A Classification of Hidden-Variable Properties

Joint work with Adam Brandenburger

Noson S. Yanofsky
Brooklyn College and The Graduate Center, CUNY
noson@sci.brooklyn.cuny.edu

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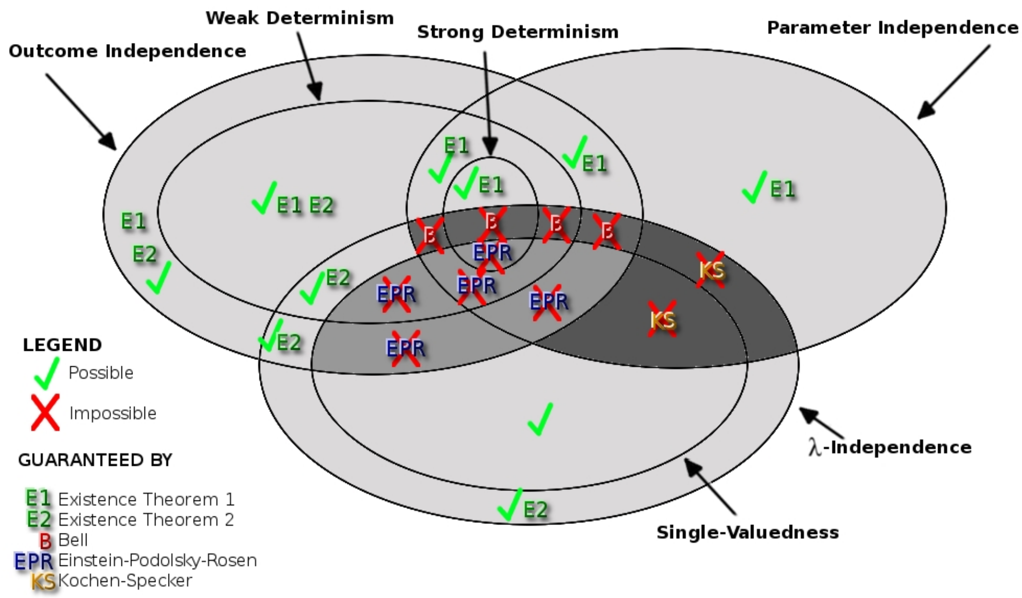
Questions

- What is the relationship between all the different no-go theorems?
- Can the no-go theorems be stated in one single formalism?
- What type of hidden variables *are* possible?
- Does one need to know a lot of physics to understand these no-go theorems?

Goal

Hidden variables are extra components added to try to banish counterintuitive features of quantum mechanics. We start with a quantum-mechanical model and describe various properties that can be asked of a hidden-variable model. We present six such properties and a Venn diagram of how they are related. With two existence theorems and three no-go theorems (EPR, Bell, and Kochen-Specker), we show which properties of empirically equivalent hidden-variable models are possible and which are not. Formally, our treatment relies only on classical probability models, and physical phenomena are used only to motivate which models to choose.

The Big Picture



Outline

1. Introduction
2. The Set-Up
3. Two Existence Theorems
4. EPR
5. Bell
6. Kochen-Specker
7. Other No-Go Theorems

Introduction

Hidden-variable theories aim to remove strange aspects of QM by building more "complete" models (in the terminology of Einstein-Podolsky-Rosen). The completed models should agree with the predictions of QM, but exhibit one or more of the desired properties of: (i) determinism, (ii) locality, and (iii) independence.

Can such models actually be built? The famous "no-go" theorems of QM show that there are severe limitations to what can be done. But it is also true that certain combinations of properties are possible.

Our modest goal is to provide a formal framework in which various properties one might ask of hidden-variable models can be stated and in which various non-existence and existence results can be organized. Almost all–if not all–of the ingredients of what we do are well known to researchers in the area. Our contribution is in putting all the ingredients into one simple setting.

The setting is classical probability spaces. The question is, given a classical probability model, whether there exists an associated hidden-variable model that is empirically equivalent to the first model and that satisfies certain properties. These properties are motivated by the literature on hidden variables in QM. The specific properties we consider–and the relationships among them–can be depicted in the Venn diagram. The diagram contains 21 regions.

Introduction

The main result of the paper is that we can give a complete account of these 21 regions. For 10 of these regions (indicated with checks), it is always possible to find an equivalent hidden-variable model with the properties in question. For the remaining 11 regions (indicated with crosses), this may not be possible. We fill in the regions via two existence results and three non-existence results. The latter three are the famous theorems of Einstein-Podolsky-Rosen (EPR), Bell, and Kochen-Specker.

It is important to understand that, formally, we make no use of physical phenomena. It is an exercise in classical probability theory alone. Of course, the probability spaces we select for the non-existence results are inspired by the physical experiments described in EPR, Bell, and Kocher-Specker. But we hope it is conceptually clarifying to present the hidden-variable question in a purely abstract setting–that is, to show how much follows from the rules of probability theory alone.

