QUANTUM MEASURE THEORY

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1. Introduction

A **measurable space** is a pair $(X, \mathcal{A})$ where X is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of $\mathcal{A}$. A (finite) **measure** on $\mathcal{A}$ is a map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ satisfying

(1) $\mu(A \dot{\cup} B) = \mu(A) + \mu(B)$ (**additivity**)

(2) If $A_i \in \mathcal{A}$ is an increasing sequence, then

$\mu(\cup A_i) = \lim \mu(A_i)$ (**continuity**)

Conditions (1) and (2) together are equivalent to

$\mu(\dot{\cup} A_i) = \sum \mu(A_i)$ (σ-**additivity**)

It follows from (2) that

(3) If $A_i \in \mathcal{A}$ is a decreasing sequence, then $\mu(\cap A_i) = \lim \mu(A_i)$

Because of quantum interference, Condition (1) fails for quantum measures. Instead we have the weaker condition

(4) $\mu(A \dot{\cup} B \dot{\cup} C) = \mu(A \dot{\cup} B) + \mu(A \dot{\cup} C) + \mu(B \dot{\cup} C) - \mu(A) - \mu(B) - \mu(C)$

We call (4) **grade**-2 **additivity** and also call (1) **grade**-1 **additivity**. There are higher grades of additivity but we will not discuss them here. If $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is grade-2 additive and satisfies (2) and (3), we call μ a q-**measure**. If μ is a q-measure on $\mathcal{A}$, then $(X, \mathcal{A}, \mu)$ is a q-**measure space**.

2. Examples

If ν is a measure then $\mu(A)=\nu(A)^2$ is a q-measure.

If ν is a complex measure (quantum amplitude) then $\mu(A)=|\nu(A)|^2$ is a q-measure. In this case

$$\begin{aligned}\mu(A\dot{\cup}B)&=|\nu(A\dot{\cup}B)|^2=|\nu(A)+\nu(B)|^2\\&=\mu(A)+\mu(B)+2\mathrm{Re}\left[\nu(A)\overline{\nu(B)}\right]\end{aligned}$$

Additivity is destroyed by the quantum interference term $2\mathrm{Re}\left[\nu(A)\overline{\nu(B)}\right]$.

More generally, a **decoherence functional** is a map $D\colon\mathcal{A}\times\mathcal{A}\to\mathbb{C}$ satisfying

(5) $D(A\dot{\cup}B,C)=D(A,C)+D(B,C)$

(6) $D(A,B)=\overline{D(B,A)}$

(7) $D(A,A)\geq 0$

(8) $|D(A,B)|^2\leq D(A,A)D(B,B)$

We say that D is **continuous** if $\mu(A)=D(A,A)$ satisfies (3) and (4). It can be shown that $\mu(A)=D(A,A)$ is then a q-measure. An example of a continuous decoherence functional from quantum measurement theory is

$$D(A,B)=\mathrm{tr}\left[WE(A)E(B)\right]$$

where W is a density operator (state) and E is a positive operator-valued measure (observable).

There are experimental reasons for considering q-measures. Let A_1 and A_2 be the two slits of a two-slit experiment. If $\mu(A_i)$ is the probability that a particle hits a small region Δ of the detection screen after going through slit A_i, $i = 1, 2$, then

$$\mu(A_1 \dot{\cup} A_2) \neq \mu(A_1) + \mu(A_2)$$

in general so μ is not additive. Recent experiments with a three-slit experiment indicate that μ is grade-2 additive so μ is a q-measure.

Example 1. Let $X = \{x_1, x_2, x_3, x_4\}$ and let $\mathcal{P}(X)$ be the power set on X. Define the measure ν on $\mathcal{P}(X)$ by $\nu(x_i) = 1/4$, $i = 1, 2, 3, 4$. We may think of X as the four outcomes of flipping a fair coin twice. Now consider the "quantum coin" with "probabilities" given by $\mu(A) = \nu(A)^2$. Then the "probability" of each sample point is $1/16$ and the "probability" that at least one head appears is $9/16$.

Example 2. Let $X = \{x_1, x_2, x_3\}$ with $\mu(\emptyset) = \mu(x_1) = 0$ and $\mu(A) = 1$ for all other $A \in \mathcal{P}(X)$. Then μ is a q-measure on $\mathcal{P}(X)$.

