is false ($\simeq$ classical logic), one is also interested in how one establishes that something is either true or false, and therefore, the *process* of proving things becomes an explicit part of the subject. At some point proof theoreticians became interested in how many times one uses a certain assumption within proofs. To obtain a clear view on this one needed to strip logic from the implicit abilities to

copy and delete premisses, that is, the rules $A \Rightarrow A, A$ and $A, B \Rightarrow A$. This resulted in the birth of *linear logic* [10], a logic in which one is not allowed to copy and delete premisses. Having in mind the *no-cloning* [11, 12] and *no-deleting* [13] theorems of quantum theory, this new 'linear logic' might be more of a 'quantum logic' than the original 'Birkhoff-von Neumann quantum logic', which according to most logicians wasn't even a 'logic'. Another important new feature of linear logic was the fact that it had a manifestly geometrical aspect to it, which translated in purely diagrammatic characterisations of linear logic proofs [14].

- There exists an algebraic structure which captures interacting processes as well as linear logic, namely, *monoidal categories*. These are a 'two-dimensional' variant of so-called *categories*, in the sense that they involve two interacting modes of 'composing processes'. Initially categories were introduced as a meta-theory for mathematical structures, which enables to import results of one area of mathematics into another. Its consequently highly abstract nature earned it the not all too flattering name 'generalised abstract nonsense'. Nonetheless, it meanwhile plays an important role in several areas of mathematical physics e.g. in a variety of approaches to quantum field theory and proposals for theories of quantum gravity. Important mathematical areas such as knot theory also naturally fit within monoidal categories. But for us their highly successful use in logic and computer science is more relevant. This success of category theory in computer science is witnessed by the fact that today most appointments involving category theoreticians are at computer science departments. To pass from categories in computer science to categories in physics, a mere substitution will start the ball rolling:

  $$\text{`computational process'} \mapsto \text{`physical process'}.$$

  Once we find ourselves in the world of monoidal categories the language becomes purely diagrammatical. Indeed, monoidal categories are the *semantics* of linear logic, and of the corresponding proof structures. Structuralism becomes picturalism, ...

All these developments together justify a new attempt for a 'better' formalism for quantum theory, say, quantum logic mark II. We are not saying that there is something wrong with the (current) predictions of quantum theory, but that the way in which we obtain these isn't great.

Contributors to quantum picturalism. The categorical axiomatisation of quantum theory and categorical reasoning about quantum phenomena, which provides the passage to a purely diagrammatic formalism, was initiated by Samson Abramsky and myself in [15], drawing inspiration from a theorem on diagrammatic reasoning for teleportation-like protocols in [16] – an alternative proof of the latter is in [17]. Other key contributions to the categorical axiomatisation of quantum theory were made by Peter Selinger in [18], and in collaborations with Ross Duncan, Eric Paquette, Dusko Pavlovic, Simon Perdrix and Jamie Vicary, in [19–24]. Categorical toy theories and the corresponding analysis of non-locality which we discuss here are due to Bill Edwards, Rob Spekkens and myself in [25]. Diagrammatic reasoning techniques for monoidal categories trace back to Penrose's work in the early 70's [26]. He used diagrams in a somewhat more informal way. Our approach substantially relied on existing work mainly done by the 'Australian School of category theory', namely by Kelly, Carboni, Walters, Joyal, Street and Lack in [27–30]. Among other things, they provided a rigourous mathematical foundation for diagrammatic reasoning. Related graphical methods have been around for a bit more than a decade now in mathematical physics and pure mathematics, for example in [31–35] and references therein. A proponent of these methods, John Baez, has several online available postings on the topic [36].

Applications of quantum picturalism. Applications currently under development include:

- We already know (as will show below) that we can do a substantial amount of quantum reasoning in diagrams, but, to which extend can Hilbert space calculus be completely replaced with purely diagrammatic reasoning? This is a question which we currently address. An obvious self-imposed constraint is that the diagrammatic calculations should be substantially simpler and more natural than those within the Hilbert space formalism. Ultimately we want to write a quantum computing textbook in which there are no Hilbert spaces anymore, only pictures.

- In case one doesn't buy into the above, there is a far more pragmatic, but also very promising use of these pictures. They constitute a genuine proof system that can be turned into a software tool that automates reasoning. Key to this is that these structures are discrete, as opposed to the continuum of complex numbers. Can we make a computer prove new theorems about quantum theory? We think so. A team of researchers in Oxford and Edinburgh is currently in the process of producing such a piece of on pictures based automated reasoning software [37]. We anxiously wait what will come out once they push to the start button, ... will they find new protocols, new algorithms, foundational structures of multipartite entangled states?
- At the same time these pictures provide a new axiomatic foundation for quantum theory, with many degrees of structural freedom. Hence it provides a canvas to study theories more general than quantum theory. This enables us to understand what makes quantum theory so special. Since this axiomatic foundation is very flexible, it also has the potential for unification of quantum theory with other theories, hence for crafting new theories of physics.

2 What do we mean by 'high-level'?

We try explain this concept with an example and with a metaphor.

2.1 *High-level methods for linear algebra 101*

Exercise: [16] In linear algebra, projectors are linear operators $\mathrm{P} : \mathcal{H} \to \mathcal{H}$ which are both *self-adjoint* i.e. $\mathrm{P}^\dagger = \mathrm{P}$, and *idempotent* i.e. $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$. They play a very important role in quantum theory since what happens to a state in a quantum measurement is described by a projector. We will now look at a special kind of projectors, namely those of the form $\mathrm{P} = |\Psi\rangle\langle\Psi|$ with

$$|\Psi\rangle := \sum_{\alpha,\beta} \omega_{\alpha\beta} |\alpha\beta\rangle$$

where $\omega_{\alpha\beta}$ are the entries in the matrix of a linear operator $\omega : \mathcal{H} \to \mathcal{H}$. We will consider four such projectors $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4$ respectively corresponding with linear operators $\omega_1, \omega_2, \omega_3, \omega_4$. Now, consider a vector described in the tensor product of three Hilbert spaces, $\Phi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, which we can think of as the state of a tripartite quantum system. Then, first apply projector P_1 to $\mathcal{H}_2 \otimes \mathcal{H}_3$, then projector P_2 to $\mathcal{H}_1 \otimes \mathcal{H}_2$, then projector P_3 to $\mathcal{H}_2 \otimes \mathcal{H}_3$, and then projector P_4 to $\mathcal{H}_1 \otimes \mathcal{H}_2$. The question is, given that $\Phi = \phi_{in} \otimes \Xi$ with $\phi_{in} \in \mathcal{H}_1$, what is resulting vector after applying all four projectors. In particular, given that the resulting vector will always be of the form $\Xi' \otimes \phi_{out}$ with $\phi_{out} \in \mathcal{H}_3$ (something which follows from the fact that the last projector is applied to $\mathcal{H}_1 \otimes \mathcal{H}_2$), what is the vector ϕ_{out}? Concisely put, can you write ϕ_{out} as a function of ϕ_{in} given that:

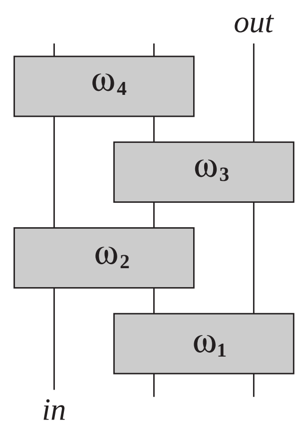


Figure 1. Diagrammatic statement of the problem. The boxes with labels ω_i represent the projectors P_i. The reason why we take ω_i as labeling rather than labelling them P_i will become clear below.

$$(\mathrm{P}_4 \otimes 1_{\mathcal{H}_3}) \circ (1_{\mathcal{H}_1} \otimes \mathrm{P}_3) \circ (\mathrm{P}_2 \otimes 1_{\mathcal{H}_3}) \circ (1_{\mathcal{H}_1} \otimes \mathrm{P}_1) \circ (\phi_{in} \otimes \Xi) = \Xi' \otimes \phi_{out}\,.$$

Solution. However complicated the problem as stated might look, the solution is simple:

$$\phi_{out} = (\omega_3 \circ \bar{\omega}_4 \circ \omega_2^T \circ \omega_3^\dagger \circ \omega_1 \circ \bar{\omega}_2)(\phi_{in})$$

where $\bar{\omega}_4$ is obtained by conjugating all matrix entries in the matrix of ω_4, where ω_2^T is the transposed of ω_2, and where $\omega_3^\dagger$ is the adjoint to ω_3. But what is more fascinating is that we can 'read' this solution directly from the graphical representation – see Figure 2. We draw a line starting from '*in*' and whenever we enter a projector at one of its two inputs, we get out via the other input, and whenever we enter a projector at one of its two outputs, we get out via the other output. The expression

$$\omega_3 \circ \bar{\omega}_4 \circ \omega_2^T \circ \omega_3^\dagger \circ \omega_1 \circ \bar{\omega}_2 \tag{1}$$

is obtained by following this line and by composing all labels we encounter on our way, in the order we encounter them, and whenever we encounter it after entering from an input we moreover conjugate all matrix entries, and whenever we encounter it while going from right to left we also take the transposed. Note that the resulting order of these labels $\omega_1, \ldots, \omega_4$ in expression (1) seems to ignore the order in which we applied the corresponding projectors $\mathrm{P}_1, \ldots, \mathrm{P}_4$.

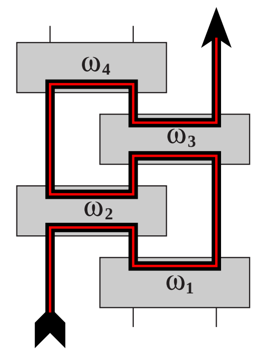


Figure 2. 'Reading' the solution of the exercise.

This exercise shows that what at first might seems to be pure 'number cracking' is governed by some beautiful 'hidden' geometry. This principle is not specific to the above four-projector situation. The same reading applies to any configuration of this kind of projectors [16].

But while at first the beauty of the geometry is appealing, the fact that it completely ignores the causal order in which we apply the projectors might be somewhat disturbing. Here we won't discuss the physical interpretation of this 'line', but just mention that the 'seemingly acausal' flow of information in this diagram has been a source of confusion e.g. [38]. A logical analysis which allows one to overcome this seemingly acausal flow of information without loosing the geometry is in [23].