Introduction

Some Comments

1. We only look at six properties. There are more.

2. We work with a single probability measure on a single space, where points in the space describe measurements on particles and outcomes of those measurements. An alternative–more conventional–approach would be to use a family of probability measures on a space describing outcomes only, with different probability measures corresponding to different measurements. In fact, all our requirements are stated in terms of conditional probabilities: If such-and-such a measurement is made, then what is the probability of a certain outcome? So, formally, our approach is more parsimonious. Yet, it does add an ingredient at the conceptual level–viz., the existence of a probability measure prior to conditioning on measurements. This measure may be thought of as representing the perspective of a super-observer who observes the experimenters as well as the outcomes of the experiments.

3. We treat only finite probability spaces. This involves a tradeoff. On the one hand, finiteness allows us to avoid all measure-theoretic issues. On the other hand, as an assumption on the space in which a hidden variable lives, finiteness is undoubtedly restrictive.

Introduction

Some Conclusion

Let us offer a comment on its conceptual meaning in QM. The main message of the no-go theorems is that in building a hidden-variable theory, some properties that might be viewed as desirable–at least, a priori–have to be given up. But there is a choice of what to give up. Arguably, it is more a matter of metaphysics than physics as to what choice to make. The point of a formal treatment is to give a precise statement of what the options are. There is a basic three-way tradeoff.

(i) **Determinism** (This comes in a strong or a weak form.) This says that randomness reflects only observer ignorance. Once hidden variables are introduced, there is no residual randomness in the universe.

(ii) **Parameter Independence** This says that when conducting an experiment on a system of particles, the outcome of a measurement on one of the particles does not depend on what measurements are performed on other particles. (The intuitive appeal of this property is that often the particles are widely separated.) This is a way of saying that the universe is local.

(iii) **λ-Independence** This says that the nature of the particles–as determined by the value of a hidden variable–does not depend on the experiment conducted. There is, in this sense, no dependence between the observer and the observed.

The Set-Up

The Models

Formally, we consider a space

$$\Psi = \{a, a', \ldots\} \times \{b, b', \ldots\} \times \{c, c', \ldots\} \cdots \times \{A, A', \ldots\} \times \{B, B', \ldots\} \times \cdots .$$

The variables $A, B, C, \ldots$ are measurements, and the variables a, b, c, $\ldots$ are associated outcomes of measurements. There might be several particles: Ann performs a measurement on her particle, Bob performs a measurements on his particle, $\ldots$. Or, Ψ might describe a case where several measurements are performed on one particle. The definitions to come apply in either case. We take each of the spaces in Ψ to be finite, and suppose that Ψ is a finite product.

Let Λ be a finite space in which a hidden variable λ lives. The overall space is then

$$\Omega = \Psi \times \Lambda.$$

Definition 1 *An* ***empirical model*** *is a pair* (Ψ, q)*, where* q *is a probability measure on* Ψ*. A* ***hidden-variable model*** *is a pair* (Ω, p)*, where* p *is a probability measure on* Ω*.*

The Set-Up

The Models

Definition 2 *An empirical model* (Ψ, q) *and a hidden-variable model* (Ω, p) *are **(empirically) equivalent** if for all* $a, b, c, \ldots, A, B, C, \ldots,$

$$q(A, B, C, \ldots) > 0 \textit{ if and only if } p(A, B, C, \ldots) > 0,$$

and when both are non-zero,

$$q(a, b, c, \ldots | A, B, C, \ldots) = p(a, b, c, \ldots | A, B, C, \ldots).$$

Here, we write "$a, b, c, \ldots$" as a shorthand for the event

$$\{(a, b, c, \ldots)\} \times \{A, A', \ldots\} \times \{B, B', \ldots\} \times \{C, C', \ldots\} \times \cdots$$

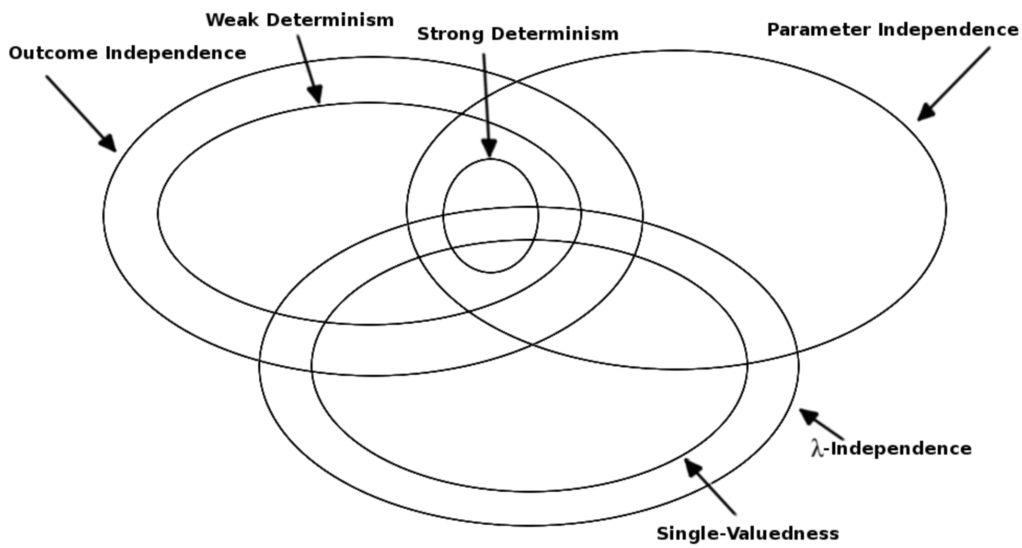
in Ψ, or the event

$$\{(a, b, c, \ldots)\} \times \{A, A', \ldots\} \times \{B, B', \ldots\} \times \{C, C', \ldots\} \times \cdots \times \Lambda$$