Example 3. Let $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_n\}$ and call (x_i, y_i), $i = 1, \dots, m$, **destructive pairs** (or **particle-antiparticle pairs**). Denoting the cardinality of a set B by $|B|$ define

$$\mu(A) = |A| - 2\,|\{(x_i, y_i) \colon x_i, y_i \in A\}|$$

for every $A \in \mathcal{P}(X)$. Thus, $\mu(A)$ is the cardinality of A after the destructive pairs in A have annihilated each other. For instance, $\mu(\{x_1, y_1, z_1\}) = 1$ and $\mu(\{x_1, y_1, y_2, z_1\}) = 2$. Then μ is a q-measure on $\mathcal{P}(X)$.

Example 4. This is a continuum version of Example 3. Let $X = [0, 1]$, let ν be Lebesgue measure on X and let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X. Define $\mu \colon \mathcal{B}(X) \to \mathbb{R}^+$ by

$$\mu(A) = \nu(A) - 2\nu\left(\{x \in A \colon x + 3/4 \in A\}\right)$$

In this case the pairs $(x, x + 3/4)$ with $x \in A$ and $x + 3/4 \in A$ act as destructive pairs. For instance $\mu([0, 1]) = 1/2$ and $\mu([0, 3/4]) = 3/4$. Again, μ is a q-measure on $\mathcal{B}(X)$.

The **symmetric difference** of sets A and B is

$$A \Delta B = (A \cap B') \cup (A' \cap B)$$

The next result is the quantum counterpart to the usual formula for measures: $\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$

3. Compatibility and the Center

Theorem 3.1. A map $\mu\colon\mathcal{A}\to\mathbb{R}^+$ is grade-2 additive if and only if μ satisfies

$$\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)+\mu(A\Delta B)-\mu(A\cap B')-\mu(A'\cap B)$$

The next result shows that grade-2 additivity can be extended to more than three mutually disjoint sets.

Theorem 3.2. If $\mu\colon\mathcal{A}\to\mathbb{R}^+$ is grade-2 additive, then for any $n\geq 3$ we have

$$\mu\left(\dot{\bigcup_{i=1}^{n}}A_i\right)=\sum_{i<j=1}^{n}\mu(A_i\dot{\cup}A_j)-(n-2)\sum_{i=1}^{n}\mu(A_i)$$

Let $(X,\mathcal{A},\mu)$ be a q-measure space. We say that $A,B\in\mathcal{A}$ are μ-**compatible** and write $A\mu B$ if $\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)$. Clearly $A\mu A$ for all $A\in\mathcal{A}$ and by Theorem 3.1, $A\mu B$ iff

$$\mu(A\Delta B)=\mu(A\cap B')+\mu(A'\cap B)$$

The μ-**center** of $\mathcal{A}$ is $Z_\mu=\{A\in\mathcal{A}\colon A\mu B \text{ for all } B\in\mathcal{A}\}$. A set $A\in\mathcal{A}$ is μ-**splitting** if $\mu(B)=\mu(B\cap A)+\mu(B\cap A')$ for all $B\in\mathcal{A}$.

Lemma 3.3. A is μ-splitting iff $A\in Z_\mu$.

Theorem 3.4. Z_μ is a sub σ-algebra of $\mathcal{A}$ and $\mu\mid Z_\mu$ is a measure. If $A_i\in Z_\mu$ are mutually disjoint, then $\mu\left[\dot{\cup}(B\cap A_i)\right]=\sum\mu(B\cap A_i)$ for every $B\in\mathcal{A}$.

In Examples 3 and 4 we have

Theorem 3.5. TFSAE (a) $A\in Z_\mu$, (b) $A\mu A'$, (c) $\mu(A)+\mu(A')=\frac{1}{2}$.

4. Characterization of Quantum Measures

A signed measure λ on $\mathcal{A}\times\mathcal{A}$ is **symmetric** if $\lambda(A\times B)=\lambda(B\times A)$ for all $A,B\in\mathcal{A}$. The next lemma shows that a symmetric signed measure λ on $\mathcal{A}\times\mathcal{A}$ is determined by its values $\lambda(A\times A)$ for $A\in\mathcal{A}$.

Lemma 4.1. If λ is a symmetric signed measure on $\mathcal{A}\times\mathcal{A}$, then for every $A,B\in\mathcal{A}$ we have

$$\begin{aligned}\lambda(A\times B)=\tfrac{1}{2}\{&\lambda\left[(A\cup B)\times(A\cup B)\right]+\lambda\left[(A\cap B)\times(A\cap B)\right]\\&-\lambda\left[(A\cap B')\times(A\cap B')\right]-\lambda\left[(A'\cap B)\times(A'\cap B)\right]\}\end{aligned}$$

A signed measure λ on $\mathcal{A}\times\mathcal{A}$ is **diagonally positive** if $\lambda(A\times A)\geq 0$ for all $A\in\mathcal{A}$.