Also, at first sight it might seem that the problem which we solved is totally artificial without any applications. But it isn't, since as we will see further, special cases of this exercise, depicted here on the right, constitute the structural core of the quantum teleportation protocol [7], the logic-gate teleportation protocol [39], and the entanglement swapping protocol [40] – missing labels stand for identities. For a full derivation of these protocols from the geometrical reading of projectors as exposed in the above exercise the reader may consult [16]. But even more so, there is a striking connection of these diagrams with the proof structures of linear logic. Because of the fact that monoidal categories provide semantics for linear logic, this prompted the development of a categorical axiomatisation of the quantum formalism.

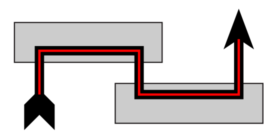


Figure 3. The structural core of quantum teleportation protocol.

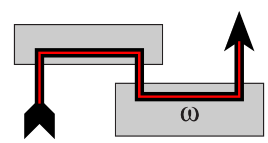


Figure 4. The structural core of logic-gate teleportation protocol.

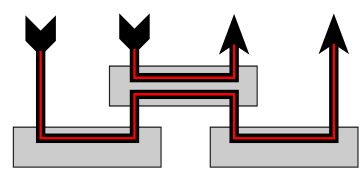
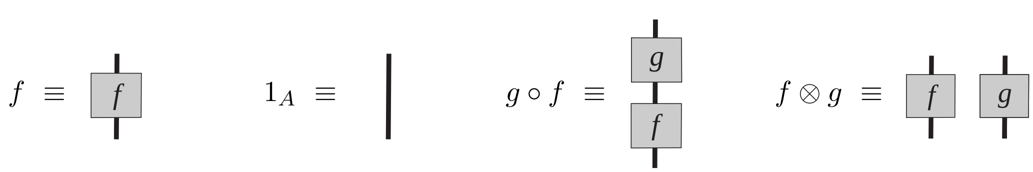


Figure 5. The structural core of entanglement swapping protocol.

The above example shows that pictures can do more than merely providing an illustration or a convenient representation: they can provide reasoning mechanisms i.e. *logic*. We now show that they can also comprehend *equational content*. The representation of linear operators as pictures which we implicitly relied on in the previous exercise went as follows:



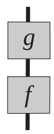
(2)

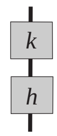
So operators are represented by boxes with and input and an output wire. In fact, we will also allow for more than one wire or none. Identities are represented by wires, composition by connecting input wires to output wires and tensor by putting boxes side-by-side.

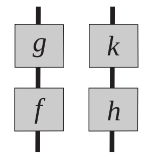
You may or you may not know that any four linear operators satisfy the equation:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \tag{3}$$

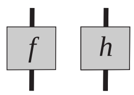
It is a easy although somewhat tedious exercise to verify this equation. How does this equation, which only involves composition and tensor, translates into pictures? We have:

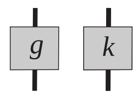
$g \circ f \equiv$ 

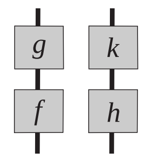
and $k \circ h \equiv$ 

so $(g \circ f) \otimes (k \circ h) \equiv$ 

On the other hand we have:

$f \otimes h \equiv$ 

and $g \otimes k \equiv$ 

so $(g \otimes k) \circ (f \otimes h) \equiv$ 

So we obtain a *tautology*! This means that the so innocent looking way in which we represented composition and tensor of linear operators as pictures already implies validity of eq.(3). Hence these simple pictures already carry non-trivial equational content.

2.2 *A metaphor: what do we look at when watching television?*

So we just saw that there is more to linear algebra than 'hacking' with matrices. Other features, namely the role played by the line in the above exercise and the tautological nature of eq.(3) show that there are structures which emerge from the underlying matrix manipulations.

Similar situations also occur in everyday life. When watching television, we don't observe the 'low-level' matrices of tiny pixels the screen is made up from, but rather the 'high-level' gestalts of each of the figuring entities (people, animals, furniture, ...) which make up the story that the images convey. These entities and their story is the essence of the images, while the matrix of pixels is just a technologically convenient representation, something which can be send as a stream of data from broadcaster to living room. What is special about this representation is that, provided the pixels are small enough, they are able to capture any image.

An alternative representation consists of a library which includes images of all figuring entities, to which we attribute coordinates. This is done in computer games. While this representation is much closer to the actual content of the images, it would be technologically unfeasible for a medium such as television. Sometimes the particular identity of these entities is also not essential for the story, but rather the overall story they convey. In a football match it is the configuration of the players relative to the ball and the changes thereof which constitute the essence.

In modern computer programming, one does not 'speak' in terms of arrays of 0s and 1s, although that's truly the data stored within the computer, but rather relies on high-level concepts about information flow. A typical example are the flow charts which are purely diagrammatic.

We sense an analogy of all of this with the status of the current quantum mechanical formalism. The way we nowadays reason about quantum theory is still very 'low-level', in terms of arrays of complex numbers and matrices which transform these arrays. Just like the pixels of the television screen, the arrays of complex numbers have the special property that they allow to represent all entities of the quantum story. So while we do obtain accurate representations of physical reality, it might not be the best way to understand it, and in particular, reason about it.

3 General compositional theories

Groups and vector spaces are examples of *algebraic structures* that are well-known to physicists. Obviously there are many other kinds of algebraic strictures. In fact, there exists an algebraic structure which is such that 'something is provable from the axioms of this algebraic structure' if and only if 'something can be derived within the above sketched diagrammatic language'.

Let us make this more precise. An algebraic structure typically consists of: (i) some *elements* $a, b, c, ...$; (ii) some *operations* such as multiplying, taking the inverse, and these operations also include special elements such as the unit; (iii) some *axioms* (or otherwise put, *laws*). For example, for a group the operations are a binary operation $-\cdot-$ which assigns to each pair of elements a, b another element $a\cdot b$, a unitary operation $(-)^{-1}$ which assigns to each element a another element a^{-1}, and a special element e. The axioms for a group are $x\cdot(y\cdot z) = (x\cdot y)\cdot z$, $x\cdot e = e\cdot x = x$, and $x^{-1}\cdot x = x\cdot x^{-1} = e$, where x, y, z are now *variables* that range over all elements of the group. These axioms tell us that the operation $-\cdot-$ is associative and has e as its unit, and that the operation $(-)^{-1}$ assigns the inverse to each element. The case of a vector space is a bit more complicated as it involves two sets of elements, namely the elements of the underlying field as well as the vectors themselves, but the idea is again more or less the same.

Let us be a bit more precise by what we mean by an axiom. By a *formal expression* we mean an expression involving both elements and operations, and typically the elements are variables. For example, in the case of a group $x\cdot(y\cdot z)^{-1}$ is such a formal expression. An axiom is an equation between two formal expressions which holds as part of the definition of the algebraic structure. But there are of course other equations between two formal expressions that hold, e.g. $x\cdot(y\cdot z)^{-1} = (x\cdot z^{-1})\cdot y^{-1}$ for groups. What we claim is that there is a certain algebraic structure defined in terms of elements, operations and axioms, such that the following holds:

(1) to each picture we can associate a unique formal expression for that algebraic structure;
(2) any equation between two pictures is derivable from the intuitive rules in the diagrammatic calculus if and only it is derivable from the axioms of the algebraic structure.

In other words, the picture calculus and the algebraic structure are essentially the same, despite the fact that at first sight they look very different. But rather than just formally defining this algebraic structure, we want to provide the reader first with an intuitive feel for it, as it is quite different from the algebraic structures physicists are used to manipulate.

Previous experiences have, somewhat surprisingly, indicated the nature of this structure, and its generality, is best conveyed without making reference to physics. Therefore we present, ...

3.1 *The algebra of cooking*

Let A be a raw potato. A admits many *states* e.g. dirty, clean, skinned, ... We want to *process* A into cooked potato B. Also B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Correspondingly, there are several ways to turn A into B e.g. boiling, frying, baking, respectively referred to as f, f' and f''. We make the fact that these cooking *process* apply to A and produce B explicit within the notation of these *processes*:

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

Our use of colours already indicated that *states* are themselves *processes* too:

$$\mathrm{I} \xrightarrow{\psi} A$$

where I stands for unspecified or unknown, i.e. we don't need to know from what *system* A has been produced, just that it is in *state* ψ and available for *processing*. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \;=\; A \xrightarrow{g\circ f} C$$

be the *composite process* of first boiling $= A \xrightarrow{f} B$ and then salting $= B \xrightarrow{g} C$ and let

$$X \xrightarrow{\mathbf{1}_X} X$$

be doing nothing to X. Clearly we have $\mathbf{1}_Y \circ \xi = \xi \circ \mathbf{1}_X = \xi$ for all *processes* $X \xrightarrow{\xi} Y$. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \qquad \text{and} \qquad C \otimes F \xrightarrow{x} M$$

respectively be boiling potato A while frying carrot D, and, mashing spiced cooked potato C and spiced cooked carrot F. The whole *process* from raw ingredients A and D to meal M is:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M \;= A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M .$$

A *recipe* is the sequence of consecutive *processes* which we apply:

$$\left(A \otimes D \xrightarrow{f \otimes h} B \otimes E\, ,\ B \otimes E \xrightarrow{g \otimes k} C \otimes F\, ,\ C \otimes F \xrightarrow{x} M\right) .$$

Of course, many *recipes* might actually result in the same *process* – cf. in a group it is possible that while $x \neq x'$ and $y \neq y'$, and hence $(x, y) \neq (x', y')$, we have $x \cdot y = x' \cdot y'$. Some equational statements may only apply to specific *recipes* while others apply at the level of formal expressions, and we refer to the latter as *laws govering receipts*. Here is one such *law governing receipts*:

$$(\mathbf{1}_Y \otimes \zeta) \circ (\xi \otimes \mathbf{1}_Z) = (\xi \otimes \mathbf{1}_U) \circ (\mathbf{1}_X \otimes \zeta) .$$

For example, for $X := A$, $Y := B$, $Z := C$, $U := D$, $\xi := f$ and $\zeta := g$ we have:

boil potato then fry carrot = fry carrot then boil potato .

