Quantum logic on finite dimensional Hilbert spaces

Tobias Hagge

U.T. Dallas

May 11, 2009

Joint work with J. Michael Dunn, Lawrence S. Moss, Zhenghan Wang

Outline

Some personal desires

- Logical foundation for quantum computation
- Logical foundation for representation theory?
- Classical logic as emergent phenomenon?

Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property

Hilbert ortholattices

The set of closed subspaces S_H of a Hilbert space H give rise to an *ortholattice* $L_H = (S_H, \wedge, \vee, \neg, \{0\}, H)$ with

- $\wedge =$ intersection,
- $\vee =$ closure of span of union,
- $\neg =$ orthogonal complement.

Axioms for an ortholattice $(S, \wedge, \vee, 0, 1)$:

- $(S, \wedge, 1)$ and $(S, \vee, 0)$ are commutative, idempotent monoids,
- $a \wedge b = a \Leftrightarrow a \vee b = b$ for all $a, b \in L$ (say $a \leq b$ if $a \vee b = b$),
- $\neg : L \rightarrow L$ is an involution such that $\neg(a \vee b) = \neg a \wedge \neg b$ for all $a, b \in L$,
- $a \vee \neg a = 1$.

(Ortho)modularity and distributivity

Hilbert lattices also satisfy the *orthomodular law*, i.e.:

$$y \leq x \implies y = x \wedge (y \vee \neg x)$$

Note: $P(y, x) := x \wedge (y \vee \neg x)$ is the projection of y onto x.
The n-dimensional Hilbert space H^n is *modular*, i.e.:

$$b \leq x \implies x \wedge (a \vee b) = (x \wedge a) \vee b$$

$L = (S, \wedge, \vee, 0, 1)$ is not necessarily distributive. One only has:

$$(a \vee b) \wedge c \geq (a \wedge c) \vee (b \wedge c)$$

L_{H^n} is distributive iff $n = 1$.
A counterexample in H^2: $a = \langle (1, 0) \rangle$, $b = \langle (0, 1) \rangle$, $c = \langle (1, 1) \rangle$.

Validity

A formula $\phi(a_1, \dots a_n)$ is a *valid* in L if $\phi(a_1, \dots a_n) = 0$ for all choices of $a_1, \dots, a_n \in L$. Let $QL(L)$ denote the set of formulas valid in L.
Note: $a = b$ iff $(a \vee b) \wedge (\neg a \vee \neg b) = 0$ iff $(a \wedge b) \vee (\neg a \wedge \neg b) = 1$.

$QL(H^n)$ becomes *weaker* as n increases. If $H^{n+1} = H^n \oplus v$, with $v = \langle v_1 \rangle$ and $\langle h, v_1 \rangle = 0$ for all $h \in H^n$, then:

$$\phi_{L_{H^{n+1}}}(a_1 \vee v, \dots a_n \vee v) = \phi_{L_{H^n}}(a_1, \dots, a_n) \vee \phi_{L_{H^1}}(v, \dots, v).$$

However, $QL(L_{H^\infty}) \neq \bigcap_{n \in \mathbb{N}} QL(L_{H^n})$ since $QL(L_{H^\infty})$ is not modular.

Axiomatization of $QL(L_{H^n})$ and $QL(L_{H^\infty})$

- We don't know how to write a nice set of axioms for $QL(L_{H^n})$ (or $QL(L_{H^\infty)})$.
- Does modular + ortholattice + n-distributive axiomatize $QL(L_{H^n})$?
- Does $QL(L_{H^n})$ have a finite axiomatization?

Decidability of $QL(H^n)$

$\exists$ undecidable MOLs (Roddy 1989), but $QL(L_{H^n})$ is decidable. Reduces to decidability of FOL of $\mathbb{R}$ (decidable by (Tarski, 1948)). Sketch:

- For each $a \in L_{H^n}$, assign a matrix M_a with kernel a.
- Build $M_{\phi(\vec{a})}$ by structural induction, introducing a new matrix variable at each stage
- ϕ is valid iff $M_{\phi(\vec{a})}$ is always the zero matrix.

A	$M_A \in M_n(\mathbb{C})$: A matrix with kernel A.
$C = A \wedge B$	$\forall d(M_A d = 0 \wedge M_B d = 0 \Leftrightarrow M_C d = 0)$
$B = \neg A$	$\forall c(M_B c = 0 \Leftrightarrow (\forall d(M_A d = 0 \Rightarrow \langle c, d \rangle = 0)))$

Decidability of $QL(L_{H^\infty})$

It is not known whether $QL(L_{H^\infty})$ is decidable.

Everything in this talk depends on the notion of dimension, which is less useful in H^∞.

Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property

Dimensional dependence part 1

$QL(L_{H^n}) \neq QL(L_{H^{n+1}})$ (Dunn, H., Moss, Wang 2004, H. 2007):
Ideas:

1. Given $s \leq 1$, and a formula ϕ, one may construct a formula $\phi|_s$ which expresses the proposition that ϕ is valid in the sublattice generated below s.
2. Failure of distributive law is not arbitrary:
   If $a = p \vee (q \wedge r)$, $b = (p \vee q) \wedge (p \vee r)$ then
   $$dim((a \vee b) \wedge (\neg a \vee \neg b)) \leq \lfloor \frac{n}{2} \rfloor.$$
3. Projections obey dimensional laws:
   $$dim(P(P(P(a, b), a), \neg b)) \leq min(dim(b), dim(\neg b) \leq \lfloor \frac{n}{2} \rfloor.$$

This was, however, not the first (or nicest) proof of this result.

CONTRIBUTIONS TO GENERAL ALGEBRA 5
Proceedings of the Salzburg Conference, Mai 29 - June 1, 1986
Verlag Hölder-Pichler-Tempsky, Wien 1987 - Verlag B. G. Teubner, Stuttgart

Michael Roddy - René Mayet

n-DISTRIBUTIVITY IN ORTHOLATTICES

We introduce an ortholattice equation which is equivalent to n-distributivity in the variety of modular ortholattices, and discuss it.

Introduction. The basic tool of von Neumann's coordinatization theorem [10] is the homogeneous n-frame, and the importance of this configuration in the variety of modular lattices has been well documented. In particular, Huhn showed that the exclusion of the n-frame for a given n defines a variety of modular lattices, the (n-1)-distributive modular lattices. He then proceeded to develop the theory of these and related lattices (in the several papers listed in the bibliography).

The definitions and the theory pertain to any class of algebras each of whose members has a modular lattice reduct, in particular to MOL (the modular ortholattices). Moreover, these ideas have natural "ortho"-generalizations. The purpose of this paper is to present some of this theory in the ortholattice setting.

One motivation for presenting this material is that the existence of complements in MOL's makes the ideas even more powerful than in the variety of modular lattices. We think that this is illustrated by their use in the proof of the following two theorems.

Theorem (0.1). ([11]). Every variety of MOL's is comparable to the variety generated by MOω.

Theorem (0.2). ([12]).

(a) There exists a finitely presented three generated MOL

Dimensional dependence part 0

An ortholattice is *n-distributive* if

$$x \vee \bigwedge_i y_i = \bigwedge_j (x \vee \bigwedge_{i \neq j} y_i) \qquad (D_n)$$

for all $x, y_1, \ldots y_n \in L$. (This equation may be replaced by an equation in three variables.)

(Roddy, Mayet 1986) showed that n-distributivity holds in a subirreducible MOL iff (among other things) L has height $\leq n$. Thus $D_n \in QL(L_{H^n})$ but $D_n \notin QL(L_{H^{n+1}})$.

Dimensional dependence part 2

Work in progress by Giuntini and Freytes puts the result in a general framework.

A lattice is *atomic* if for every $a \in L$, $a = \bigvee_i a_i$, where for each a_i, $b < a_i \implies b = 0$.
Atomic lattices are the most general framework for making dimension arguments.

Let $L_a = \{b \in L | b \leq a\}$.
Then $a < b \implies QL(L_a) \supset QL(L_b)$ in an atomic MOL L iff L is irreducible.

Outline

Finite submodel property

(We'll say) $QL(L)$ has the *finite submodel property* if for any formula $\phi \notin QL(L)$, L has a finite sublattice L' such that $\phi \notin QL(L')$.

Theorem

$QL(L_{H^n})$ does not have the finite submodel property. Furthermore, for $n \geq 3$ there exists $\phi \notin QL(L_{H^n})$ such that $\phi \in QL(L)$ for any sublattice L which is not dense in $QL(L_{H^n})$.

I do not know the full generality in which this theorem holds.

"Classical" conjunctions

Given a formula $\phi(\overrightarrow{a})$, we can construct $\widetilde{\phi}(\overrightarrow{a}, \overrightarrow{b})$ such that:

1. If $\phi(\overrightarrow{a}) = 0$ then $\widetilde{\phi}(\overrightarrow{a}, \overrightarrow{b}) = 0$ for all $\overrightarrow{b}$,
2. If $\phi(\overrightarrow{a}) \neq 0$, then for some $\overrightarrow{b}$, $\widetilde{\phi}(\overrightarrow{a}, \overrightarrow{b}) = 1$.