in Ω, and similarly for other expressions. We will adopt this shorthand throughout.

The Set-Up

The Venn Diagram of the Properties



The Set-Up

The Properties

Definition 3 *A hidden-variable model* (Ω, p) *satisfies* ***Single-Valuedness*** *if* Λ *is a singleton.*

This condition says that the hidden variable can take on only one value. In effect, this condition doesn't allow hidden variables. We include it because EPR will be usefully formulated this way.

Definition 4 *A hidden-variable model* (Ω, p) *satisfies* λ***-Independence*** *if for all* $A, A', B, B', C, C', \ldots, \lambda$,

$$p(\lambda | A, B, C, \ldots) = p(\lambda | A', B', C', \ldots).$$

The condition says that the process determining the value of the hidden variable is independent of which measurements are chosen.

Remark 1 *If a hidden-variable model satisfies Single-Valuedness, then it satisfies* λ*-Independence.*

The Set-Up

The Properties

Definition 5 *A hidden-variable model* (Ω, p) *satisfies* ***Strong Determinism*** *if, for every* A, λ*, whenever* $p(A, \lambda) > 0$*, there is an* a *such that* $p(a|A, \lambda) = 1$*, and similarly for* B, λ, b*, etc.*

Definition 6 *A hidden-variable model* (Ω, p) *satisfies* ***Weak Determinism*** *if, for every* $A, B, C, \ldots, \lambda$*, whenever* $p(A, B, C, \ldots, \lambda) > 0$*, there is a tuple* $a, b, c, \ldots$ *such that* $p(a, b, c, \ldots |A, B, C, \ldots, \lambda) = 1$*.*

Determinism is a basic condition in the literature. But we are careful to make a distinction between a strong and weak form. We will see that various results are true for one form but false for another. Broadly, the condition is that the hidden variable determines (almost surely) the outcomes of measurements. But, Strong Determinism says this holds measurement-by-measurement, while Weak Determinism says this holds only once all measurements are specified. There is a one-way implication:

Lemma 1 *If a hidden-variable model satisfies Strong Determinism, then it satisfies Weak Determinism.*

The Set-Up

The Properties

Definition 7 *A hidden-variable model* (Ω, p) *satisfies* ***Outcome Independence*** *if for all* $a, b, c, \ldots, A, B, C, \ldots, \lambda$,

$$p(a|A, B, C, \ldots, b, c, \ldots, \lambda) = p(a|A, B, C, \ldots, \lambda), \tag{2.1}$$

and similarly with a *and* b *interchanged, etc.*

Outcome Independence says that conditional on the value of the hidden variable and the measurements undertaken, the outcome of any one measurement is (probabilistically) unaffected by the outcomes of the other measurements.

Lemma 2 *A hidden-variable model* (Ω, p) *satisfies Outcome Independence if and only if for all* $a, b, c, \ldots, A, B, C, \ldots, \lambda$,

$$p(a, b, c, \ldots|A, B, C, \ldots, \lambda) = p(a|A, B, C, \ldots, \lambda) \times p(b|A, B, C, \ldots, \lambda) \times p(c|A, B, C, \ldots$$

Lemma 3 *If a hidden-variable model satisfies Weak Determinism, then it satisfies Outcome Independence.*

The Set-Up

The Properties

Definition 8 *A hidden-variable model* (Ω, p) *satisfies* ***Parameter Independence*** *if for all* $a, A, B, C, \ldots, \lambda$*, whenever* $p(A, B, C, \ldots, \lambda) > 0$*,*

$$p(a|A, B, C, \ldots, \lambda) = p(a|A, \lambda), \tag{2.3}$$

and similarly for $b, A, B, C, \ldots, \lambda$*, etc.*

Parameter Independence says that, conditional on the value of the hidden variable, the outcome of any one measurement depends (probabilistically) only on that measurement and not on the other measurements.

Lemma 4 *If a hidden-variable model satisfies Strong Determinism, then it satisfies Parameter Independence.*

The Set-Up

Derived Properties

Definition 9 *A hidden-variable model* (Ω, p) *satisfies* ***Locality*** *if for all* $a, b, c, \ldots, A, B, C, ,\ldots, \lambda$*, whenever* $p(A, B, C, \ldots, \lambda) > 0$*,*

$$p(a, b, c, \ldots |A, B, C, \ldots, \lambda) = p(a|A, \lambda) \times p(b|B, \lambda) \times p(c|C, \lambda) \times \cdots . \tag{2.4}$$

Proposition 1 *A hidden-variable model satisfies Locality if and only if it satisfies Outcome Independence and Parameter Independence.*

Non-Contextuality, due to Kochen-Specker, is a property of an empirical model. It says that the probability of obtaining a particular outcome of a measurement does not depend on the other measurements performed.