This law is only an instance of a more general law on recipes, namely

$$(\zeta \circ \xi) \otimes (\kappa \circ \omega) = (\zeta \otimes \kappa) \circ (\xi \otimes \omega) ,$$

which in the particular case of $\xi := f$, $\zeta := g$, $\kappa := k$ and $\omega := h$ reads as:

boil potato then salt potato, while, fry carrot then pepper carrot

||

boil potato while fry carrot, then, salt potato while pepper carrot

Note in particular that we rediscover eq.(3) of the previous section, which was then a tautology within the picture calculus, and is now a general low on cooking processes.

It should be clear to the reader that in the above we could easily have replaced cooking processes, by either biological or chemical processes, or mathematical proofs or computer programs, or, obviously, *physical processes*. So eq.(3) is a general principle that applies whenever we are dealing with any kind of *systems* and *processes* thereon. The mathematical structure of these is a bit more involved than that of a group. While for a group we had elements, operations, and laws i.e. equations between formal expressions, here:

(C1) Rather than an underlying set of elements, as in the case of a group, we have two sorts of things, one to which we referred as *systems*, and the other to which we referred as *processes*.

(C2) There is an operation $- \otimes -$ on *systems* as well as an operation $- \otimes -$ on *processes*, with respective units I and 1_{I}. Both of these are very similar to the multiplication of the group. In addition to this operation, there is also an operation $- \circ -$ on *processes*, but for two *processes* $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, their composite $g \circ f$ exists if and only if we have $B = C$.

(C3) The way in which $-\otimes-$ and $-\circ-$ interact with each other is given by the laws:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \qquad \text{and} \qquad 1_{A\otimes B} = 1_A \otimes 1_B\,.$$

The items (C1), (C2) and (C3), up to some subtleties for which we refer the reader to [41–43], define what it means to be a *monoidal category*, a mathematical structure which has been around now for some 45 years [44]. It has become prominent in computer science, and is gaining prominence in physics. *Systems* are typically referred to as *objects*, *processes* are referred to as *morphisms*, the operation $-\circ-$ as *composition*, and the operation $-\otimes-$ as the *tensor*.

The words then and while we used to refer to $-\circ-$ and $-\otimes-$ are clearly connected to the 'time-like' and 'space-like' separation one has in relativistic spatio-temporal causal structure. Put differently, we can compose processes both 'sequentially' and 'in parallel'. We will refer to such a theory of systems and processes thereon, in which we have two interacting compositions in the above described sense, as *compositional theories*. Hence we distinguish between the theory itself and the mathematical model, that is, *monoidal categories*, in which we describe it.

Remark: At several occasions it was pointed out to us that theories in the above sense do not cover all possible physical theories, namely: (1) field theories have no clear concept of system due to creation and annihilation operators; (2) in a theory of quantum gravity there should be no strict distinction between space-like and time-like degrees of freedom. These are not valid criticisms. While the particular discourse conducted above relies on a clearly defined notion of system, this was only done in order to convey a clear story to the reader. This is not that essential, and can in fact be undone by adding extra structures, as the many formalisations of field theories in terms of monoidal categories demonstrate. The clear separation between space-like and time-like degrees of freedom can also be undone by adding extra structure.

3.2 *Another metaphor: why does a tiger have stripes and a lion doesn't?*

One strategy for finding an answer to this question would be to dissect the tiger and the lion. Maybe the explanation is hidden in the nature of the building blocks which these two animals are made up from. We find intestines but they seem to be very much the same in both cases. With a tiny knife we further dissect till we identify a smaller kind of building block we now refer to as a 'cell'. Again, no obvious difference for tigers and lions at this level. We need to go even smaller, till we discover DNA and this constituent truly reveals a difference. So yes, now we know why tigers have stripes and lions don't. Do we really? It seems to us that the real *explanation* for the fact that tigers have stripes and lions don't is the *process*

$$\text{prey} \otimes \text{predator} \otimes \text{environment} \xrightarrow{\text{hunt}} \text{dead prey} \otimes \text{eating predator}$$

which represents the successful challenge of a predator, operating within a certain environment, on a certain prey. Key to the likeliness that such a challenge will be successful is the predator's camouflage. Lions hunt in sandy savanna while tigers hunt in the forest and it is relative to this environment that stripes happen to be adequate camouflage for tigers and plain sandy colours happen to be adequate camouflage for lions. The fact that this difference is encoded in their respective DNA is an evolutionary consequence of this, via the process of natural selection.

This example clearly illustrates that there are different levels of structural description that apply to a certain situation, and that some of these might be more relevant than others. Rather than looking at the individual structure of entities, and their constituents, above we looked at how they interact with others, namely environment and prey. It is this passage which is enabled by monoidal categories; more traditional structures such as groups and C*-algebras are intrinsically monolithic. Philosophically this passage enables us to (at least to some extend) consider other perspectives than a purely reductionistic one. In particular, for quantum theory, it enables us to put more emphasis on the way in which quantum systems interact. Let it be clear that we do not want to take a particular philosophical stance here, since the results we will present are totally independent of it. But one should be aware that certain paradigms or perspectives, such as reductionism, might lead one to not properly understand important things.

3.3 *Compositional theories ≡ picture calculi*

We already introduced some basics of the diagrammatic language in eqs.(2). For example, on the right is the diagrammatic representation of

$$l \circ (g \otimes 1) \circ (f \otimes h \otimes k)\,,$$

or, by eq.(3) which both applies to pictures and compositional theories,

$$l \circ ((g \circ (f \otimes h)) \otimes k)\,,$$

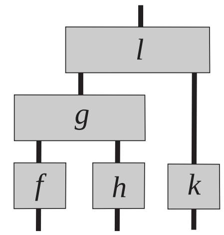
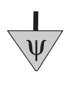
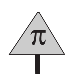
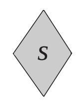


Figure 6. Compound processes as pictures

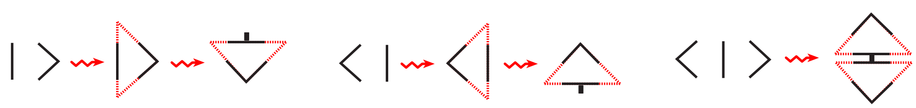
where we relied on $1 \circ k = k$. We represent the 'unspecified' system I by 'nothing', that is, no wire. We represent *states* (cf. kets), *effects* (cf. bras), and *numbers* (e.g. bra-kets) by:

$\mathrm{I} \xrightarrow{\psi} A \equiv$ 

$A \xrightarrow{\pi} \mathrm{I} \equiv$ 

$\mathrm{I} \xrightarrow{\pi} \mathrm{I} \equiv$ 

Note how these triangles and diamonds are essentially the same as Dirac notation:



Hence the graphical language builds further on something physicists already know very well. Within the mathematical definition of a monoidal category these special morphisms state, effect and number are subject to some equational constraints, but in the graphical calculus this is completely accounted for by the fact that it corresponds to 'no wire'.

Sometimes one wishes to have a process

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

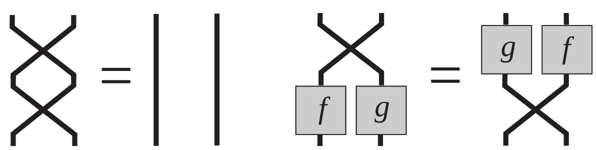


Figure 7. Laws on 'swapping systems'.

that *swaps* systems in compositional theories. Again this can be made mathematically precise and is captured by the mathematical notion of *symmetric monoidal category*. This involves substantially more equational requirements but each of these is again intuitively evident in diagrammatic terms e.g. above on the right we depicted:

$$\sigma_{B,A} \circ \sigma_{A,B} = 1_{A,B} \qquad \text{and} \qquad \sigma_{A,B} \circ (f \otimes g) = (g \otimes f) \circ \sigma_{A,B}\,.$$

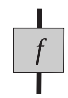
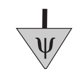
Theorem 3.1: [29] *The graphical calculus for monoidal categories and symmetric monoidal categories is such that an equational statement between formal expressions in the language of (symmetric) monoidal categories holds if and only if it is derivable in the graphical calculus.*

The theory of graphical languages for a variety of different species of monoidal categories, including so-called *braided* ones, is surveyed in a recent paper by Selinger [45].

4 Picture calculus for quantum theory I: lots from little

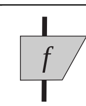
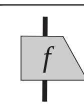
In standard quantum theory processes are described by certain linear maps. Therefore the symmetric monoidal category which has Hilbert spaces as objects, (bounded/Hilbert-Schmidt) linear maps as morphisms, and the tensor product as the tensor, plays an important role for us. We denote it by **Hilb**. When restricting to finite dimensional Hilbert spaces we write **FHilb**. Up to issues to do with redundancy of global phases, which can be dealt with categorically [46] but which we ignore here, **FHilb** can be interpreted as the compositional theory of all post-selected[1] processes for quantum systems with finite degrees of freedom.

In **FHilb** we have $\mathrm{I} := \mathbb{C}$, since for any Hilbert space, when conceiving $\mathbb{C}$ as a one-dimensional Hilbert space, that $\mathcal{H} \otimes \mathbb{C} \simeq \mathcal{H}$. Consequently, states are linear maps $\psi : \mathbb{C} \to \mathcal{H}$. If we know $\psi(1) \in \mathcal{H}$, then the whole map is determined by linearity. Hence these linear maps are in bijective correspondence with the vectors in $\mathcal{H}$. Similarly one shows that the linear maps $\psi : \mathbb{C} \to \mathbb{C}$ are in bijective correspondence with the complex numbers.

pics:				
cats:	object A	morphism $A \xrightarrow{f} B$	I	$\mathrm{I} \xrightarrow{\psi} B$
FHilb:	Hilbert space $\mathcal{H}$	linear map $f : \mathcal{H} \to \mathcal{H}'$	$\mathbb{C}$	$\psi \in \mathcal{H}$

So **FHilb** is 'a naive version of quantum theory' recast as a compositional theory, but by still explicitly referring to Hilbert spaces. What we truly would like to do is to describe quantum theory purely in diagrammatic terms, without reference to Hilbert space. In the remainder of this section we will adjoin two intuitively natural features to the graphical language which bring us substantially closer to the fundamental concepts of the quantum realm, and will already allow for some protocol derivation. These results appeared in a joint paper with Abramsky [15].