Theorem 4.2. A map $\mu\colon\mathcal{A}\to\mathbb{R}^+$ is a q-measure iff there exists a diagonally positive, symmetric signed measure λ on $\mathcal{A}\times\mathcal{A}$ such that $\mu(A)=\lambda(A\times A)$ for all $A\in\mathcal{A}$. Moreover, λ is unique.

Idea of Proof. Uniqueness follows from Lemma 4.1. If $\mu(A)=\lambda(A\times A)$, it is easy to check that μ is a q-measure. Conversely, let μ be a q-measure. For $A,B\in\mathcal{A}$ define

$$\lambda(A\times B)=\tfrac{1}{2}\left[\mu(A\cup B)+\mu(A\cap B)-\mu(A\cap B')-\mu(A'\cap B)\right]$$

Show that λ extends to a signed measure on $\mathcal{A}\times\mathcal{A}$. $\square$

5. A Quantum Integral

Let $(X, \mathcal{A}, \mu)$ be a q-measure space. We first discuss how not to define a q-integral. If f is a measurable simple function then f has a unique canonical representation $f = \sum c_i \chi_{A_i}$ where $c_i \neq c_j$, $A_i \cap A_j = \emptyset$, $i \neq j$, $A_i \in \mathcal{A}$, $i = 1, \ldots, n$. Following Lebesgue we define the naive integral $N \int f d\mu = \sum c_i \mu(A_i)$. One problem is that $N \int d\mu$ is ambiguous. Unlike the Lebesgue integral if we represent f in a noncanonical way $f = \sum d_i \chi_{B_i}$, then in general $N \int f d\mu \neq \sum d_i \mu(B_i)$. Another problem is that the usual limit laws do not hold so there seems to be no way to extend this naive integral to arbitrary measurable functions. For instance, in Example 2, $N \int 1 d\mu = 1$. If we define the functions

$$f_n = \chi_{\{x_1, x_2\}} + \left(1 - \frac{1}{n}\right) \chi_{\{x_3\}}$$

then f_n is an increasing sequence converging to 1. But

$$N \int f_n d_\mu = \mu\left(\{x_1, x_2\}\right) + \left(1 - \frac{1}{n}\right) \mu(x_3) = 2 - \frac{1}{n}$$

Hence,

$$\lim_{n \to \infty} N \int f_n d\mu = 2 \neq 1 = N \int 1 d\mu$$

We overcome these difficulties by defining

$$\int f d\mu = \int_0^\infty \mu\left[f^{-1}(\lambda,\infty)\right] d\lambda - \int_0^\infty \mu\left[f^{-1}(-\infty,-\lambda)\right] d\lambda$$

if both the integrals on the right are finite and $d\lambda$ is Lebesgue measure. We then say that f is **integrable**. Any measurable function has the unique representation $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ are measurable and $f_1 f_2 = 0$. In fact, $f_1 = \max(f,0)$, $f_2 = -\min(f,0)$. We then have

$$\int f d\mu = \int f_1 d_\mu - \int f_2 d\mu$$

Because of this we can usually study the properties of the integral by considering nonnegative functions.

Some properties of the integral are: $\int cf d\mu = c\int f d\mu$,
$\int (c+f)d\mu = c\mu(X) + \int f d\mu = \int c d\mu + \int f d\mu$ and $\int f d\mu \geq 0$ if $f \geq 0$.
The integral is not linear. For example, if $\mu(A\dot{\cup}B) \neq \mu(A) + \mu(B)$ we have

$$\int (\chi_A + \chi_B) d\mu = \int \chi_{A\dot{\cup}B} d\mu = \mu(A\dot{\cup}B) \neq \mu(A) + \mu)(B) = \int \chi_A d\mu + \int \chi_B d\mu$$

Theorem 5.1. If $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $A_i \cap A_j = \emptyset$, $i \neq j$, and $0 < \alpha_1 < \cdots < \alpha_n$, then

$$\begin{aligned}
&\int f d\mu \\
&= \alpha_1\left[\mu(A_1\dot{\cup}A_2) + \cdots + \mu(A_1\dot{\cup}A_n) - (n-2)\mu(A_1) - \mu(A_2) - \cdots - \mu(A_n)\right] \\
&+ \alpha_2\left[\mu(A_2\dot{\cup}A_3) + \cdots + \mu(A_2\dot{\cup}A_n) - (n-3)\mu(A_2) - \mu(A_3) - \cdots - \mu(A_n)\right] \\
&\vdots \\
&+ \alpha_{n-1}\left[\mu(A_{n-1}\dot{\cup}A_n) - \mu(A_n)\right] + \alpha_n \mu(A_n)
\end{aligned}$$