Definition 10 *An empirical model* (Ψ, q) *satisfies* ***Non-Contextuality*** *if for all* $a, A, B, B', C, C', \ldots$*, whenever* $q(A, B, C, \ldots) > 0$ *and* $q(A, B', C', \ldots) > 0$*,*

$$q(a|A, B, C, \ldots) = q(a|A, B', C', \ldots).$$

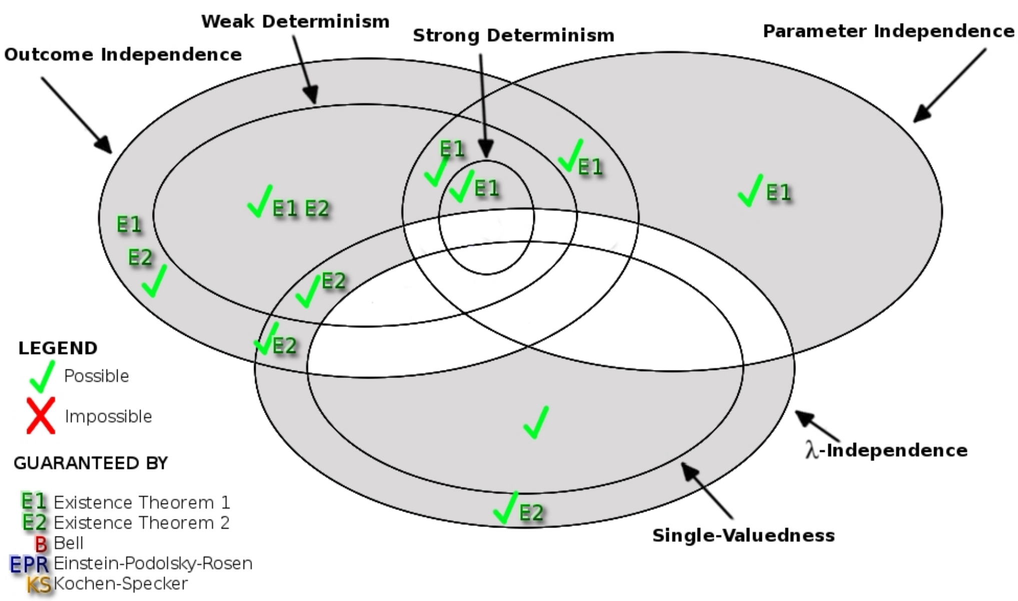
Also, the corresponding conditions must hold for $b, A, A', B, C, C', \ldots$*, etc.*

Proposition 2 *If a hidden-variable model* (Ω, p) *satisfies* λ*-Independence and Parameter Independence, then any equivalent empirical model* (Ψ, q) *satisfies Non-Contextuality.*

Two Existence Theorems

Theorem 1 *Given an empirical model* (Ψ, q)*, there is an equivalent hidden-variable model* (Ω, p) *which satisfies Strong Determinism.*

Theorem 2 *Given an empirical model* (Ψ, q) *with rational probabilities, there is an equivalent hidden-variable model* (Ω, p) *which satisfies Weak Determinism and* λ*-Independence.*



(The region for Single-Valuedness alone also has a check. The existence of an equivalent hidden-variable model satisfying Single-Valuedness alone is immediate–it is essentially just the given empirical model.)

Two Existence Theorems

First Proof

The proof is basically a mathematical trick. We simply take the hidden variable to be all the information possible. This means, in particular, that the hidden variable would have to 'know' the probabilities for different measurements and outcomes. With this huge hidden variable, we can build up the probability measure p from the given measure q. This construction is physically unsatisfying, of course–but not ruled out by the general concept of a hidden variable. It is also rather obvious.

Proof. We give the proof for the case that Ψ is a 4-way product, but the extension to a general (finite) product will be clear. Set

$$\Lambda = \{a, a', \ldots\} \times \{b, b', \ldots\} \times \{A, A', \ldots\} \times \{B, B', \ldots\},$$

and define p in stages, as follows. For any pair A, B, set

$$p(A, B) = q(A, B). \tag{3.1}$$

For any pair A, B, and $\lambda = (\tilde{a}, \tilde{b}, \tilde{A}, \tilde{B})$, set

$$p(\lambda|A, B) = \begin{cases} q(\tilde{a}, \tilde{b}|A, B) & \text{if } \tilde{A} = A \text{ and } \tilde{B} = B, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Two Existence Theorems

First Proof

For pairs a, b and A, B, and $\lambda = (\tilde{a}, \tilde{b}, \tilde{A}, \tilde{B})$, set

$$p(a, b|A, B, \lambda) = \begin{cases} 1 & \text{if } \tilde{a} = a,\ \tilde{b} = b,\ \tilde{A} = A,\ \tilde{B} = B, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

This defines a measure p on Ω using

$$p(a, b, A, B, \lambda) = p(a, b|A, B, \lambda) \times p(\lambda|A, B) \times p(A, B).$$

(Note that $p(\cdot, \cdot|A, B, \lambda)$ is not a measure if $A \neq \tilde{A}$ or $B \neq \tilde{B}$. But then $p(\lambda|A, B) = 0$, so there is no difficulty.)