4.1 *Concepts derivable from flipping boxes upside-down*

pics:		
cats:	$A \xrightarrow{f} B$	$B \xrightarrow{f^\dagger} A$
FHilb:	linear map	its adjoint

Assume that for each graphical element there is a corresponding one obtained by flipping it upside-down. To make this visible in the graphical calculus we introduce asymmetry. In the case of **FHilb** we can interpret this 'flipping' in terms of the linear-algebraic adjoint, obtained by transposing a matrix and conjugating its entries. Therefore we also denote such a 'flipping' operation by † in arbitrary monoidal categories. We call a monoidal category with such a flipping operations a *dagger monoidal category.*

Again, while in the graphical language we can simply define this operation by saying that we flip things upside-down, in category-theoretic terms we have to specify several equational requirements, for example, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ and $1_A^\dagger = 1_A$.

So what do adjoints buy us? They let us define the following in any dagger monoidal category:

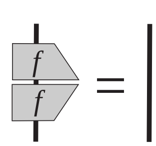
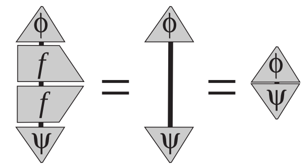
Definition 4.1: The *inner product* of two states $\mathrm{I} \xrightarrow{\psi} A$ and $\mathrm{I} \xrightarrow{\phi} A$ is $\phi^\dagger \circ \psi$. A morphism $A \xrightarrow{f} B$ is *unitary* iff $f^\dagger = f^{-1}$ where $B \xrightarrow{f^{-1}} A$ is defined by $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. A morphism $A \xrightarrow{f} A$ is *self-adjoint* iff $f = f^\dagger$ and it is a *projector* if moreover $f \circ f = f$.

The names of these concepts are justified by the fact that in **FHilb** they coincide with the usual ones. Hence, for example, self-adjointness of a linear operator translates in diagrammatic

[1]Post-selected means that in measurements we condition on the outcome. Arbitrary linear maps can be realised by means of post-selected logic-gate teleportation [39]. Arbitrary amplitudes can be obtained by amplification.

terms as 'invariance under flipping it upside-down'. In any dagger monoidal category we can derive the more usual definition of unitarity in terms of preservation of the inner-product:

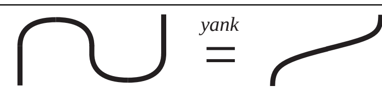
Proposition 4.2: *Unitary morphisms preserve inner-products.*

pics:		
cats:	$f^\dagger \circ f = 1_B$	$(f \circ \phi)^\dagger \circ (f \circ \psi) = \phi^\dagger \circ (f^\dagger \circ f) \circ \psi = \phi^\dagger \circ \psi$
FHilb:	f is isometry	$\langle f(\phi) \vert f(\psi)\rangle = \langle \phi \vert (f^\dagger \circ f)(\psi)\rangle = \langle \phi \vert \psi \rangle$

The proof of this proposition is depicted in the table on the right. Recall here that f is unitary if and only if both f and $f^\dagger$ are isometries, and that a linear map $f : \mathcal{H} \to \mathcal{H}'$ is an isometry iff $f^\dagger \circ f = 1_{\mathcal{H}}$. Also the notion of *positivity* generalises to dagger monoidal categories, but more interesting is the notion of *complete positivity*. In standard quantum theory completely positive maps, roughly speaking, assign to each density matrix another density matrix in such a way that mixtures of pure states are preserved. They are of key importance to describing noisy processes, open systems, and decohence. It turns out that they can already be defined at the general level of dagger monoidal categories, such that in the case of **FHilb** we obtain the usual notion. We only mention this result here, and refer the reader to [18, 47] for a detailed discussion of this generalised notion of completely positive maps, and applications thereof.

4.2 *Concepts derivable from* ∪*-turns*

We adjoin new graphical elements to the graphical calculus, a ∪-shaped and a ∩-shape wire.

	element 1	**element 2**	**rule**
pics:			
cats:	$\mathrm{I} \xrightarrow{\eta_A} A \otimes A$	$A \otimes A \xrightarrow{\epsilon_A} \mathrm{I}$	$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A$
FHilb:	$\sum_i \vert ii\rangle$	$\sum_i \langle ii \vert$	$(\sum_i \langle ii \vert \otimes 1_{\mathcal{H}})\,(1_{\mathcal{H}} \otimes \sum_i \vert ii\rangle) = 1_{\mathcal{H}}$

We refer to ∪'s as *Bell-states* and to ∩'s as *Bell-effects*. These ∪'s and ∩'s obey an intuitive graphical rule. While symbolically this rule is quite a mouthful, graphically it is so simple that it looks somewhat silly: a line involving ∪'s and ∩'s can always been 'straightened' or 'yanked'. The reason for depicting the identity as  will become clear when we use this rule in applications. In Figure 8 we show how the diagrammatic and the symbolic descritions of this rule relate. States and effects satisfying this property do exist in **FHilb** and indeed correspond to Bell-states and Bell-effects. Depending on one's taste one can depict a Bell-state either as  or as ; we pick the latter.

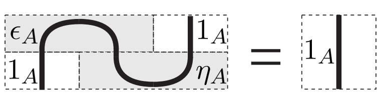
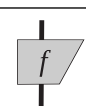
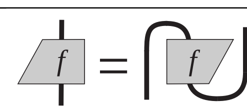


Figure 8. Comparison of the diagrammatic and the category-theoretic description of 'straightening/yanking'.

These ∪'s and ∩'s turn out to capture a surprising amount of linear-algebraic structure. They for example allow to generalise the linear-algebraic notion of *transpose* to arbitrary compositional theories. Graphically we denote this adjoint by rotating the box representing

pics:		
cats:	$A \xrightarrow{f} B$	$f^* = B \xrightarrow{(\epsilon_B \otimes 1_A)\circ(1_B \otimes f \otimes 1_A)\circ(1_B \otimes \eta_A)} A$
FHilb:	linear map	its transposed

the morphism by 180 degrees. This choice is not at all arbitrary. As shown in Figure 9, the definition of the transpose together with the yanking axiom for the ∪'s and ∩'s allows us to prove that we can 'slide' boxes along these ∪'s and ∩'s, which indeed exactly corresponds to rotating the box 180 degrees. We'll see further how this principle alone will allow us to derive several quantum informatic protocols.

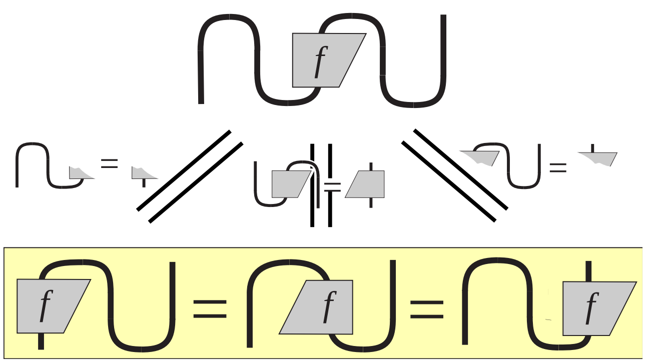
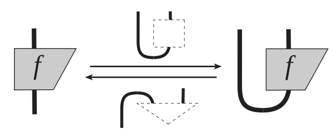


Figure 9. Proof of the sliding rule. We apply yanking to the picture at the top to obtain bottom-left and bottom right. The bottom-middle picture follows by the definition of the transpose.

These cups and caps also generalise so-called 'map-state' duality to arbitrary compositional theories. Let us recall what this map-state duality is about. To a linear map $f : \mathcal{H} \to \mathcal{H}'$ with matrix $(m_{ji})_{ji}$ in basis $\{|i\rangle\}_i$ of $\mathcal{H}$ and basis $\{|j\rangle'\}_j$ of $\mathcal{H}'$ we can always associate a bipartite vector $\Psi_f := \sum_{ji} m_{ji} \cdot |i\rangle \otimes |j\rangle' \in \mathcal{H} \otimes \mathcal{H}'$. This correspondence between linear maps from $\mathcal{H}$ to $\mathcal{H}'$ and vectors in $\mathcal{H} \otimes \mathcal{H}'$ is a bijective one. In particular, we can write this bipartite state in terms of f itself and a Bell-state, namely as $\Psi_f = (1_\mathcal{H} \otimes f) \sum_{ii} |ii\rangle$, and this expression straightforwardly lifts to the graphical calculus. So does the bijective correspondence

 (4)

due to the yanking rule, and it also lifts to completely positive maps, resulting in a generalisation of the Choi-Jamiolkowski Isomorphism. Other concepts of linear algebra which can be expressed in terms of ∪'s and ∩'s are the *trace*, the *partial trace*, and the *partial transpose*, which all play an important role in quantum theory.

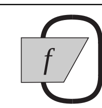
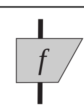
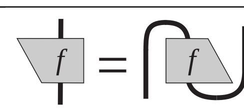
pics:	
cats:	$tr(f) = \mathrm{I} \xrightarrow{\eta_A \circ (f \otimes 1_A) \circ \epsilon_A} \mathrm{I}$
FHilb:	trace i.e. $\sum_i m_{ii}$

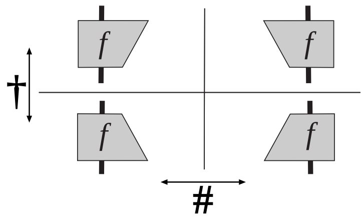
Figure 10. Also the trace allows a diagrammatic presentation in terms of ∪'s and ∩'s. It's abstract category-theoretic axiomatisation is in [48].

Remark: Rather than defining ∪'s as morphisms $A \otimes A \xrightarrow{\epsilon_A} \mathrm{I}$, like we did above, there are good reasons to define ∪'s as morphisms $A^* \otimes A \xrightarrow{\epsilon_A} \mathrm{I}$, where A^* is referred to as the *dual*. For example, when we take $\mathcal{H}^*$ to be the dual Hilbert space of a Hilbert space $\mathcal{H}$ (i.e. the space of functionals) then the Bell-states are basis independent, and hence so are the trace and transposed. We won't go into this issue any further and refer the reader to [15, 23].