For example, we see from Theorem 5.1 that

$$\int \alpha_1 \chi_{A_1} d\mu = \alpha_1 \mu(A_1)$$

$$\int (\alpha_1 \chi_{A_1} + \alpha_2 \chi_{A_2}) d\mu = \alpha_1 \left[\mu(A_1 \dot\cup A_2) - \mu(A_2)\right] + \alpha_2 \mu(A_2)$$

$$\begin{aligned}\int (\alpha_1 \chi_{A_1} + \alpha_2 \chi_{A_2} + \alpha_3 \chi_{A_3}) d\mu \\ &= \alpha_1 \left[\mu(A_1 \dot\cup A_2) + \mu(A_1 \dot\cup A_3) - \mu(A_1) - \mu(A_2) - \mu(A_3)\right] \\ &+ \alpha_2 \left[\mu(A_2 \dot\cup A_3) - \mu(A_3)\right] + \alpha_3 \mu(A_3)\end{aligned}$$

We conclude from Theorem 5.1 that if μ happens to be a measure, then $\int f d\mu = \sum \alpha_i \mu(A_i) = N \int f d\mu$. Thus, the q-integral generalizes the Lebesgue integral.

Theorem 5.2. If f, g and h are integrable functions with disjoint support, then

$$\begin{aligned}\int (f + g + h) d\mu = \int (f + g) d\mu + \int (f + h) d\mu + \int (g + h) d\mu \\ - \int f d\mu - \int g d\mu - \int h d\mu\end{aligned}$$

By induction, Theorem 5.2 extends to n integrable functions $f_, \ldots, f_n$ with mutually disjoint support

$$\int \sum_{i=1}^{n} f_i d\mu = \sum_{i<j=1}^{n} \int (f_i + f_j) d\mu - (n-2) \sum_{i=1}^{n} \int f_i d\mu$$

For $A \in \mathcal{A}$ we define $\int_A f d\mu = \int f \chi_A d\mu$.

Corollary 5.3. If f is integrable, then

$$\int_{A\dot{\cup}B\dot{\cup}C} f d\mu = \int_{A\dot{\cup}B} f d\mu + \int_{A\dot{\cup}C} f d\mu + \int_{B\dot{\cup}C} f d\mu - \int_A f d\mu - \int_B f d\mu - \int_C f d\mu$$

Theorem 5.2 does not extend to arbitrary integrable functions f, g and h. Consider Example 1 of a "quantum coin." Let $A = \{x_1, x_2\}$, $B = \{x_2, x_3\}$, $C = \{x_3, x_4\}$ and let $f = \chi_A$, $g = \chi_B$, $h = \chi_C$. Since

$$f + g + h = \chi_{\{x_1,x_4\}} + 2\chi_{\{x_2,x_3\}}$$

by Theorem 5.1 we have

$$\int (f + g + h)d\mu = \mu(X) - \mu(B) + 2\mu(B) = 5/4$$

However,

$$\int (f+g)d\mu + \int (f+h)d\mu + \int (g+h)d\mu - \int f d\mu - \int g d\mu - \int h d\mu$$
$$= \int \left(\chi_{\{x_1,x_3\}} + 2\chi_{\{x_2\}}\right) d\mu + \int 1 d\mu + \int \left(\chi_{\{x_2,x_4\}} + 2\chi_{\{x_3\}}\right) d\mu - 3/4$$
$$= \mu\left(\{x_1, x_2, x_3\}\right) - \mu(x_2) + 2\mu(x_2) + 1 + \mu\left(\{x_2, x_3, x_4\}\right) - \mu(x_3)$$
$$+ 2\mu(x_3) - 3/4 = 3/2$$

These do not coincide. For another example of a q-integral let f be the number of heads. Then $f(x_1) = 2$, $f(x_2) = f(x_3) = 1$, $f(x_4) = 0$ so $f = \chi_{\{x_2,x_3\}} + 2\chi_{\{x_1\}}$. Hence,

$$\int f d\mu = \mu\left(\{x_1, x_2, x_3\}\right) - \mu(x_1) + 2\mu(x_1) = 5/8$$

Compare this to

$$N \int f d\mu = \mu\left(\{x_2, x_3\}\right) + 2\mu(x_1) = 3/8$$

6. Convergence Theorem

For measurable functions f, g on X we say that g μ-**dominates** f if $\mu\left[g^{-1}(\lambda, \infty)\right] \geq \mu\left[f^{-1}(\lambda, \infty)\right]$ for every $\lambda \in \mathbb{R}$.