From (3.1), $p(A, B) > 0$ if and only if $q(A, B) > 0$. If both are positive, then from Figure 3.2,

$$p(a, b|A, B) = 1 \times q(a, b|A, B),$$

so that equivalence is satisfied.

Two Existence Theorems

First Proof

It remains to verify that (Ω, p) satisfies Strong Determinism. So, suppose $p(A, \lambda) > 0$. Writing $\lambda = (\tilde{a}, \tilde{b}, \tilde{A}, \tilde{B})$, we therefore assume $A = \tilde{A}$. Using (3.1)-(3.3),

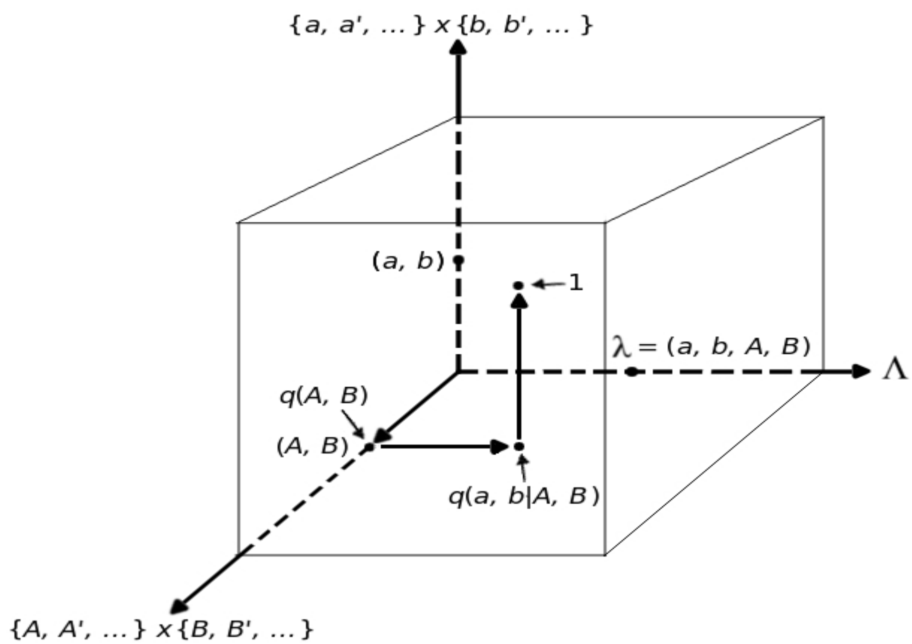
$$p(a, A, \lambda) = p(a, \tilde{A}, \lambda) = \sum_{b', B'} p(a, b', \tilde{A}, B', \lambda) = p(a, \tilde{b}, \tilde{A}, \tilde{B}, \lambda) = q(a, \tilde{b}, \tilde{A}, \tilde{B}) \times \chi_{\{a=\tilde{a}\}},$$

$$p(A, \lambda) = p(\tilde{A}, \lambda) = \sum_{a', b', B'} p(a', b', \tilde{A}, B', \lambda) = p(\tilde{a}, \tilde{b}, \tilde{A}, \tilde{B}, \lambda) = q(\tilde{a}, \tilde{b}, \tilde{A}, \tilde{B}),$$