4.3 $2 \times 2 = 4$

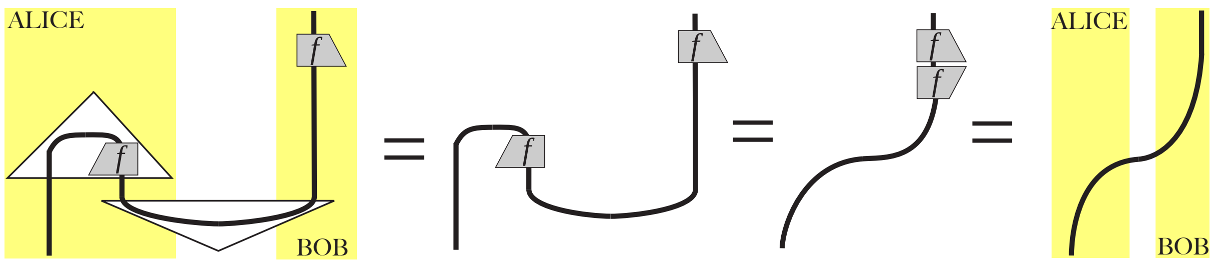
If we combine the structures introduced in the previous two sections we can construct the transposed of the adjoint, or equally, as is obvious from the graphical calculus, the adjoint of the transposed. In **FHilb** this corresponds to conjugating matrix

pics:		
cats:	$A \xrightarrow{f} B$	$f^\sharp = A \xrightarrow{(\epsilon_A \otimes 1_B) \circ (1_A \otimes f^\dagger \otimes 1_B) \circ (1_A \otimes \eta_B)} B$
FHilb:	linear map	its conjugate

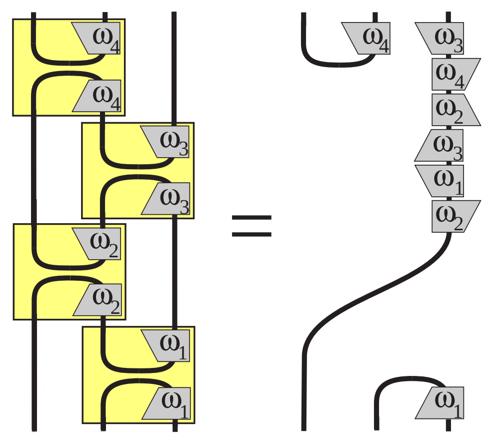


entries. On the left we summarise the graphical representation of the adjoint, the conjugate and the transposed, and the ways they relate to each other. This is the setting in which things start to become interesting, and we can start exploring the area quantum informatic protocols. All the results apply to arbitrary compositional theories in which we can flip boxes upside-down and have ∪'s and ∩'s, so in particular, to **FHilb**.

First we derive the quantum teleportation protocol. Assume that f is a *unitrary morphism* i.e. its adjoint is equal to its inverse. Physically it represents a reversible operation. We have:

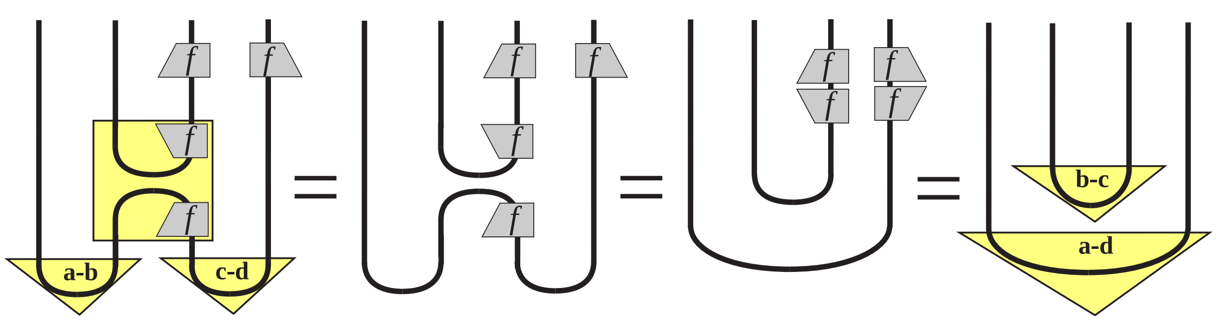


The picture on the left describes the setup. Alice and Bob share a Bell-state (= the white triangle at the bottom). Alice also possesses another qubit in unknown state (= the leftmost black wire at the bottom). She performs a bipartite measurement on her two qubits for which the resulting corresponding effect is the remaining triangle, that is, symbolically, $(\Psi_{f^*})^\dagger$ in the notation of the previous section. By map-state duality we know that any bipartite-effect can be represented in this manner for some f. The fact that here f is unitary guarantees that the effect is maximally entangled. Finally Bob performs the adjoint to f on his qubit. The picture on the right shows that the overall result of doing all of this is that Alice's qubit ends up with Bob. Importantly, the fact that Alice and Bob's operation are labelled by the same symbol f implies that Alice needs to communicate what her f is (i.e. her measurement outcome) to Bob – and hence this process does not violate no-faster-than-light-communication imposed by special relativity.



On the right you find the solution to the exercise we presented in Section 2.1. Indeed, that's all there is to it. Since **FHilb** is an example of a compositional theory this general proof implies the result for the specific case of linear algebra.

We also derive the entanglement swapping protocol:



The four qubits involved, a, b, c, d, are initially in two Bell-states, a-b and c-d. By performing a (non-destructive) measurement on b and c (= yellow square), and performing corresponding unitaries on c and d, we get a situation where the Bell-states are now a-d and b-c.

Dagger symmetric monoidal categories in which each object comes with a ∪ and ∩, subject to certain conditions which make all of these live happily together, are *dagger compact categories*. Theorem 3.1 extends to dagger compact categories.

Theorem 4.3 : [18, 27] *The graphical calculus for dagger compact categories is such that an equational statement between formal expressions in the language of dagger compact categories holds if and only if it is derivable in the graphical calculus.*

But in fact, now there is even more. As mentioned before, **FHilb** is an example of a dagger compact category, but of course there are also many other ones. To give two examples [42, 49]:

- Taking sets as objects, relations as morphisms, the cartesian product as tensor, and relational converse as the dagger, results in a dagger compact category **Rel**.
- Taking closed $n-1$-dimensional manifolds as objects, n-dimensional manifolds connecting these as morphisms (= *cobordisms*), the disjoint union of these manifolds as the tensor, and reversal of the manifold as the dagger, results in a dagger compact category **nCob**.

Remark: Following Atiyah in [54], *topological quantum field theories* can be succinctly defined as *monoidal functors* from **nCob** into **FHilb**, i.e. maps that send objects to objects and morphisms to morphisms, and which preserve both composition and tensor – details are in [42, 49, 50].

The dagger compact categories **Rel** and **nCob**, in particular the latter, are radically different from **FHilb**. This would make one think that there is nothing special about **FHilb** within the context of dagger compact categories. But in fact, **FHilb** is very special as a dagger compact category, as the following result due to Selinger demonstrates, inspired by an earlier result in [51] due to Hasegawa, Hofmann and Plotkin for a related kind of monoidal categories:

Theorem 4.4 : [52] *An equational statement between formal expressions in the language of dagger compact categories holds if and only if it holds in the dagger compact category* **FHilb***.*

Let us spell out what this exactly means. Obviously, any statement provable for dagger compact categories carries over to **FHilb** since the latter is an example of a dagger compact category. So anything that we prove in the graphical calculus automatically applies Hilbert spaces and linear maps. But this theorem now tells us that the converse is also true, that is, if some equational statement happens to hold for Hilbert spaces and linear maps, which is expressible in the language of dagger compact categories, then we can always derive it in the graphical language. This of course does not mean that all that we can prove about quantum theory can be proven diagrammatically. But all those statements involving identities, adjoints, (partial) transposes, conjugates, (partial) traces, composition, tensor products, Bell-states and Bell-effects, and also Hilbert spaces, numbers, states and linear maps as variables, can be proven diagrammatically. For dagger compact categories such as **Rel** and **nCob** there does not exist an analogous result.

A current challenge is to extend this so-called *completeness* theorem to richer graphical languages, e.g. the one presented in the next section of this paper, which capture even more of the Hilbert space structure. The ultimate challenge would be to find a graphical language which captures the complete Hilbert space structure, if that is even possible of course. These results are very important for the automation of quantum reasoning, discussed in the introduction. They tell us the space of theorems which a 'theorem prover' based on a certain logic is able to prove.

Obviously most of the results in quantum informatics use a much richer language than that of dagger compact categories. But that doesn't necessarily mean that it could not be formulated merely in this restrictive language. An example is the no-cloning theorem. While usually stated in linear-algebraic terms, the no-cloning theorem can in fact already be proven for arbitrary dagger compact categories, a result due to Abramsky that reads as follows:

Theorem 4.5 : [53] *If in a dagger compact category there exists a universal cloning morphism then this dagger compact category must be a trivial one. In other words, there are no non-trivial dagger compact categories which admit a universal cloning morphism.*

5 Picture calculus for quantum theory II: complementary observables and phases

The aim is now to further refine our graphical language to the extend that we can describe arbitrary linear maps within it, hence the whole of quantum theory. This will enable to perform more sophisticated computations diagrammatically, and study important quantum phenomena such as non-locality in a high-level manner. This in fact only requires few additional concepts.

5.1 *Observables as pictures*

Here are things not expressible in the graphical language of dagger compact categories:

- In our graphical description of teleportation of the previous section we mentioned that the fact that f appears both at Alice's and Bob's site implied that they needed to communicate with each other. A comprehensive diagrammatic presentation of this protocol should therefore have a second kind of wire which represents such a *classical channel*.
- The graphical description of teleportation included effects labelled by f and we mentioned that f may vary due to the non-deterministic nature of measurements. But we didn't express which such effects together make up a measurement. In other words, we have no diagrammatic descriptions of the projector spectra and eigenstates of observables.

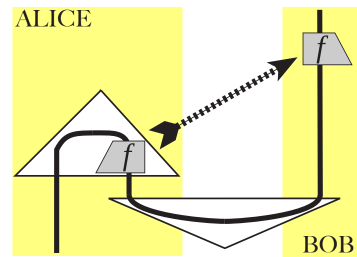
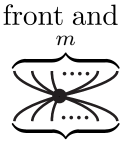


Figure 11. We want to depict a classical channel, here indicated by a dotted arrow, also by wires, different from a quantum channel of course.