Then $\widetilde{\phi} \wedge \chi$ is valid in L_{H^n} iff for every $\overrightarrow{a}$, either $\phi(\overrightarrow{a}) = 0$ or $\chi(\overrightarrow{a}) = 0$.
Construction sketch:

- If $dim(\phi(\overrightarrow{a})) = m > 0$ we can choose b, c so that $P(P(\phi(\overrightarrow{a}), b), c)$ is any arbitrary subspace of dimension $\leq m$.
- $P(P(\phi(\overrightarrow{a}), b_1), c_1) \vee \ldots \vee P(P(\phi(\overrightarrow{a}), b_n), c_n)$ can take any value with appropriate choices of $b_1, \ldots, b_n$ and $c_1 \ldots c_n$.

A weaker statement is possible in $QL(L_{H^\infty})$.

Dimensional restriction

$\widetilde{D_k|_a}(a, \overrightarrow{b}, \overrightarrow{c}) = 0$ if $dim(a) \leq k$, gives arbitrary values from choices of $\overrightarrow{b}$ and $\overrightarrow{c}$ otherwise.

Let

$R_k(a, \overrightarrow{b_1}, \overrightarrow{c_1}, \overrightarrow{b_2}, \overrightarrow{c_2}) = \widetilde{D_{k-1}|_a}(a, \overrightarrow{b_1}, \overrightarrow{c_1}) \wedge \widetilde{D_{n-k-1}|_{\neg a}}(a, \overrightarrow{b2}, \overrightarrow{c2})$.

$R_k(a, \ldots) = 0$ unless $dim(a) = k$, arbitrary values if $dim(a) = k$.

$R_k(a, \ldots) \wedge \phi(a, \overrightarrow{d})$ is valid iff $\phi(a, \overrightarrow{d})$ is valid when $dim(a) = k$.

Proof of theorem

By dimensional restriction, let $a, b_1, b_2, b_3, c_1, c_2$ be two dimensional subspaces of H_3 such that for $i, j \in [1 \ldots 3]$ and $k, l \in [1, 2]$ the following hold:

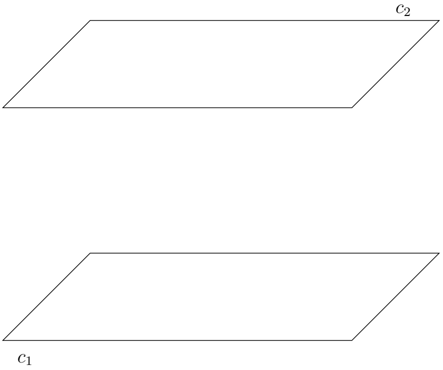
1. $dim(a \wedge b_i) = dim(a \wedge c_k) = 1$,
2. $dim(b_i \wedge b_j) = 1$,
3. $dim(b_i \wedge b_j \wedge a) = 1$,
4. $dim(c_k \wedge c_l) = 1$,
5. $dim(c_k \wedge c_l \wedge a) = 1$,
6. $dim(b_i \wedge c_k \wedge a) = 0$.

Let $\bar{b}_i$ and $\bar{c}_k$ denote the intersections of the b_i and c_k respectively with an affine $\mathbb{C}$-plane a' parallel to a. Then

1. $\bar{b}_1, \bar{b}_2$ and $\bar{b}_3$ are parallel complex lines,
2. $\bar{c}_1$ and $\bar{c}_2$ are parallel complex lines,
3. Each $\bar{b}_i$ and $\bar{c}_k$ intersect in a single point $p_{i,k}$.

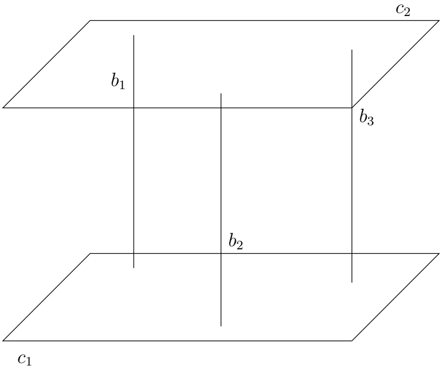
Proof of theorem (continued)

One gets the following:



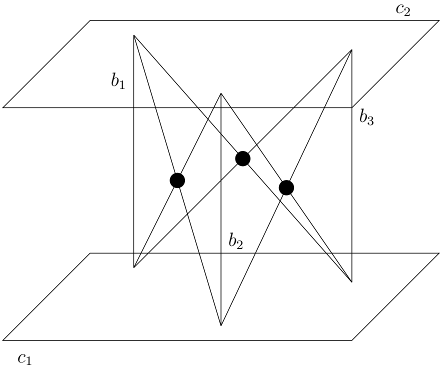
Proof of theorem (continued)

One gets the following:



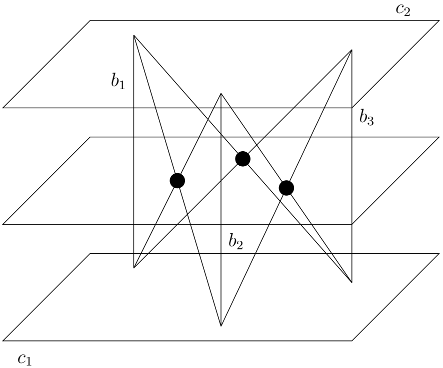
Proof of theorem (continued)

One gets the following:



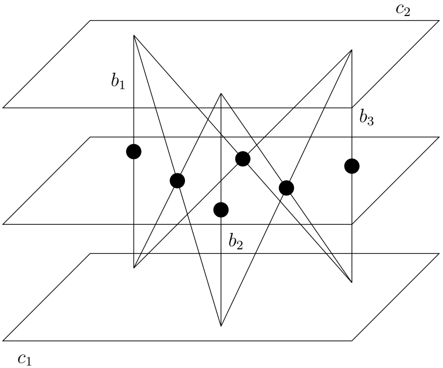
Proof of theorem (continued)

One gets the following:



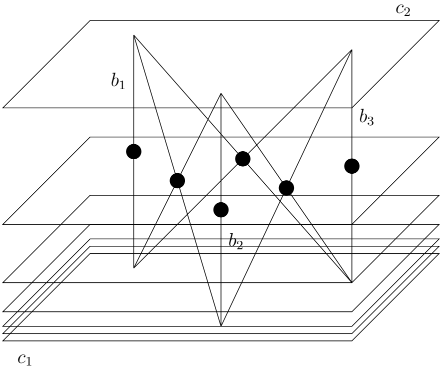
Proof of theorem (continued)

One gets the following:



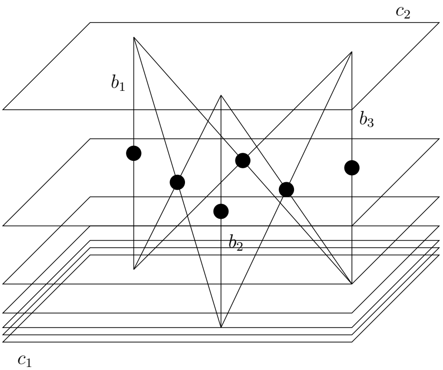
Proof of theorem (continued)

One gets the following:



Proof of theorem (continued)

One gets the following:



Continuing, one gets a dense set of points in R^3.

Proof of theorem (conclusion)

Let ϕ be a formula which is not valid in $QL(L_{H^3})$.
Let

$$\chi(\ldots) = R_2(a, \overrightarrow{d_a}) \bigwedge_i R_2(b_i, \overrightarrow{d_{b_i}}) \bigwedge_k R_2(c_k, \overrightarrow{d_{c_k}})$$
$$\bigwedge_{i \neq j} R_1(b_i \wedge b_j, \overrightarrow{d_{b_i \wedge b_j}}) \bigwedge_{i \neq j} R_1(b_i \wedge b_j \wedge a, \overrightarrow{d_{b_i \wedge b_j \wedge a}})$$
$$\wedge R_1(c_1 \wedge c_2, \overrightarrow{d_{c_1 \wedge c_2}}) \wedge R_1(c_1 \wedge c_2 \wedge a, \overrightarrow{d_{c_1 \wedge c_2 \wedge a}})$$
$$\bigwedge_{i,k} R_0(b_i \wedge c_k \wedge a, \overrightarrow{d_{b_i \wedge c_k \wedge a}}) \wedge \phi(\overrightarrow{e})$$

Then χ is not valid in $QL(L_{H^3})$, but is valid in any non-dense sublattice. By restricting to three dimensional subspaces with appropriate intersection properties, one may construct similar formulas for L_{H^n}.

Thanks!

References:

- J. Michael Dunn, Tobias Hagge, Lawrence Moss, and Zhenghan Wang. Quantum logic as motivated by quantum computing. J. Symbolic Logic 70 (2005), no. 2, 353359. Arxiv:math/0412144.
- Roberto Giuntini, Hector Freytes. Dimensional intervals of modular ortholattices. In preparation.
- Tobias Hagge. QL(Cn) determines n. J. Symbolic Logic 72 (2007), no. 4, 11941196. Arxiv:math/0604373.
- Michael Roddy, René Mayet. n-distributivity in ortholattices. Contributions to general algebra 5, Mai 29 - June 1, 1986.
- Michael Roddy. On the word problem for orthocomplemented modular lattices. Can. J. Math., Vol XLI, No. 6, 1989, pp. 961-1004.