so that

$$p(a|A, \lambda) = \chi_{\{a=\tilde{a}\}},$$

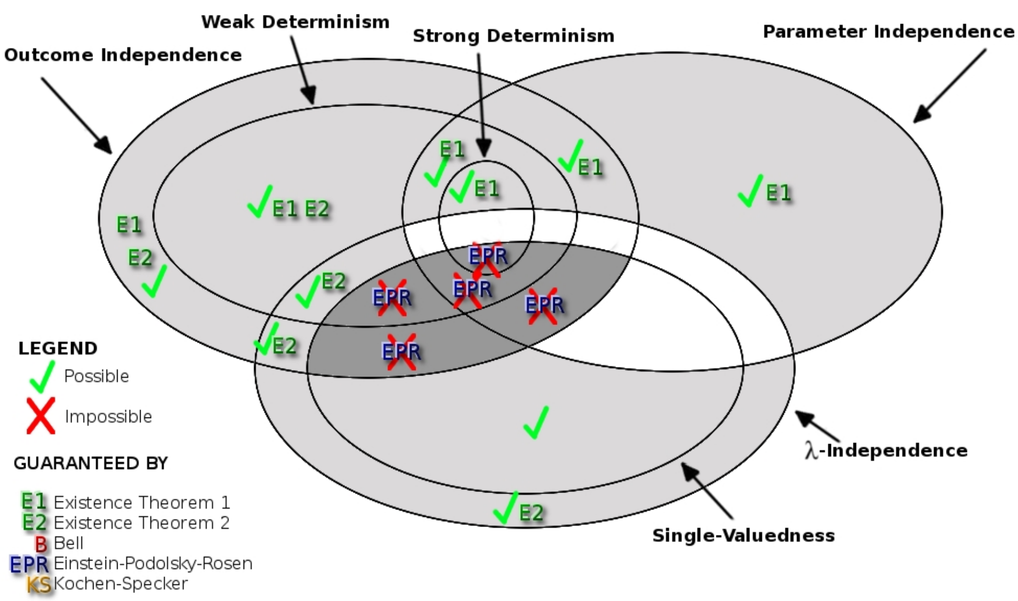
which is Strong Determinism. ■



EPR

The Statement

Theorem 3 *There is an empirical model* (Ψ, q) *for which there is no equivalent hidden-variable model* (Ω, p) *which satisfies Single-Valuedness and Outcome Independence.*



EPR

The Proof

Proof. We let $\Psi = \{+_a, -_a\} \times \{+_b, -_b\} \times \{A\} \times \{B\}$ and define q as

	$+_b$	$-_b$
$+_a$	0	$\frac{1}{2}$
$-_a$	$\frac{1}{2}$	0

$q(\cdot, \cdot|A, B)$

Now suppose, contra hypothesis, there is an equivalent hidden-variable model (Ω, p) satisfying Single-Valuedness and Outcome Independence. Let $\Lambda = \{\lambda\}$. Then we must have

$$p(+_a, -_b|A, B, \lambda) = p(-_a, +_b|A, B, \lambda) = \frac{1}{2},$$

from which

$$p(+_a|A, B, \lambda) = p(+_a, +_b|A, B, \lambda) + p(+_a, -_b|A, B, \lambda) = 0 + \frac{1}{2},$$

and

$$p(+_a|A, B, -_b, \lambda) = \frac{p(+_a, -_b|A, B, \lambda)}{p(-_b|A, B, \lambda)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1,$$

contradicting Outcome Independence. ■

EPR

Comments

The conditions of EPR are tight. We cannot drop Outcome Independence. By the existence theorems we cannot drop Single-Valuedness. Here is a specific construction–for the EPR empirical model–of an equivalent hidden-variable model satisfying Strong Determinism (so, certainly Outcome Independence) and even λ-Independence. Let $\Lambda = \{\lambda^1, \lambda^2\}$, and set $p(\lambda^1) = p(\lambda^2) = \frac{1}{2}$ and

$$p(+_a, -_b|A, B, \lambda^1) = 1, \qquad p(-_a, +_b|A, B, \lambda^2) = 1.$$

Using $p(A, B) = 1$, we see that the stated conditions hold.

At the level presented here, the EPR argument doesn't need any quantum effects. It could be realized entirely classically. Von Neumann gave a nice example of classical action at a distance:

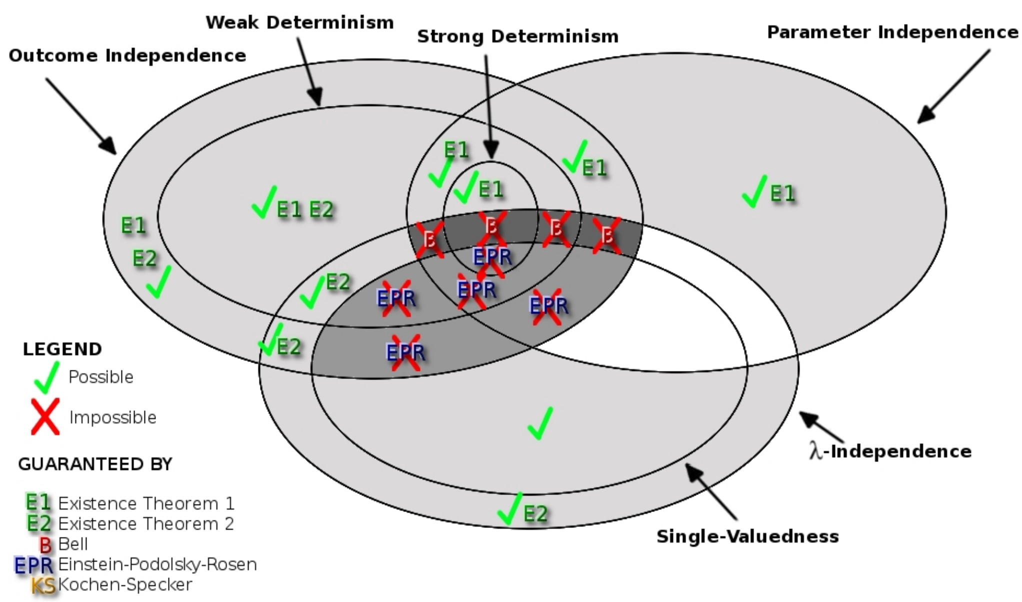
> Let S_1 and S_2 be two boxes. One knows that $1,000,000$ years ago either a white ball had been put into each or a black ball had been placed into each but one does not know which color the balls were. Subsequently one of the boxes (S_1) was buried on Earth, the other (S_2) on Sirius .... Now one digs S_1 on Earth out, opens it and sees: the ball is white. This action on Earth changes instantaneously the S_2 statistic on Sirius.