We only need one kind of additional graphical element to be able to articulate each of these graphically. There are two complementary presentations of it, each pointing at distinct features. To one we refer to as *spiders*, and the other as a *copying-deleting-pair*. The results presented here appeared in joint papers with Pavlovic, Vicary and Paquette [20, 22, 55].

5.1.1 Spider presentation

A *non-degenerate observable* or *basis* for an object A in a dagger symmetric monoidal category is a family of *spiders* with n front and m back legs, one for each $n, m \in \mathbb{N}$, depicted as  and denoted by $A^{\otimes n} \xrightarrow{\delta_n^m} A^{\otimes m}$.

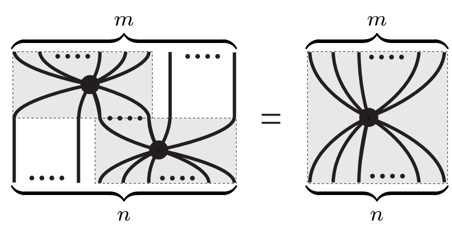


Figure 12. Rule for composing spiders. It is essential that the spiders 'shake hands/legs' i.e. the two dots corresponding to the spiders' heads need to be connected via a wire.

The composition axiom which governs these spiders is depicted on the right. In words, whenever we have two spiders such that at least one leg of spider 1 is connected to a leg of spider 2 then we can *fuse* them into a single spider. We also require δ_1^1 to be the identity, and that the set of spiders in invariant under upside-down flipping and leg-swapping. Spiders and their composition rules generalise the cup's, cap's and their yanking rule of the previous section. Indeed, when comparing Figure 8 and Figure 12 one sees that one obtains cup's, cap's and their yanking rule by interpreting $\delta_0^2 =$  as the cup and $\delta_2^0 =$  as the cap. So if on an object we have a non-degenerate observable then we also have cup's and cap's.

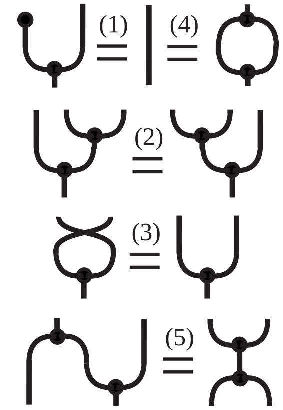
You may rightfully ask yourself what the hell these spiders have to do with the observables of quantum theory. The answer is given by the following not so trivial theorem.

Theorem 5.1: [22] *In* **FHilb** *we have that non-degenerate observables* $\{\mathcal{H}^{\otimes n} \xrightarrow{\delta_n^m} \mathcal{H}^{\otimes m}\}_{n,m}$ *in the above sense exactly correspond with orthonormal bases on the underlying Hilbert space* $\mathcal{H}$.

So on a Hilbert space $\mathcal{H}$ in **FHilb** these non-degenerate observables in terms of spiders and orthonormal bases are one-and-the-same thing. To establish which orthonormal basis on a Hilbert space $\mathcal{H}$ corresponds to a given non-degenerate observable $\{\mathcal{H}^{\otimes n} \xrightarrow{\delta_n^m} \mathcal{H}^{\otimes m}\}_{n,m}$ we will first pass to an alternative but equivalent presentation of non-degenerate observables in dagger symmetric monoidal categories. From a pictorial point of view this alternative presentation is less attractive, but both from a physical and an algebraic point of view it makes more sense.

5.1.2 Copying-deleting-pair presentation

A *non-degenerate observable* or *basis* for an object A in a dagger symmetric monoidal category consists of a *copying* operation $A \xrightarrow{\delta} A \otimes A$ and a *deleting* operation $A \xrightarrow{\varepsilon} \mathrm{I}$ which satisfy the following axioms:

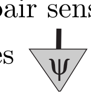


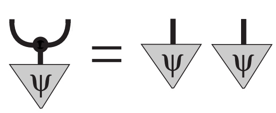
(1) ε is a *unit* for (the *comultiplication*) δ i.e. $(\varepsilon \otimes 1_A) \circ \delta = 1_A$;
(2) δ is *coassociative* i.e. $(1_A \otimes \delta) \circ \delta = (\delta \otimes 1_A) \circ \delta$;
(3) δ is *cocommutative* i.e. $\sigma_{A,A} \circ \delta = \delta$;
(4) δ is an *isometry* i.e. $\delta^\dagger \circ \delta = 1_A$;
(5) δ satisfies the *Frobenius law* introduced in [28] i.e.:

$$(\delta^\dagger \otimes 1_A) \circ (1_A \otimes \delta) = \delta \circ \delta^\dagger .$$

When introducing graphical objects $\delta =$  and $\varepsilon =$  we obtain the graphical rules depicted on the right. In standard mathematical jargon all of these together mean that (A, δ, ε) is a so-called *special dagger Frobenius commutative comonoid.* It is quite remarkable that this set of axioms exactly corresponds to the spiders discussed above. To pass from spiders to a copying-deleting-pairs we set $\delta := \delta_1^2$ and $\varepsilon := \delta_1^0$. Conversely, from the above axioms it follows that any composite of δ's, ε's, their adjoints, identities, by using both composition and tensor, and provided its graphical representation is connected, only depends on the number of inputs n and outputs m [30, 55]. The spider δ_n^m then represents this unique morphism.

So how does such a copying-deleting-pair (and hence also the spiders) encode a basis? Given a basis $\{|i\rangle\}_i$ of a Hilbert space $\mathcal{H}$ we define the copying operations as the linear map which 'copies these basis vectors' i.e. $\delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H} :: |i\rangle \mapsto |ii\rangle$, while the deleting operation is the linear map which 'uniformly deletes these basis vectors' i.e. $\varepsilon : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1$. That these maps faithfully encode this basis, and no other basis, follows directly from the no-cloning theorem [11, 12]: as the only vectors that can be copied by such an operation have to be orthogonal, they can only be the basis vectors we started from. Explicitly put, with the above prescription of δ the only non-zero vectors $|\psi\rangle \in \mathcal{H}$ satisfying the equation $\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle$ are the basis vectors $\{|i\rangle\}_i$. Eigenvectors obtained in this way are the same as those of the self-adjoint operator $\sum_i a_i |i\rangle\langle i|$, which is the representation of this observable in the usual quantum mechanical formalism.

More generally, in any dagger symmetric monoidal category one can define *eigenstates* (or *eigenvectors*) for an observable in the copying-deleting-pair sense as a state that is copied by δ. Graphically this means that these generalised eigenstates  satisfy the equation:

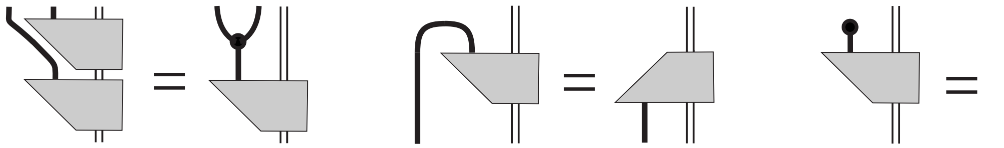


(5)

This is a strong property since it means that the 'connected' picture on the left can be replaced by the 'disconnected' one on the right. Obviously this has major implications in computations.

The copying-deleting-pair presentation also points at a physical interpretation of non-degenerated our observables. The copying and deleting maps witness those states that can be copied, and hence, again by the no-cloning theorem, those that happily live together within a classical realm. This indicates a picture of the classical-quantum distinction which is somewhat opposite to the usual one: rather than constructing a quantum version of a classical theory via quantisation, here we extract a classical version out of a quantum theory, say *classicization.* We won't go any deeper in this issue and the philosophical speculations this raises.

So the observables defined in terms of spiders are all non-degenerated. But one can define degenerated counterparts to these which, in fact, more clearly elucidates their conceptual significance. The main idea is that given a spider on A we define arbitrary not-necessarily degenerated observables as certain morphisms $B \xrightarrow{m} A \otimes B$. In this case B stands for the quantum system and A stands for the classical data (i.e. the *measured values* or the *spectrum*) for the observable. Since B appears both before and after the measurement we are considering non-demolition measurements here. We define *measurements* as those such morphism obeying:



(6)

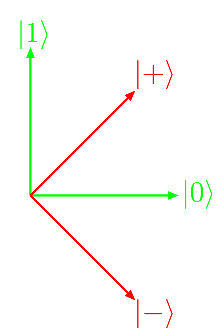
where the single wire stands for the classical data A while the double wire stands for the quantum system B – the structural reason for this single-double distinction is in [24]. The first of these conditions states: if after a measurement we perform the same measurement again then this is the same as copying the outcome we obtained in the first measurement. This of course is the same as: we obtain the same outcome in the second measurement as in the first one.

Theorem 5.2 : [20] *Linear maps* $f : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{H}_1$ *in* **FHilb** *satisfying eqs.(6) relative to a chosen basis (cf. Theorem 5.1) exactly correspond to projector spectra of self-adjoint operators on* $\mathcal{H}_1$ *for which the number of non-equal eigenvalues is equal to the dimension of* $\mathcal{H}_2$.

One can verify that for the non-degenerated observables defined as triples (A, δ, ε) the morphism $A \xrightarrow{\delta} A \otimes A$ provides an example of such a measurement, with $B := A$. The fact that both the classical data and the quantum system are represented by the same symbol might look a bit weird at first but poses no structural problem: the classical values are represented by the triple (A, δ, ε) and not by A alone. The analogy in Hilbert space quantum mechanics is that we think of the Hilbert space as the quantum system while the pair consisting of a Hilbert space and an observable 'thereon' represents the classical values for that observable. Moreover, to avoid conceptual confusion we can represent the quantum system by an isomorphic copy of A.

So now we've got ourselves a graphical representation of arbitrary observables at hand. As already mentioned at the beginning of this section this also allows us to reason about classical data flow diagrammatically – cf. the caption of Figure 11. We won't discuss this here but refer the interested reader to [20, 24], where also the role of *decoherence* in measurements is discussed, which applies to all measurements in dagger symmetric monoidal categories. Rather than focussing on the interaction of the classical and the quantum relative to a given observable, we will investigate how different observables interact, all still within the diagrammatic realm.