Lemma 6.1. If μ is a measure, then $f \leq g$ a.e. $[\mu]$ implies that g μ-dominates f.

The converse of Lemma 6.1 does not hold. For example, let $X = [0,1]$ and let μ be Lebesgue measure on X. Let $f = \frac{1}{2}\chi_{[1/2,1]}$ and $g = \chi_{[0,1/2]}$. Then g μ-dominates f but $f \not\leq g$ a.e. $[\mu]$.

Theorem 6.2. (q-dominated monotone convergence theorem) If $f_i \geq 0$ is an increasing sequence of measurable functions on a q-measure space $(X, \mathcal{A}, \mu)$ converging to f and there exists an integrable function g with g μ-dominating f_i for all i, then $\lim_{i\to\infty} \int f_i d\mu = \int f d\mu$.

Let $f\colon X \to \mathbb{R}^+$ be measurable and suppose there exists a Lebesgue integrable function $g\colon \mathbb{R} \to \mathbb{R}^+$ such that for every $A \in \mathcal{A}$, $\mu\left(\{x \in A\colon f(x) > \lambda\}\right) \leq g(\lambda)$ for all $\lambda \in \mathbb{R}$. Define $\mu_1(A) = \int_A f d\mu$. Then

$$\mu_1(A) = \int \mu\left(\{x \in A\colon f(x) > \lambda\}\right) d\lambda \leq \int g(\lambda) d\lambda < \infty$$

It follows from Corollary 5.3 that μ_1 is grade-2 additive.

Theorem 6.3. μ_1 is a q-measure and $\mu_1 \ll \mu$ (i.e. $\mu(A) = 0$ implies $\mu_1(A) = 0$).

Theorem 6.3 suggests a q-Radon-Nikodym theorem but alas, there is no such theorem even when X is finite.

Example. As in Example 2, let $X = \{x_1, x_2, x_3\}$ with $\mu(\emptyset) = \mu(x_1) = 0$ and $\mu(A) = 1$ for all other $A \in \mathcal{P}(X)$. Let ν be the measure on $\mathcal{P}(X)$ given by $\nu(x_1) = 0$, $\nu(x_2) = \nu(x_3) = 1$, so that $\nu(\{x_2, x_3\}) = \nu(X) = 2$ and $\nu(\{x_1, x_2\}) = \nu(\{x_1, x_3\}) = 1$. Then $\nu \ll \mu$. Suppose there exists an f such that $\nu(A) = \int_A f d\mu$ for every $A \in \mathcal{P}(X)$. Then

$$f(x_2) = f(x_2)\mu(x_2) = \int_{\{x_2\}} f d\mu = \nu(x_2) = 1$$
$$f(x_3) = f(x_3)\mu(x_3) = \int_{\{x_3\}} f d\mu = \nu(x_3) = 1$$

Hence,

$$2 = \nu(\{x_2, x_3\}) = \int_{\{x_2, x_3\}} f d\mu = \mu(\{x_2, x_3\}) = 1$$

which is a contradiction.

7. Quantum Lebesgue Measure

Let $X = [0,1]$ and let ν be Lebesgue measure on $\mathcal{B}(X)$. Define the q-measure μ on $\mathcal{B}(X)$ by $\mu(A) = \nu(A)^2$. We call μ q-Lebesgue measure. In the sequel y will denote a fixed element of X. We now compute integrals of common functions.

$$\int_0^y 1 d\mu = \mu\left([0,y]\right) = y^2$$

$$\int_0^y x d\mu(x) = \int_0^\infty \nu\left(\{x\colon x\chi_{[0,y]}(x) > \lambda\}\right)^2 d\lambda = \int_0^y (y-\lambda)^2 d\lambda$$

$$= -\frac{(y-\lambda)^3}{3}\Bigg|_0^y = \frac{y^3}{3}$$

In general, for any nonnegative integer n

$$\int_0^y x^n d\mu(x) = \frac{2}{(n+2)(n+1)}\, y^{n+2}$$

Surprisingly, we have

$$\int_0^y (x^2+x) d\mu(x) = \frac{1}{6}\, y^4 + \frac{1}{3}\, y^3 = \int_0^y x^2 d\mu(x) + \int_0^y x d\mu(x)$$

We call the next result the q-fundamental theorem of calculus.

Theorem 7.1. If f is differentiable and monotone on $(0,y)$, then

$$\frac{1}{2}\,\frac{d^2}{dy^2}\int_0^y f(x)d\mu(x) = f(y)$$