EPR's conclusion was that the theory of QM needed to be "completed." This leads to the question of whether a construction like the one we just gave is always possible. This then leads to Bell's Theorem.

Bell

The Statement

Theorem 4 *There is an empirical model* (Ψ, q) *for which there is no equivalent hidden-variable model* (Ω, p) *which satisfies* λ*-Independence, Parameter Independence, and Outcome Independence.*



Bell

Set-Up of the Proof

Proof. We let

$$\Psi = \{+_a, -_a\} \times \{+_b, -_b\} \times \{A_1, A_2, A_3\} \times \{B_1, B_2, B_3\},$$

and define q with $q(A_i, B_j) = \frac{1}{9}$ for all i, j and

	$+_b$	$-_b$
$+_a$	0	$\frac{1}{2}$
$-_a$	$\frac{1}{2}$	0

$q(\cdot, \cdot | A_i, B_i)$

	$+_b$	$-_b$
$+_a$	$\frac{3}{8}$	$\frac{1}{8}$
$-_a$	$\frac{1}{8}$	$\frac{3}{8}$

$q(\cdot, \cdot | A_i, B_j)$ for $j \neq i$

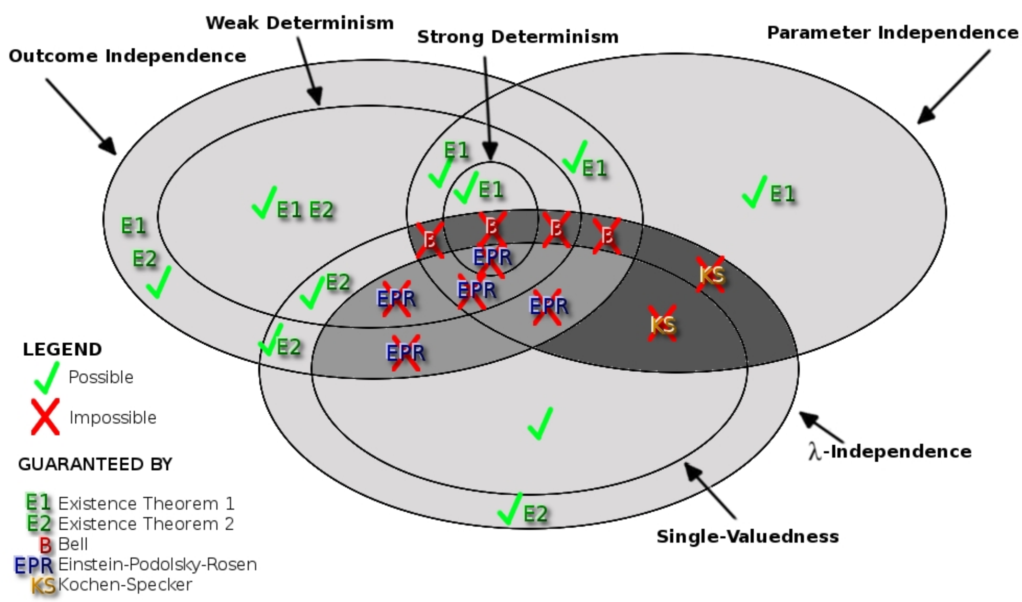
■

Kochen-Specker

The Statement

Theorem 5 *There is an empirical model* (Ψ, q) *for which there is no equivalent hidden-variable model that satisfies* λ*-Independence and Parameter Independence.*

Kochen-Specker demonstrated the existence of a QM model that fails Non-Contextuality: Whether or not their particle has spin in a certain direction is dependent on which other directions are also measured.



Kochen-Specker

The Background for the Proof.

To prove Kochen-Specker in our probabilistic framework, we will need to adapt the concept of exchangeability from probability theory. To give our definition, we consider the special case where the spaces of possible measurements are all the same, as are the spaces of possible outcomes:

$$\{A, \ldots\} = \{B, \ldots\} = \cdots = \{X_1, X_2, \ldots, X_m\},$$

$$\{a, \ldots\} = \{b, \ldots\} = \cdots = \{x_1, x_2, \ldots, x_n\},$$

for integers m, n. We will consider a permutation map π:

$$(A, B, \ldots) \mapsto (\pi(A), \pi(B), \ldots),$$
$$(a, b, \ldots) \mapsto (\pi(a), \pi(b), \ldots).$$

Note that we use π twice (despite the different domains), because we want to consider the same permutation on the two sequences.