5.2 *A pair of complementary observables in pictures*



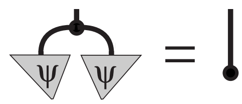
Now that we have a graphical representation for observables at hand a natural question arises: are there diagrammatic laws which describe *complementary* [56] (or *unbiased* [57]) observables?

This question was addressed by Duncan and the author in [21]. The most famous example of complementary observables are obviously the *position* and *momentum* observables. The simplest example are the Z- and X-observables for a qubit. For the Z-observable the eigenstates are $|0\rangle$ and $|1\rangle$ while for the Z-observable the eigenstates are $|+\rangle := \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle)$ and $|-\rangle := \frac{1}{\sqrt{2}} \cdot (|0\rangle - |1\rangle)$. What it means for the Z- and X-observables to be complementary is that the eigenstates for one are *unbiased* for the other. By *unbiasedness* of a (normalised) vector $|\psi\rangle$ relative to an observable we mean that $|\langle\psi|e\rangle|^2 = \frac{1}{N}$

where N is the dimension of the Hilbert space and $|e\rangle$ is any eigenstate of the observable. This in particular means that when the system is in state $|\psi\rangle$ and we measure the observable, all outcomes are equally probable, hence the term 'unbiased'. One could alternatively say that such a pair of observables are 'maximally non-classical', that is, 'maximally quantum', in that the eigenstates of one fail to be an eigenstate of the other in an 'extremal manner'. Hence one would expect a substantial chunk of quantum mechanical structure to be captured by complementary observables, and a diagrammatic account on these would substantially boost the power of the graphical calculus.

observable	**eigenstates**	**unbiased states**
Z	$\|0\rangle, \|1\rangle$	$\|0\rangle + e^{i\alpha}\|1\rangle$ e.g. $\|+\rangle, \|-\rangle$
X	$\|+\rangle, \|-\rangle$	$\|+\rangle + e^{i\alpha}\|-\rangle$ e.g. $\|0\rangle, \|1\rangle$
(A, δ, ϵ)	$\|\psi\rangle$ in eq.(5)	$\|\psi\rangle$ in eq.(7)

Unbiasedness of a state for an observable (A, δ, ϵ) can be expressed in arbitrary dagger symmetric monoidal categories and depicts as follows:



(7)

The states obeying this equation won't be normalised, but have the square-root of the dimension as length. This can be easily overcome by introducing the dimension in the lefthandside of the equation, but we won't elaborate on this issue here. What this equation says in the case of **FHilb** is easily computed: taking δ to be the linear map which copies the vectors in $\{|i\rangle\}_{i=1}^{i=n}$ then $\delta^\dagger : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} :: |ii\rangle \mapsto |i\rangle; |ij(\neq i)\rangle \mapsto 0$ so for $|\psi\rangle = (\psi_1, \ldots, \psi_n)$ we obtain

$$\delta^\dagger(|\psi\rangle \otimes \overline{|\psi\rangle}) = (\psi_1\overline{\psi_1}, \ldots, \psi_n\overline{\psi_n}) = \left(|\langle\psi|1\rangle|^2, \ldots, |\langle\psi|n\rangle|^2\right) \quad \text{while} \quad \varepsilon^\dagger = (1, \ldots, 1)\,.$$

Hence $\delta^\dagger(|\psi\rangle \otimes \overline{|\psi\rangle}) = \varepsilon^\dagger$ indeed implies unbiasedness of state $|\psi\rangle$ relative to observable $(\mathcal{H}, \delta, \varepsilon)$.

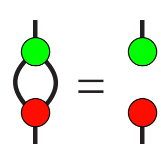
Since we now both know what 'eigenstate' –cf. eq.(5)– and 'unbiased' –cf. eq.(7)– mean in arbitrary dagger symmetric monoidal categories we can define complementarity for them:

Definition 5.3: Two observables $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ in a dagger symmetric monoidal category are *complementary* if the eigenstates of one are unbiased for the other.

Graphically, to distinguish between two observables we will use colouring, green and red.

Theorem 5.4: [21] *If a dagger symmetric monoidal category has 'enough states' then two observables* $(A, \delta_Z, \epsilon_Z)$ *and* $(A, \delta_X, \epsilon_X)$ *are complementary if and only if they satisfy:*

$$\delta_Z^\dagger \circ \delta_X = \varepsilon_Z \circ \varepsilon_X^\dagger \qquad \textit{i.e.}$$



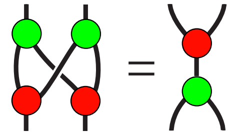
. (8)

Observe the radical topology change from the lefthandside to the righthandside of the equation. The reader can easily verify that for the Z- and the X-observables, respectively defined as:

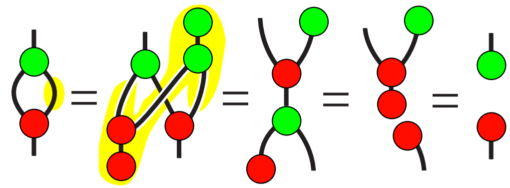
$$\delta_Z :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \qquad \varepsilon_Z :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 1 \end{cases} \qquad \delta_Z :: \begin{cases} |+\rangle \mapsto |++\rangle \\ |-\rangle \mapsto |--\rangle \end{cases} \qquad \varepsilon_Z :: \begin{cases} |+\rangle \mapsto 1 \\ |-\rangle \mapsto 1 \end{cases}$$

this equation indeed holds, up to a scalar multiple that is. But one verifies that also:

$$(\delta_Z^\dagger \otimes \delta_Z^\dagger) \circ (1_{\mathcal{H}} \otimes \sigma_{\mathcal{H},\mathcal{H}} \otimes 1_{\mathcal{H}}) \circ (\delta_X \otimes \delta_X) = \delta_X \circ \delta_Z^\dagger \quad \textit{i.e.} \tag{9}$$



holds for these Z- and the X-observables. In fact this equation holds not just for the Z- and the X-observables but seems to hold for all pairs of complementary observables in Hilbert spaces of arbitrary dimension that one encounters in the literature.[1] It is in fact a strictly stronger statement than eq.(8) as the proof on the right exposes; in the first step we rely on spiders to rewrite the wire with the yellow dot as everything in the second picture within the yellow region, the second step uses eq.(9), the third step uses the fact that the adjoint to the deleting operation for one observable is an eigenstate for the other observable, a choice which we can always make [21], and the last step again uses spiders.

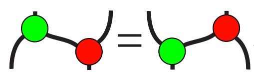
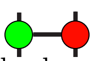
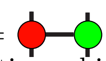


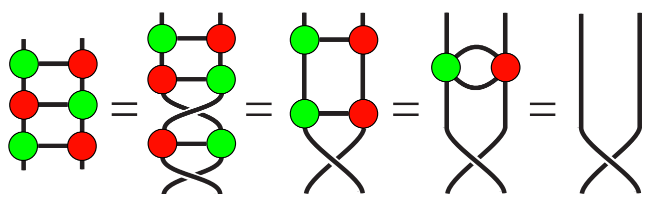
It should be mentioned that eq.(9) has also appeared in the study of so-called *bialgebras*, of which *quantum groups* are special examples [58]. The connections between their occurrence in this algebraic context and in that of complementary observables isn't completely clear yet.

5.2.1 *Example: computing with quantum logic gates*

An important gate in quantum computing is the two-qubit *controlled-not* gate:

$$\mathsf{cnot} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\mathsf{not}(x)\rangle \end{cases} \quad \text{with} \quad \mathsf{not} : \mathcal{H} \to \mathcal{H} :: \begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{cases}.$$

For the Z- and X-observable one can verify that , which we therefore can depict as . This is fact exactly cnot. We denote $\sigma_{\mathcal{H},\mathcal{H}} \circ \mathsf{cnot} \circ \sigma_{\mathcal{H},\mathcal{H}} =$  by cnot^σ. The standard result that $\mathsf{cnot} \circ \mathsf{cnot}^\sigma \circ \mathsf{cnot} = \sigma_{\mathcal{H},\mathcal{H}}$, of which the computation which usually proceeds by multiplying three four by four matrices, can now be derived from graphical laws:

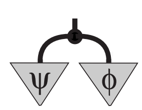
$\mathsf{cnot} \circ \mathsf{cnot}^\sigma \circ \mathsf{cnot} =$  $= \sigma_{\mathcal{H},\mathcal{H}}$

where the 2nd diagrammatic step uses eq.(9), the 3rd relies on spiders and the 4th uses eq.(8).

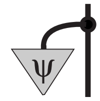
5.3 *Phases in pictures*

The general notion of observable (A, δ, ε) in dagger symmetric monoidal categories comes with a corresponding notion of (relative) phase. We denote the set of all states $\psi : \mathrm{I} \to A$ which are unbiased for (A, δ, ε) as $\mathcal{S}(A, \delta, \varepsilon)$. For two such states we moreover set

$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi) =$$

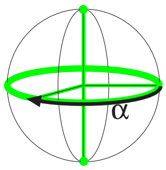


[1] The full scope of this equation is subject to ongoing investigations, as are many questions on complementary observables.

Given any state $\psi : \mathrm{I} \to A$ we can also consider $U_\psi := \delta^\dagger \circ (\psi \otimes 1_A) =$ 

and one can show that U_ψ is unitary (i.e. its adjoint is equal to its inverse) if and only if ψ is unbiased for (A, δ, ε). We denote the set of all these unitary morphisms of the form $U_\psi := \delta^\dagger \circ (\psi \otimes 1_A)$ by $\mathcal{U}(A, \delta, \varepsilon)$.

Theorem 5.5: [21] *For any observable (A, δ, ε) in a dagger symmetric monoidal category $(\mathcal{S}(A, \delta, \varepsilon), \odot, \epsilon)$ and $(\mathcal{U}(A, \delta, \varepsilon), \circ, 1_A)$ are isomorphic abelian groups. For $\mathcal{S}(A, \delta, \varepsilon)$ the inverses are provided by conjugates and for $(\mathcal{U}(A, \delta, \varepsilon), \circ, 1_A)$ the inverses are provided by adjoints.*



For a qubit in **FHilb** and basis $\{|0\rangle, |1\rangle\}$ we obtain $|0\rangle + e^{i\alpha}|1\rangle$ as the elements of $\mathcal{S}(A, \delta, \varepsilon^\dagger)$ and the unitaries with matrices $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$ as the elements of $\mathcal{U}(A, \delta, \varepsilon)$. So we indeed obtain *phases* and hence refer to this group as the *phase group*. Since $(|0\rangle + e^{i\alpha}|1\rangle) \odot (|0\rangle + e^{i\alpha'}|1\rangle) = |0\rangle + e^{i(\alpha+\alpha')}|1\rangle$ the multiplication in the group corresponds to adding angles, and since $(|0\rangle + e^{i\alpha}|1\rangle) \odot (|0\rangle + e^{-i\alpha}|1\rangle) = |0\rangle + |1\rangle = \varepsilon^\dagger$ the inverse in the group corresponds to reverses angles. To emphasise this connection of unbiased states for the particular case of qubits in **FHilb** with angles we will from now on denote unbiased states for non-degenerated observables in arbitrary dagger symmetric monoidal categories as α. For the same reasons we from now on denote the group's multiplication $\odot$ as $+$.

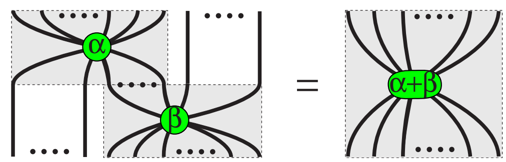
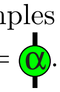


Figure 13. Spiders decorated with phases can still be fuzed together provided we add the phases.

Due to the fact that these *generalised phases* are derivable from a non-degenerate observable in a dagger symmetric monoidal category, that is, a family of spiders, they interact particularly well with these spiders. In fact, we obtain a much richer family of spiders, of which the heads are now *decorated* with these generalised phases. Strictly speaking the heads of these spiders shouldn't be symmetrical since they are not invariant under conjugation, but given that we depict them in a particular way, i.e. as circles with a Greek letter, it should be clear that to the reader that they change under conjugation. Special examples of decorated spiders are unbiased states $\alpha =$ 

and *generalised phase gates* $\delta^\dagger \circ (\alpha \otimes 1_A) =$ 

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5.3.1 *Example: information flows in quantum computational models*

By $\mathbf{FHilb}_2$ we mean the restriction of the category **FHilb** to Hilbert spaces of which the objects are restricted to powers of two-dimensional Hilbert spaces. In other words, objects correspond to n qubits where n takes values within $\mathbb{N}$.

Theorem 5.6: *Every linear map in $\mathbf{FHilb}_2$ can be expressed in the language of a pair of complementary observables and the corresponding phases. In other words, it can be written down using only red and green decorated spiders.*

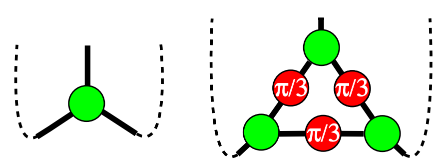
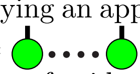
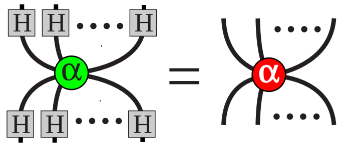


Figure 14. The GHZ-state and the W-state [59] expressed in terms of decorated spiders. The latter crucially involves three $\pi/3$ phases.

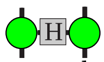
Proof. Any unitary operation from n qubits to n qubits can be expressed in terms of the two-qubit cnot gate and one-qubit phase gates [60]. Above we showed how the cnot gate can be expressed in terms of a green and a red (non-decorated) spider and that phase gates arise as special cases of decorated spiders. So we can express any unitary operation on qubits using only red and green decorated spiders. By applying an appropriate unitary to an n-qubit state which we can represent as spiders, e.g. $|+\ldots+\rangle =$ , we can obtain an arbitrary n-qubit state. Finally, via $\cap$'s, which are also special cases of spiders, we can obtain any linear maps from m to k qubits from an $m + k$-qubit state, by relying on the map-state duality of eq.(4). $\square$

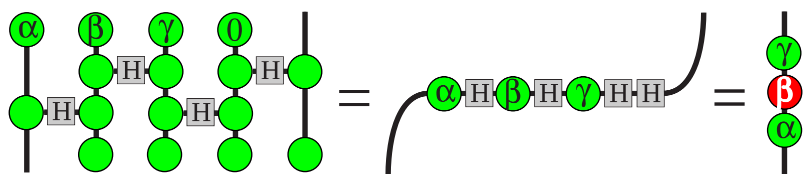


Hence the phase group structure enables us to diagrammatically represent all of quantum computing with finite spectra.

Jointly with Duncan we showed in [21] how to reason about algorithms and a variety of quantum computational models in this diagrammatic language. We present one such example here. For this purpose it proved to be useful to assume that there is an operation which 'changes colours' cf. the picture on the left. For qubits in **FHilb** the *Hadamard gate*, which has $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ as its matrix, plays this role. Since $\mathsf{cphase} = (1 \otimes H) \circ \mathsf{cnot} \circ (1 \otimes H)$ with

$$\mathsf{cphase} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\mathsf{phase}(x)\rangle \end{cases} \quad \text{with} \quad \mathsf{phase} : \mathcal{H} \to \mathcal{H} :: \begin{cases} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto -|1\rangle \end{cases} .$$

we have $\mathsf{cphase} =$ .



As can be seen on the right, we can now easily derive how we can implement an arbitrary one qubit unitary gate when the available resources are qubits in the $|+\rangle$-state, cphase-gates and postselected measurements (i.e. effects) of $|0\rangle + e^{i\alpha}|1\rangle$. The first step fuses decorated spiders and the second step is just the action of the colour changer. The righthandside represents an arbitrary one qubit unitary gate in terms of its Euler angle decomposition. This example is important in the context of so-called *measurement base quantum computing* [61] since the portion of the lefthandside picture consisting of the $|+\rangle$-states and the cphase-gates is a so-called *cluster state*. The computation shows that via postselected measurements of $|0\rangle + e^{i\alpha}|1\rangle$ applied to this entangled state we can obtain arbitrary one-qubit unitary gates, which is essential for showing universality of measurement base quantum computing as a computational model. Implicitly the diagrammatic calculation also transforms the measurement based setup into a circuit.

While this is of course a very simple example, it indicates the potential for simplifying far more complicated configurations. It also indicates the potential for translating implementations in one quantum computational model, to other quantum computational models.

Since these computations only involve discrete ingredients,[1] as compared to the continuum of the complex numbers, one can hope to automate these. At the moment we do not have a clear view on what the equational statements are that we can derive from the laws on complementary observables and the fusing rules for decorated spiders. Having a better view on this is crucial for knowing which equational statements a 'theorem prover' may be able to find.

5.3.2 Example: the origin of quantum non-locality

We report on recent work, with Edwards and Spekkens in [25], in which we trace back non-locality of theories to phase groups, and nothing but phase groups. In [62] Spekkens proposed a *toy theory* which looked remarkably similar to quantum theory. More precisely, there is an important restriction of quantum theory, its *stabeliser* fragment, in which for a qubit we only consider the eigenstates of the Z-, X- and Y-observables, but which already carries an important fragment of quantum theory. In particular, it is *non-local*. It is this fragment of quantum theory which Spekkens' toy theory aims to mimic, and at first sight it does this on-the-nose. However, there is a crucial difference: Spekkens' toy theory happens to be *local*. Given that these two theories seem so similar, what makes one local and the other one non-local?

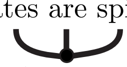
It turns out that the here discussed framework for general compositional theories provides a very precise answer to this question. In [63] Spekkens' toy theory was succinctly recast as a

[1]The derivations do not require to specify what the phase group concretely is.

dagger symmetric monoidal category, called **Spek**. The same can easily be done for the stabiliser fragment of quantum theory, to which we refer as **Stab**. One can show that the observables for a qubit, now of course in our more generalised sense, exactly match in these theories. In particular, in both a qubit has three mutually complementary observables (in our generalised sense).

There are many other correspondences between these, namely in both cases all qubit states are either an unbiased state or an eigenstate for any qubit observables. There is a notion of GHZ-state and GHZ-correlations that applies to arbitrary dagger symmetric monoidal categories. For three qubits there are tripartite GHZ-states both in **Spek** and in **Stab**. Hence we can speak of *GHZ-correlations* in both theories, that is, which outcomes one can simultaneously obtain in measurements of each of the qubits in a GHZ-state. It is here that things become interesting.

Theorem 5.7: [25] *In all dagger symmetric monoidal categories, observables and GHZ-states are in a canonical bijective corresponce. Moreover, if for a certain object all states are either eigenstates or unbiased relative to an observable, then the phase group of that observable completely determines the GHZ-correlations for the corresponding GHZ-states.*

GHZ-states are spiders with three front and no back legs i.e. $\delta_0^3 =$ . We also know that phase groups are commutative groups. Both for the qubits in **Spek** and **Stab** these contain four elements, and in fact, there are exactly two four element groups, namely Z_4 and $Z_2 \times Z_2$. For observables on qubits in **Spek** the phase group happens to be $Z_2 \times Z_2$, while for observables on qubits in **Stab** the phase group is Z_4. One can moreover show that having Z_4 as a phase group is enough for a theory to be non-local. This result (at least to some extend) unveils which 'piece' of the Hilbert space puzzle causes non-locality.

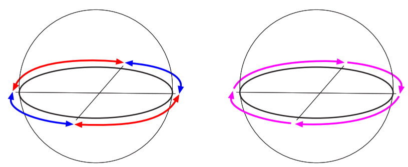


Figure 15. The different phase groups for the compositional theories **Spek** and **Stab**.

6 Experimental verification: kindergarten quantum mechanics

In physics and science in general, traditionally, claims have to be substantiated by experiments. Is there any way we can substantiate our claims concerning the low-levelness of the quantum mechanical formalism via actual experiments? Here is a sketch for such an experiment.

Experiment. Consider ten children of ages between six and ten and consider ten high-school teachers of physics and mathematics. The high-school teachers of physics and mathematics will have all the time they require to refresh their quantum mechanics background, and also to update it with regard to recent developments in quantum information. The children on the other hand will have quantum theory explained in terms of the graphical formalism. Both teams will be given certain set of questions, for the children formulated in diagrammatic language, and for the teachers in the usual quantum mechanical formalism. Who solves the most problems and solves them in the fastest time wins. If the diagrammatic language is much more intuitive, it should be possible for the children to win.

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