Definition 11 *An empirical model* (Ψ, q) *satisfies* ***Exchangeability*** *if for any indices* $i_1, i_2, \ldots \in \{1, 2, \ldots, m\}$ *and* $j_1, j_2, \ldots \in \{1, 2, \ldots, n\}$,

$$q(A = X_{i_1}, B = X_{i_2}, \ldots) > 0 \text{ if and only if } q(\pi(A) = X_{i_1}, \pi(B) = X_{i_2}, \ldots) > 0,$$

for any permutation π*, and when both are non-zero,*

$$q(a = x_{j_1}, b = x_{j_2}, \ldots | A = X_{i_1}, B = X_{i_2}, \ldots) = \\ q(\pi(a) = x_{j_1}, \pi(b) = x_{j_2}, \ldots | \pi(A) = X_{i_1}, \pi(B) = X_{i_2}, \ldots).$$

In words, the requirement is that if we swap any number of measurements, then, as long as we swap the outcomes in the same way, the overall probability is unchanged. Thus, let q be the probability that Ann gets the outcome x_{j_1} and Bob gets the outcome x_{j_2}, if Ann performs measurement X_{i_1} on her particle and Bob performs measurement X_{i_2} on his particle. Let q' be the probability that Ann gets the outcome x_{j_2} and Bob gets the outcome x_{j_1}, if Ann performs measurement X_{i_2} on her particle and Bob performs measurement X_{i_1} on his particle. Exchangeability says that $q' = q$. Likewise, for several measurements on a single particle. This is similar to exchangeability à la de Finetti, though with a conditioning component.

Kochen-Specker

The Proof

Proof.

We follow Cabello, Estebaranz, and Garc a simple treatment which results in the 4×9 array. For various tuples of four orthogonal directions in 4-space (from a total of 18 directions), we ask whether or not the particle has spin in each of these directions. In each case, the answer will be that we get three directions without spin and only one direction with spin.

A	E_1	E_1	E_8	E_8	E_2	E_9	E_{16}	E_{16}	E_{17}
B	E_2	E_5	E_9	E_{11}	E_5	E_{11}	E_{17}	E_{18}	E_{18}
C	E_3	E_6	E_3	E_7	E_{13}	E_{14}	E_4	E_6	E_{13}
D	E_4	E_7	E_{10}	E_{12}	E_{14}	E_{15}	E_{10}	E_{12}	E_{15}

Consider an empirical model where

$$\{A, \ldots\} = \{B, \ldots\} = \{C, \ldots\} = \{D, \ldots\} = \{E_1, E_2, \ldots, E_{18}\},$$

$$\{a, \ldots\} = \{b, \ldots\} = \{c, \ldots\} = \{d, \ldots\} = \{0, 1\}.$$

Exchangeability is assumed to hold, and q assigns positive probability to each of the nine tuples of measurement settings.

Kochen-Specker

The Proof

Finally, for any column, the empirical model has the property that precisely one of the following holds:

$$q(1,0,0,0|E_{i_1},E_{i_2},E_{i_3},E_{i_4}) = 1, \tag{6.1}$$

$$q(0,1,0,0|E_{i_1},E_{i_2},E_{i_3},E_{i_4}) = 1, \tag{6.2}$$

$$q(0,0,1,0|E_{i_1},E_{i_2},E_{i_3},E_{i_4}) = 1, \tag{6.3}$$

$$q(0,0,0,1|E_{i_1},E_{i_2},E_{i_3},E_{i_4}) = 1. \tag{6.4}$$

Now suppose, contra hypothesis, that there is an equivalent hidden-variable model satisfying λ-Independence and Parameter Independence — that is satisfies Non-Contextuality.

Next, take, say, the first column. If

$$q(0,1,0,0|E_1,E_2,E_3,E_4) = 1, \tag{6.5}$$

then certainly

$$q(b=1|E_1,E_2,E_3,E_4) = 1.$$

Kochen-Specker

The Proof

Since $(E_2, E_5, E_{13}, E_{14})$ is non-null, so is $(E_5, E_2, E_{13}, E_{14})$, by Exchangeability. Using Non-Contextuality, we therefore have

$$q(b = 1|E_5, E_2, E_{13}, E_{14}) = 1,$$

from which, by Exchangeability again,

$$q(a = 1|E_2, E_5, E_{13}, E_{14}) = 1.$$

Now use (6.1)-(6.4) to get

$$q(1, 0, 0, 0|E_2, E_5, E_{13}, E_{14}) = 1, \tag{6.6}$$

which tells us about the fifth column.

We therefore get a coloring problem: We try to color precisely one entry in each column–corresponding to the measurement that yields a 1. For example, suppose we color the entry E_2 in the first column–corresponding to (6.5). Then (6.6) tells us that we must color the entry E_2 in the fifth column. However, this is impossible. Each E_i appears an even number of times in the Table, and there is an odd number of columns. Thus, the table cannot be colored. ■

Other No-Go Theorems

1. Gleason's Theorem (We do not work in Hilbert space.)
2. Conway and Kochen (Relaxing Parameter Independence.)
3. Bohmian mechanics (The two existence theorems 'predict' the possibility of Bohmian mechanics–though not its specific content, of course.)